

ON GUARANTEED ESTIMATION OF THE MEAN OF AN AUTOREGRESSIVE PROCESS¹

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This paper considers the problem of sequential point estimation of the mean of a stable autoregressive process with unknown scale and autoregressive parameters. The construction of a sequential procedure makes use of special stopping rules and some modifications of least-squares estimates. The procedure enables estimating the mean with prescribed mean-square accuracy uniformly in nuisance parameters. The uniform asymptotic normality and the asymptotic minimaxity of the sequential estimate are established. The asymptotic formula for the mean sample size is obtained.

1. Introduction. Consider the autoregressive AR(p) process

$$(1.1) \quad x_n - \mu = \lambda_1(x_{n-1} - \mu) + \cdots + \lambda_p(x_{n-p} - \mu) + d\varepsilon_n,$$

where $\{\varepsilon_n\}$ is a sequence of independent identically distributed (i.i.d.) random variables with $E\varepsilon_n = 0$ and $E\varepsilon_n^2 = 1$. The vector of initial values $\zeta = (x_0, \dots, x_{-p+1})'$ is assumed to be stochastically independent of the sequence $\{\varepsilon_n\}$; the prime denotes the transposition. The constant parameters $\mu, \lambda_1, \dots, \lambda_p, d$ are unknown with the vector $\lambda = (\lambda_1, \dots, \lambda_p)'$ belonging to the stability region Λ of the process (1.1); that is, all roots of the characteristic equation

$$(1.2) \quad z^p - \lambda_1 z^{p-1} - \cdots - \lambda_p = 0$$

lie inside the unit circle. The problem is to estimate the parameter μ with preassigned mean-square accuracy in the presence of nuisance parameters $\lambda_1, \dots, \lambda_p, d$ by observations of the process x_n . The mean μ in the model (1.1) is customarily estimated by the sample average

$$(1.3) \quad \hat{\mu}_n = \frac{\sum_{i=1}^n x_i}{n}.$$

Asymptotic properties of estimate (1.3) for model (1.1) and more general stationary processes have been studied in detail [see, e.g., Anderson (1984)]. This estimate has several merits: it is unbiased; there are explicit equations for its variance; it is asymptotically normal. One can note, however, that there are some difficulties connected with using the estimate (1.3) if the parameters

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$\lambda_1, \dots, \lambda_p, d$ are unknown. For example, in the case of an autoregression of the first order, the variance of $\hat{\mu}_n$ is

$$(1.4) \quad E_{\theta}(\hat{\mu}_n - \mu)^2 = \frac{d^2 \sum_{l=1}^n \sum_{k=1}^n \lambda^{|l-k|}}{n^2(1 - \lambda^2)},$$

$\theta = (\mu, \lambda, d)$; E_{θ} denotes the average by the distribution of the process (1.1) with given parameters μ, λ and d . When the parameter λ is unknown, one should use estimate (1.3) very carefully because it is no longer guaranteed in the sense that its mean-square accuracy is unknown. Furthermore, (1.4) implies that

$$(1.5) \quad \sup_{|\lambda| < 1} E_{\theta}(\hat{\mu}_n - \mu)^2 = \infty,$$

that is, for any $n \geq 1$, one cannot guarantee the quality of estimate (1.3) of parameter μ without additional a priori information about parameter λ .

There arises the following natural question: is it possible in principal to estimate the mean of an autoregressive process AR(p) with preassigned mean-square accuracy when parameters $\lambda_1, \dots, \lambda_p$ and d are unknown? A similar problem arose, probably for the first time, when estimating the mean of a normal population with unknown variance. The solution of this problem has been obtained by Stein (1945) who proposed the well-known two-stage sequential procedure. Since then estimation theory has been a fast-growing subject and many fundamental results have been proved [we refer to Govindarajulu (1987), Sen (1982), Siegmund (1985) and Woodroffe (1982)]. We cannot go into detail here and would like only to stress that most of the available literature considers the situation when observations are independent and there is very little known for processes with dependent values. Though the results on sequential estimation for the i.i.d. observation scheme cannot be applied directly to the above-stated problem, they are helpful in realizing that dependence of the mean-square accuracy of estimates of the mean μ on nuisance parameters $\lambda_1, \dots, \lambda_p$ and d can be overcome only by employing sequential procedures. This paper proposes one of the approaches to estimation of the parameter μ in the process (1.1), which makes use of the results on point sequential estimates of parameters λ_i obtained by Konev and Pergamenshchikov (1981). The results on sequential estimation of parameters λ_i in autoregressive schemes and further references can be founded in Borisov and Konev (1977), Lai and Siegmund (1983), Sriram (1988), Konev and Lai (1995) and Dmitrienko and Konev (1994). Different approaches to estimation of the mean of an autoregressive process, taking into account the cost of observation, were proposed by Aras (1990), Fakhre-Zakeri and Lee (1993) and Sriram (1987).

The remainder of this paper is arranged as follows. In Section 2 we consider fixed precision estimators of the parameter μ of model (1.1) with unknown parameters $\lambda_1, \dots, \lambda_p, d$. The sequential plan of estimating μ is a pair $(T(H), \tilde{\mu}(H))$, in which $T(H)$ is a special stopping rule, $\tilde{\mu}(H)$ is a sequential estimate of the parameter μ at the moment $T(H)$ and H is a positive parameter. It is shown (Theorem 2.1) that this plan has the following properties: for

any $0 < L < \infty$ and $0 < H < \infty$,

$$(1.6) \quad \begin{aligned} T(H) < \infty, & \quad P_\theta \text{ a.s.}, \\ \sup_{\theta \in \Theta} E_\theta(\tilde{\mu}(H) - \mu)^2 & \leq \kappa/H, \end{aligned}$$

where $\Theta = \{\theta = (\mu, \lambda, d): |\mu - \mu_0| \leq q, \lambda \in \Lambda, d \neq 0\}$, κ and μ_0 are known constants.

By the second inequality (1.6), parameter μ in model (1.1) can be estimated with any preassigned mean-square accuracy by making use of sequential plan $(T(H), \tilde{\mu}(H))$ for an appropriate value of the procedure parameter H . The construction of sequential estimates in Section 2 is rather cumbersome but there is one excuse for it. By (1.1), information about the parameter μ has to be extracted from observations x_n , satisfying the following equations:

$$(1.7) \quad \begin{aligned} x_n &= m + \lambda_1 x_{n-1} + \cdots + \lambda_p x_{n-p} + d\varepsilon_n, \\ m &= \mu(1 - \lambda_1 - \cdots - \lambda_p). \end{aligned}$$

From here one can see that information about the parameter μ in the sample (x_1, \dots, x_n) may be small if the sum $\lambda_1 + \cdots + \lambda_p$ is close to 1. Therefore even good estimates of parameters $\lambda_1, \dots, \lambda_p$ do not ensure by themselves fixed precision estimation of μ and specific stopping rules are needed.

In Section 3 the asymptotic behavior of the mean duration of the proposed sequential plan has been studied under the assumption that the parameter λ belongs to some compact set in the stability region Λ (Theorem 3.1). We refer the reader to Shiryaev and Spokoiny (1993) for the other results on asymptotic properties of stopping moments associated with sequential estimation of parameters in autoregression. In Section 4 uniform asymptotic normality of the estimate $\tilde{\mu}(H)$ is shown (Theorem 4.1). In Section 5 we prove asymptotic optimality in the minimax sense of the proposed estimate in a certain class of sequential and nonsequential procedures (Theorem 5.1).

2. Fixed-accuracy estimate for parameter μ . We consider model (1.1) assuming that

$$(2.1) \quad \mu \in (\mu_0 - q, \mu_0 + q),$$

where $-\infty < \mu_0 < \infty$, $q > 0$, μ_0 and q are known constants; the distribution function of the noise ε_n has a bounded density $f(x)$, that is,

$$(2.2) \quad f(x) \leq f^*, \quad x \in (-\infty, \infty).$$

The mean μ will be estimated by observations $y_n = x_n - \mu_0$, which according to (1.7) satisfy the equations

$$(2.3) \quad y_n = m + \lambda_1 y_{n-1} + \cdots + \lambda_p y_{n-p} + d\varepsilon_n,$$

$$(2.4) \quad m = (\mu - \mu_0)(1 - 1'_p \lambda);$$

$1_p = (1, \dots, 1)'$ is the vector of dimension p . The basic idea of estimation of μ is to sample until enough information is gathered and to use least-squares estimates of parameters m and λ_i , $i = \overline{1, \dots, p}$. To realize it, we need a system of guaranteed estimates of these parameters. At first we assume that there exists a sequence $\tilde{a}_k = (\tilde{m}_k, \tilde{\lambda}'_k)'$, $k = 1, 2, \dots$ of estimates of the vector $a = (m, \lambda)'$ such that

$$(2.5) \quad \sup_{-\infty < \mu < \infty, \lambda \in \Lambda, d \neq 0} E_\theta \|\tilde{a}_k - a\|^2 \leq e_k,$$

where $\{e_k\}$ is a numerical sequence with

$$(2.6) \quad \rho_1 = \sum_{k \geq 1} e_k < \infty;$$

$\|a\|^2 = a'a$. Then (2.4) may be rewritten as

$$(2.7) \quad \tilde{m}_k = (\mu - \mu_0)\tilde{\beta}_k + \tilde{\Delta}_k,$$

where

$$(2.8) \quad \begin{aligned} \tilde{\beta}_k &= 1 - 1'_p \tilde{\lambda}_k, \\ \tilde{\Delta}_k &= \tilde{m}_k - m + (\mu - \mu_0)1'_p(\tilde{\lambda}_k - \lambda). \end{aligned}$$

Here $\tilde{\Delta}_k$ is an unobservable sequence (noise). Now we apply the sequential least-squares method to estimate the parameter μ from (2.7). Define a stopping rule

$$(2.9) \quad \nu = \nu(H) = \inf \left\{ l \geq 1: \sum_{k=1}^l \tilde{\beta}_k^2 \geq H \right\}, \quad 0 < H < \infty, \\ (\inf \{\emptyset\} = +\infty)$$

and the sequential estimate for the parameter μ at the moment ν (on the set where $\nu < +\infty$)

$$(2.10) \quad \tilde{\mu} = \tilde{\mu}(H) = \sum_{k=1}^{\nu} \tilde{\beta}_k \tilde{m}_k / \sum_{k=1}^{\nu} \tilde{\beta}_k^2 + \mu_0.$$

According to (2.9) the sample volume is measured in terms of accumulated energy of "the useful signal" $\tilde{\beta}_k$.

PROPOSITION 2.1. *Let a sequence of estimates $\tilde{a}_k = (\tilde{m}_k, \tilde{\lambda}'_k)'$, $k = 1, 2, \dots$ of the vector $a = (m, \lambda)'$ in (2.3) satisfy conditions (2.5), (2.6).*

Then for any $0 < q < \infty$, $-\infty < \mu_0 < \infty$ and $0 < H < \infty$, the sequential plan (2.9), (2.10) has the following properties:

$$(2.11) \quad \begin{aligned} \nu(H) &< \infty, & P_\theta\text{-a.s.}, & \lambda \in \Lambda; \\ \sup_{|\mu - \mu_0| \leq q, \lambda \in \Lambda} E_\theta (\tilde{\mu}(H) - \mu)^2 &\leq \kappa/H, & \kappa &= (1 + pq^2)\rho_1. \end{aligned}$$

PROOF. Under conditions (2.5) and (2.6), the estimate \tilde{a}_k is strongly consistent and for all $\mu \in (-\infty, \infty)$ and $\lambda \in \Lambda$,

$$(2.12) \quad \lim_{k \rightarrow \infty} \tilde{\beta}_k = 1 - \lambda_1 - \dots - \lambda_p > 0 \quad \text{a.s.}$$

Therefore the first inequality, (2.11), holds. From (2.9) and (2.10) by the Cauchy–Bunyakovskii inequality, we obtain

$$\begin{aligned} E_\theta(\tilde{\mu}(H) - \mu)^2 &= E_\theta \left(\frac{\sum_{k=1}^v \tilde{\beta}_k \tilde{\Delta}_k}{\sum_{k=1}^v \tilde{\beta}_k^2} \right)^2 \\ &\leq E_\theta \sum_{k=1}^v \tilde{\Delta}_k^2 / H \leq \kappa / H, \quad |\mu - \mu_0| \leq q, \quad \lambda \in \Lambda. \end{aligned}$$

Hence Proposition 2.1. \square

To complete the construction of a sequential estimate for the parameter μ , it remains to choose a sequence of estimates $\{\tilde{a}_k, k \geq 1\}$ satisfying conditions (2.5), (2.6). Note that ordinary least-square estimates of parameters $m, \lambda_1, \dots, \lambda_p$ do not fit, because in the general case their mean-square accuracies depend on the values of these parameters. We will make a choice of estimates \tilde{a}_k with the required properties by exploiting the method of guaranteed estimation of autoregressive parameters proposed by Konev and Pergamshchikov (1981) and its modification for the case of unknown variance [Dmitrienko and Konev (1994)]. This method is realized in two steps.

STEP 1. Let us introduce a system of stopping moments $\{\tau(z), z > 0\}$ by

$$(2.13) \quad \begin{aligned} \tau(z) &= \inf \left\{ l > r(z): \sum_{j=r(z)+1}^l \|Y\|^2 \geq zR(z) \right\}, \quad z > 0, \\ Y_j &= (1, y_{j-1}, \dots, y_{j-p})', \\ R(z) &= \sum_{j=1}^{r(z)} y_j^2 / r(z) + 1, \quad r(z) = 6 + [z/l(z)], \end{aligned}$$

where $[a]$ denotes the whole part of a number a . Here $l: [0, \infty) \rightarrow [0, \infty)$ is a slowly increasing function in the sense that

$$(2.14) \quad \lim_{z \rightarrow \infty} l(z) = \infty, \quad \lim_{z \rightarrow \infty} l(z)/z^\gamma = 0$$

for any $\gamma > 0$ and the function $z/l(z)$ is also assumed to be increasing [for example, $l(z) = \ln(z + e)$].

For each moment $\tau(z)$ we define the modified least-square estimate as

$$(2.15) \quad \alpha(z) = \begin{pmatrix} m(z) \\ \lambda(z) \end{pmatrix} = G^+(z) \left(\sum_{j=r(z)+1}^{\tau(z)-1} Y_j y_j + \alpha(z) Y_{\tau(z)} y_{\tau(z)} \right),$$

where

$$G(z) = \sum_{j=r(z)+1}^{\tau(z)-1} Y_j Y_j' + \alpha(z) Y_{\tau(z)} Y_{\tau(z)}';$$

G^+ denotes the inverse of the matrix G if $\det G > 0$ and $G^+ = 0$ otherwise; $\alpha(z)$ is a weight multiplier which is determined from the equation

$$(2.16) \quad \sum_{j=r(z)+1}^{\tau(z)-1} \|Y_j\|^2 + \alpha(z) \|Y_{\tau(z)}\|^2 = zR(z).$$

Note that $0 < \alpha(z) \leq 1$.

REMARK. The estimate (2.15) for the vector $a = (m, \lambda)'$ is based on the sample (Y_1, \dots, Y_τ) of random size τ . One can see that the initial sample (Y_1, \dots, Y_r) of nonrandom size $r(z)$ is used only to calculate the function $R(z)$ in (2.13). This function is due to the unknown variance of the noise in model (1.1). [see Dmitrienko and Konev (1994)]. The estimate (2.15) coincides with the corresponding LSE of the vector a based on the sample (Y_{r+1}, \dots, Y_τ) if $\alpha(z) = 1$. The weight multiplier $\alpha(z)$ and the function $l(z)$ are needed to ensure some nice properties of the estimates.

The system of estimates (2.13), (2.15) is the basis for construction of the sequential estimate for the vector a . We need also the following function:

$$(2.17) \quad b(z) = \begin{cases} R^{-2}(z)z^{-2}\|G^{-1}(z)\|^{-2}, & \text{if } \det G(z) > 0; \\ 0, & \text{otherwise;} \end{cases}$$

$$\|A\|^2 = \text{tr } AA'.$$

STEP 2. Let $\{c_k\}$ be a nondecreasing sequence of positive constants such that

$$(2.18) \quad \rho_2 = \sum_{k \geq 1} 1/c_k < \infty.$$

By putting $z = c_k$ in (2.13), (2.15) and (2.17), we obtain

$$(2.19) \quad \tau_k = \tau(c_k), \quad a_k = a(c_k), \quad b_k = b(c_k), \quad r_k = r(c_k).$$

Define the sequential estimate of the vector a as a weighted average

$$(2.20) \quad a^*(h) = \frac{\sum_{k=1}^{\sigma} b_k a_k}{\sum_{k=1}^{\sigma} b_k},$$

where σ determines the number of estimates a_k used in the average. Define σ as

$$(2.21) \quad \sigma = \sigma(h) = \inf \left\{ l \geq 1: \sum_{k=1}^l b_k \geq h \right\}, \quad 0 < h < \infty.$$

The sample size which is required for calculating the estimate (2.20) is

$$(2.22) \quad N(h) = \tau(c_\sigma).$$

Further, we shall need the property of guaranteed accuracy of estimates (2.20), (2.22) contained in the following assertion.

PROPOSITION 2.2. *Let process (1.1) be stable, that is, $\lambda \in \Lambda$, and $\{\varepsilon_n\}$ be a sequence of i.i.d. random variables with $E\varepsilon_n = 0$, $E\varepsilon_n^2 = 1$ and the density $f(x)$ satisfying (2.2).*

Then for any $h > 0$

$$(2.23) \quad N(h) < \infty, \quad P_\theta\text{-a.s.},$$

$$(2.24) \quad \sup_{-\infty < \mu < \infty, d \neq 0, \lambda \in \Lambda} E_\theta \|a^*(h) - a\|^2 \leq \rho_3/h,$$

where

$$(2.25) \quad \rho_3 = \pi(f^*)^2 \sum_{k \geq 1} g_k/c_k, \quad g_k = \frac{r_k^2}{(\Gamma(r_k/2))^{2/r_k}(r_k - 2)},$$

$$\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t) dt, \quad a > 0.$$

PROOF. It is well known [see, for example, Anderson (1984)] that for $-\infty < \mu < \infty$ and $\lambda \in \Lambda$, the limiting relationship

$$(2.26) \quad \lim_{N \rightarrow \infty} \frac{\sum_{i=1}^N Y_i Y_i'}{N} = F$$

holds. Here

$$(2.27) \quad F = F(\theta) = \begin{bmatrix} 1 & \vdots & (\mu - \mu_0)1_p' \\ \dots & \vdots & \dots \\ (\mu - \mu_0)1_p & \vdots & (\mu - \mu_0)^2 1_p 1_p' + F_0 \end{bmatrix},$$

$$F_0 = F_0(\theta) = \sum_{k \geq 0} A^k B (A')^k,$$

$$A = \begin{bmatrix} \lambda_1 & \dots & \lambda_p \\ I_{p-1} & ; & 0 \end{bmatrix}, \quad B = \begin{bmatrix} d^2 & 0 \dots 0 \\ \dots & \dots \\ 0 & 0 \dots 0 \end{bmatrix};$$

I_p is a unit matrix of order p ; F is a positive definite matrix. From (2.17), (2.19) and (2.26) we have

$$(2.28) \quad \sum_{i \geq 1} \|Y_i\|^2 = \infty, \quad P_\theta\text{-a.s.},$$

$$\lim_{z \rightarrow \infty} b(z) = b_0 = b_0(\theta) = (\text{tr } F \|F^{-1}\|)^{-2}, \quad P_\theta\text{-a.s.}$$

From here and (2.19), (2.21) and (2.22), we obtain (2.23). From (2.15), (2.19) and (2.3) it follows that

$$(2.29) \quad \begin{aligned} \|\alpha^* - \alpha\|^2 &\leq \left(\sum_{k=1}^{\sigma} b_k \|a(c_k) - \alpha\|/h \right)^2 \\ &\leq \sum_{k=1}^{\sigma} b_k \|a(c_k) - \alpha\|^2 / h \leq d^2 \sum_{k=1}^{\infty} \|M(c_k)\|^2 / c_k^2 R^2(c_k), \end{aligned}$$

where

$$(2.30) \quad M(z) = \sum_{j=r(z)+1}^{\tau(z)-1} Y_j \varepsilon_j + \alpha(z) Y_{\tau(z)} \varepsilon_{\tau(z)}.$$

Taking into account (2.16) we have

$$\begin{aligned} E_{\theta}(\|M(z)\|^2 | Y_1, \dots, Y_{r(z)+1}) \\ \leq E_{\theta} \left(\sum_{j=r(z)+1}^{\tau(z)-1} \|Y_j\|^2 + \alpha(z) \|Y_{\tau(z)}\|^2 \middle| Y_1, \dots, Y_{r(z)+1} \right) = zR(z). \end{aligned}$$

Combining this estimate with (2.28) we obtain

$$(2.31) \quad \begin{aligned} E_{\theta} \|\alpha^* - \alpha\|^2 &\leq \frac{1}{h} \sum_{k \geq 1} \frac{1}{c_k} E_{\theta} \frac{d^2}{R(c_k)} \\ &\leq \frac{1}{h} \sum_{k \geq 1} \frac{r_k}{c_k} E_{\theta} \frac{d^2}{\sum_{j=1}^{r_k} y_j^2} \leq \frac{\pi(f^*)^2 \sum_{k \geq 1} g_k / c_k}{h}. \end{aligned}$$

The last inequality is due to condition (2.2) and can be proved along the lines of Proposition 2 in Dmitrienko and Konev (1994). Hence Proposition 2.2. \square

REMARK. The sequence g_k in (2.25) is convergent:

$$\lim_{k \rightarrow \infty} g_k = 2e$$

and is bounded:

$$(2.32) \quad \max_{k \geq 1} g_k \leq g_0, \quad g_0 = 3 \exp(1 + 1/4\pi e).$$

Now we can define the sequence of estimates $\{a_k\}$ satisfying conditions (2.5), (2.6). Let

$$(2.33) \quad \tilde{a}_k = (\tilde{m}_k, \tilde{\lambda}'_k)' = \alpha^*(h_k) = (m^*(h_k), (\lambda^*(h_k))')',$$

where $\{h_k\}$ is a nondecreasing sequence of positive numbers such that

$$(2.34) \quad \rho_4 = \sum_{k \geq 1} 1/h_k < \infty.$$

The sequence of estimates (2.33) satisfies (2.5), (2.6) with

$$(2.35) \quad e_k = \rho_3 / h_k.$$

If one uses the estimate of the mean μ of the process (1.1), defined by (2.9), (2.10) and (2.33), then the total duration of the estimation procedure is

$$(2.36) \quad T(H) = N(h_{\nu(H)}) = \tau(c_{\sigma(H)}), \quad H > 0.$$

THEOREM 2.1. *Let process (1.1) be stable and $\{\varepsilon_n\}$ be a sequence of i.i.d. random variables with density $f(x)$ satisfying (2.2) and $E\varepsilon_n = 0$, $E\varepsilon_n^2 = 1$. Then the sequential plan $(T(H), \tilde{\mu}(H))$ defined by (2.9), (2.10), (2.33) and (2.36) possesses the properties: for any $0 < H < \infty$ and $0 < q < \infty$,*

$$(2.37) \quad T(H) < \infty, \quad P_{\theta}\text{-a.s.}, \quad \lambda \in \Lambda,$$

$$(2.38) \quad \sup_{\theta \in \Theta} E_{\theta}(\tilde{\mu}(H) - \mu)^2 \leq \kappa_1/H,$$

where $\Theta = \{\theta = (\mu, \lambda', d): |\mu - \mu_0| \leq q, \lambda \in \Lambda, d \neq 0\}$, $\kappa_1 = (1 + pq^2)\rho_3\rho_4$.

The assertion of Theorem 2.1 follows directly from Propositions 2.1, 2.2.

3. The mean duration of the procedure. In this section we will study asymptotic properties of the moment (2.36) as $H \rightarrow \infty$. In addition to conditions imposed in Sections 1 and 2 on the distribution of the noise ε_n in (1.1), we assume that

$$(3.1) \quad E\varepsilon_1^8 < \infty, \quad E\|\zeta\|^8 < \infty.$$

Also the sequences $\{c_k\}$ and $\{h_k\}$ will be chosen as functions of the parameter H . Let

$$(3.2) \quad c_k = c_k(H) = \begin{cases} H, & k \leq n_0(H); \\ c_k^*, & k > n_0(H); \end{cases}$$

$$h_k = h_k(H) = c_k(H);$$

where $c_k^* = k^{1+\delta}$, $0 < \delta < \sqrt{2}-1$; $n_0(H) = [Hl(H)]$ and the function $l(H)$ satisfies (2.14). Such choice of these sequences is associated with the asymptotic behavior of the basic least-squares estimates of the vector a and their sequential modifications (2.15). As a result, the two-step procedure (2.20), (2.22) transforms, as will be shown below, into the sequential estimate (2.15) with $z = H$ and for large H ,

$$a^*(H) \approx a(H), \quad T(H) \approx \tau(H).$$

The following theorem assesses the performance of the procedure duration $T(H)$.

THEOREM 3.1. *Under the conditions of Theorem 2.1 and (3.1), the stopping moment (2.36) satisfies the limiting relationship*

$$(3.3) \quad \lim_{H \rightarrow \infty} \sup_{\theta \in \mathcal{K}} E_{\theta} \left| \frac{T(H)}{H} - \frac{d_1}{\text{tr } F} \right| = 0,$$

for any $-\infty < \mu_0 < \infty$, and $q > 0$, where \mathcal{K} is an arbitrary compact set in the region

$$(3.4) \quad \begin{aligned} \Theta &= \{\theta = (\mu, \lambda', d): |\mu - \mu_0| \leq q, \lambda \in \Lambda, d \neq 0\}; \\ d_1 &= 1 + \langle F \rangle_{22}. \end{aligned}$$

F is a positive definite matrix defined by (2.27); $\langle F \rangle_{ij}$ is the (i, j) th element of the matrix F .

REMARK. By Theorem 3.1 for any fixed $\mathcal{K} \subset \Theta$ and sufficiently large H ,

$$E_\theta T(H) \approx \frac{d_1 H}{\text{tr } F}.$$

To prove Theorem 3.1 we need several auxiliary propositions. First we will establish some properties of stopping moments $\sigma(H)$ and $\nu(H)$. The moment $\sigma(H)$, defined in (2.21), depends on the function $b(z)$, satisfying (2.28). The following lemma gives the rate of convergence of $b(z)$ to its limit.

LEMMA 3.1. Under the conditions of Theorem 3.1,

$$(3.5) \quad \limsup_{z \rightarrow \infty} \sup_{\theta \in \mathcal{K}} z^2 E_\theta (b(z) - b_0(\theta))^4 < \infty.$$

Proof of Lemma 3.1 is given in the Appendix.

PROPOSITION 3.1. Let the conditions of Theorem 3.1 hold. Then for sufficiently large H and for all $m \geq n_0(H)$,

$$(3.6) \quad \sup_{\theta \in \mathcal{K}} P_\theta(\sigma(H) > m) \leq Ll(H)(n_0^3(H)/Hm^4 + m^{-2-\delta}),$$

where $0 < L < \infty$, $l(H)$ is defined in (2.14), δ is the same as in (3.2).

PROOF. By definition of $\sigma(H)$ and (2.28) we have

$$(3.7) \quad \begin{aligned} P_\theta(\sigma(H) > m) &\leq P_\theta\left(\sum_{k=1}^m b(c_k) < H\right) \\ &\leq P_\theta\left(\sum_{k=1}^m |b(c_k) - b_0(\theta)| > mb_* - H\right), \end{aligned}$$

where

$$b_* = \inf_{\theta \in \mathcal{K}} b_0(\theta).$$

Further note that for sufficiently large H and all $m \geq n_0(H)$,

$$mb_*/2 - H \geq 0.$$

This inequality follows from (3.2) and (2.14). From here and (3.7), with the help of the Chebyshev and Hölder inequalities, we obtain

$$\begin{aligned} P_\theta(\sigma(H) > m) &\leq P_\theta\left(\sum_{k=1}^m |b(c_k) - b_0(\theta)| > \frac{mb_*}{2}\right) \\ &\leq \frac{16E_\theta(\sum_{k=1}^m |b(c_k) - b_0(\theta)|)^4}{(b_*m)^4} \\ &\leq \frac{16}{(b_*m)^4} \left(\sum_{k=1}^m \left(\frac{1}{c_k}\right)^{1/3}\right)^3 \sum_{k=1}^m \frac{E_\theta(\sqrt{c_k}|b(c_k) - b_0(\theta)|)^4}{c_k}. \end{aligned}$$

By applying Lemma 3.1 and taking into account the choice of the sequence c_k in (3.2) we come to (3.6). Hence Proposition 3.1. \square

In order to prove a similar property for the moment $\nu(H)$, we need the following result.

LEMMA 3.2. *Under the conditions of Theorem 3.1 the fourth moment of the deviation of the estimate (2.20) satisfies the inequality*

$$(3.8) \quad \sup_{\theta \in \mathcal{X}} E_\theta \|a^*(h) - a\|^4 \leq Ll^2(H)/h^2$$

for all $h > 0$ and sufficiently large H .

The proof of Lemma 3.2 is given in the Appendix.

PROPOSITION 3.2. *Under the conditions of Theorem 3.1 for sufficiently large H and all $m \geq n_0(H)$,*

$$(3.9) \quad \sup_{\theta \in \mathcal{X}} P_\theta\{\nu(H) > m\} \leq Ll^3(H)(n_0^3(H)/m^4H + m^{-2-\delta}).$$

PROOF. From the definition of the stopping moment $\nu(H)$ in (2.9),

$$\begin{aligned} P_\theta\{\nu(H) > m\} &= P_\theta\left\{\sum_{k=1}^m \tilde{\beta}_k^2 < H\right\} = P_\theta\left\{m\beta^2 + \sum_{k=1}^m (\tilde{\beta}_k^2 - \beta^2) < H\right\} \\ &= P_\theta\left\{m\beta^2 + \sum_{k=1}^m (\tilde{\beta}_k - \beta)^2 + 2\beta \sum_{k=1}^m (\tilde{\beta}_k - \beta) < H\right\} \\ &\leq P_\theta\left\{m\beta^2 + 2\beta \sum_{k=1}^m (\tilde{\beta}_k - \beta) < H\right\} \\ &\leq P_\theta\left\{2\beta \sum_{k=1}^m |1'_p(\tilde{\lambda}_k - \lambda)| > m\beta_* - H\right\}, \end{aligned}$$

where

$$\beta_* = \inf_{\lambda \in \mathcal{X}_1} \beta^2(\lambda), \quad \mathcal{X}_1 = \{\lambda \in \Lambda: \theta = (\theta, \lambda', d) \in \mathcal{X}\};$$

$\beta(\lambda)$ and $\tilde{\lambda}_k$ are defined by (2.12) and (2.33). Note that in view of (2.8) and (2.14) for sufficiently large H and all $m \geq n_0(H)$,

$$m\beta_* - H \geq m\beta_*/2.$$

Therefore we have

$$\begin{aligned} P_\theta\{\nu(H) > m\} &\leq P_\theta\left\{2\sqrt{p} \sum_{k=1}^m \|\tilde{\lambda}_k - \lambda\| > m\beta_*/2\right\} \\ &\leq P_\theta\left\{4\sqrt{p} \sum_{k=1}^m \|a^*(h_k) - a\| > m\beta_*\right\}. \end{aligned}$$

By applying the Chebyshev inequality, Lemma 3.1 and substituting (3.2) we obtain (3.9). Hence Proposition 3.2. \square

Further, we need the following technical result.

LEMMA 3.3. *Let y_n be defined by (2.3). Then for any $n \geq 1$ and all $\theta \in \mathcal{K}$,*

$$(3.10) \quad \sum_{k=1}^n y_k^2 \leq L \left(\sum_{k=1}^n \varepsilon_k^2 + n + \|\zeta\|^2 \right),$$

where L is some constant which does not depend on θ .

Proof of Lemma 3.3 is given in the Appendix.

PROPOSITION 3.3. *Under the conditions of Theorem 3.1 for some $\gamma > 0$,*

$$(3.11) \quad \limsup_{H \rightarrow \infty} \sup_{\theta \in \mathcal{K}} E_\theta(T(H)/H)^{(1+\gamma)} < \infty.$$

PROOF. For notational simplicity we shall write $v = v(H)$ instead of $c_{\sigma(h_{\nu(H)})}(H)$. By (2.13),

$$(3.12) \quad \tau(H) \leq HR(H) + r(H) + 1$$

and, hence, by (2.36),

$$T(H) \leq vR(v) + r(v).$$

Therefore it suffices to show that

$$(3.13) \quad \limsup_{H \rightarrow \infty} \sup_{\theta \in \mathcal{K}} E_\theta(r(v)/H)^{(1+\gamma)} < \infty,$$

$$(3.14) \quad \limsup_{H \rightarrow \infty} \sup_{\theta \in \mathcal{K}} E_\theta(vR(v)/H)^{(1+\gamma)} < \infty$$

for some $\gamma > 0$.

Since $r(z) \leq z + 6$, then the inequality (3.13) holds if

$$(3.15) \quad \limsup_{H \rightarrow \infty} \sup_{\theta \in \mathcal{K}} E_\theta(v(H)/H)^{(1+\gamma)} < \infty.$$

The proof of this inequality is given in the Appendix. Consider (3.14). By definition of $R(v)$ in (2.13) and Lemma 3.2,

$$vR(v) \leq L \left(v \sum_{j=1}^{r(v)} \varepsilon_j^2 / r(v) + v + \|\zeta\|^2 v / r(v) \right) \leq L \left(l(v) \sum_{j=1}^{r(v)} \varepsilon_j^2 + v + \|\zeta\|^2 v / r(v) \right).$$

Therefore it suffices to verify that

$$(3.16) \quad \limsup_{H \rightarrow \infty} \sup_{\theta \in \mathcal{X}} E_\theta \left(l(v) \sum_{j=1}^{r(v)} \varepsilon_j^2 / H \right)^{(1+\gamma)} < \infty,$$

$$(3.17) \quad \limsup_{H \rightarrow \infty} \sup_{\theta \in \mathcal{X}} E_\theta (l(v) \|\zeta\|^2 / H)^{(1+\gamma)} < \infty.$$

We have

$$(3.18) \quad \begin{aligned} E_\theta \left(l(v) \sum_{j=1}^{r(v)} \varepsilon_j^2 \right)^{(1+\gamma)} &= (1 + \gamma) \int_0^\infty t^\gamma P_\theta \left\{ l(v) \sum_{j=1}^{r(v)} \varepsilon_j^2 > t \right\} dt = I_1 + I_2; \\ I_1 &= (1 + \gamma) \int_0^\infty t^\gamma P_\theta \left\{ l(v) \sum_{j=1}^{r(v)} \varepsilon_j^2 > t, v \leq t/2 \right\} dt; \\ I_2 &= (1 + \gamma) \int_0^\infty t^\gamma P_\theta \{ v > t/2 \} dt. \end{aligned}$$

Taking into account (2.14), we obtain

$$\begin{aligned} P_\theta \left\{ l(v) \sum_{j=1}^{r(v)} \varepsilon_j^2 > t, v \leq t/2 \right\} &\leq P_\theta \left\{ l(t/2) \sum_{j=1}^{r(t/2)} \varepsilon_j^2 > t \right\} \\ &= P_\theta \left\{ r(t/2)l(t/2) + l(t/2) \sum_{j=1}^{r(t/2)} (\varepsilon_j^2 - 1) > t \right\} \\ &\leq P_\theta \left\{ t/2 + 6l(t/2) + l(t/2) \sum_{j=1}^{r(t/2)} (\varepsilon_j^2 - 1) > t \right\} \\ &\leq \frac{2^4 l^4(t/2) E_\theta (\sum_{j=1}^{r(t/2)} (\varepsilon_j^2 - 1))^4}{(t - 12l(t/2))^4} \\ &\leq \frac{2^4 l^4(t/2) r^2(t/2) E_\theta (\varepsilon_1^2 - 1)^4}{(t - 12l(t/2))^4} \\ &\leq \frac{2^4 l^2(t/2) (t + 6l(t/2))^2 E_\theta (\varepsilon_1^2 - 1)^4}{(t - 12l(t/2))^4}. \end{aligned}$$

Let $t_0 > 0$ be such that

$$t - 12l(t/2) \geq t/2$$

for all $t \geq t_0$. Then I_1 can be estimated as

$$(3.19) \quad I_1 \leq (1 + \gamma) \int_0^{t_0} t^\gamma dt + L \int_{t_0}^\infty \frac{l^2(t)(t^{2+\gamma} + t^\gamma l^2(t))}{t^4} dt < \infty$$

for $0 < \gamma < 1$. Further we note that

$$(3.20) \quad I_2 = 2^{1+\gamma} E_\theta v^{1+\gamma}.$$

By making use of estimates (3.19), (3.20) in (3.18) and applying (3.15) we obtain inequality (3.16). The next inequality (3.17) follows from conditions (2.14) and (3.1). Therefore (3.14) holds. Hence Proposition 3.3. \square

This completes the proof of the auxiliary propositions.

PROOF OF THEOREM 3.1. By Proposition 3.3, the family $\{T(H)/H, H > 0\}$ is uniformly integrable. Therefore it suffices to show that for any $\delta > 0$,

$$\limsup_{H \rightarrow \infty} \sup_{\theta \in \mathcal{K}} P_\theta \{|T(H)/H - d_1/\text{tr } F| > \delta\}.$$

From the definition of $T(H)$ in (2.36) and from (3.1) it follows that

$$\begin{aligned} P_\theta \{|T(H)/H - d_1/\text{tr } F| > \delta\} &\leq P_\theta \{|\tau(H)/H - d_1/\text{tr } F| > \delta\} \\ &\quad + P_\theta \{\sigma(H) > n_0(H)\} + P_\theta \{\nu(H) > n_0(H)\}. \end{aligned}$$

In view of Propositions 3.1 and 3.2, it remains to verify that

$$(3.21) \quad \limsup_{H \rightarrow \infty} \sup_{\theta \in \mathcal{K}} P_\theta \{|\tau(H)/H - d_1/\text{tr } F| > \delta\} = 0.$$

This relationship is proved in the Appendix. Hence Theorem 3.1. \square

4. Uniform asymptotic normality of estimates (2.10), (2.33). The following is a key result for studying the asymptotic distribution of the sequential estimates $\tilde{\mu}(H)$.

PROPOSITION 4.1 [Lai and Siegmund (1983)]. *Let $x_n, \varepsilon_n, n = 0, 1, \dots$ be random variables adapted to the increasing sequence of σ -algebras $\mathcal{F}_n, n = 0, 1, \dots$. Let $\{P_\theta, \theta \in \Theta\}$ be a family of probability measures such that under every P_θ the following hold:*

- (i) $\varepsilon_1, \varepsilon_2, \dots$ are i.i.d. with $E_\theta \varepsilon_1 = 0, E_\theta \varepsilon_1^2 = 1$;
- (ii) $\sup_{\theta \in \Theta} E_\theta \{\varepsilon_1^2, |\varepsilon_1| > a\} \rightarrow 0, \text{ as } a \rightarrow \infty$;
- (iii) ε_n is independent of \mathcal{F}_{n-1} for each $n \geq 1$;
- (iv) $P_\theta \{\sum_{i \geq 1} x_i^2 = \infty\} = 1$;
- (v) $\sup_{\theta \in \Theta} P_\theta \{x_n^2 > a\} \rightarrow 0 \text{ as } a \rightarrow \infty \text{ for each } n \geq 0$;
- (vi) for each $\delta > 0$

$$\lim_{m \rightarrow \infty} \sup_{\theta \in \Theta} P_\theta \left\{ x_n^2 \geq \delta \sum_{i=1}^{n-1} x_i^2 \text{ for some } n \geq m \right\} = 0.$$

For $c > 0$ let

$$N_c = \inf \left\{ n \geq 1: \sum_{i=1}^n x_{i-1}^2 \geq c \right\}, \quad \inf \{\emptyset\} = \infty.$$

Then uniformly in $\theta \in \Theta$ and $-\infty < t < \infty$,

$$P_\theta \left\{ c^{-1/2} \sum_{i=1}^{N_c} x_{i-1} \varepsilon_i \leq t \right\} \rightarrow \Phi(t) \quad \text{as } c \rightarrow \infty,$$

where $\Phi(t)$ is the standard normal distribution function.

First we shall prove asymptotic normality of the sequential least-squares estimate (2.15) of the vector of unknown parameters $a = (m, \lambda)'$.

PROPOSITION 4.2. *Let process (1.1) be stable, $\{\varepsilon_n\}$ be i.i.d. with $E\varepsilon_1 = 0$, $E\varepsilon_1^2 = 1$, $E\varepsilon_1^8 < \infty$ and (2.2) be satisfied. Then*

$$(4.1) \quad \lim_{H \rightarrow \infty} \sup_{\theta \in \mathcal{K}} |P_\theta \{ \sqrt{H} V'(a(H) - a) \leq t \} - \Phi(t/d_2(V))| = 0,$$

where \mathcal{K} is an arbitrary compact set in the region Θ , $V \in R^{p+1}$; $d_2^2(V) = d^2 V' F^{-1} V \operatorname{tr} F/d_1$.

The proof of Proposition 4.2 is given in the Appendix.

Now we can show uniform asymptotic normality of the estimate (2.10), (2.33).

THEOREM 4.1. *Let the conditions of Theorem 2.1 and (3.3) hold and the sequences c_k and h_k be defined by (3.1). Then*

$$(4.2) \quad \lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \sup_{-\infty < t < \infty} \left| P_\theta \left\{ \sqrt{E_\theta T(H)} (\tilde{\mu}(H) - \mu) \leq t \right\} - \Phi((1 - \lambda' 1_p)t/d) \right| = 0;$$

\mathcal{K} is an arbitrary compact set in Θ .

PROOF. By (3.1) the estimate $\tilde{\mu}$, defined by (2.10) and (2.33), can be written on the intersection of sets

$$(4.3) \quad \Gamma(H) = \{ \sigma(H) \leq n_0(H) \} \cap \{ \nu(H) \leq n_0(H) \}$$

as follows:

$$\tilde{\mu}(H) = \frac{m(H)}{1 - 1'_p \lambda(H)}.$$

From here and (2.15) we have on Γ :

$$(4.4) \quad \sqrt{E_\theta T(H)} (\tilde{\mu}(H) - \mu) = \sqrt{E_\theta T(H)} \frac{V'_0(a(H) - a)}{1 - 1'_p \lambda(H)},$$

where $V_0 = (1, (\mu - \mu_0) 1'_p)'$.

Propositions 3.1 and 3.2 [under $m = n_0(H)$] imply

$$(4.5) \quad \lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} P_\theta \{ \Gamma^c(H) \} = 0$$

for any compact set $\mathcal{K} \in \Theta$. By applying Proposition 4.2 and Theorem 3.1 to (4.4) with $V = V_0$ and noting that $d_2^2(V_0) = d^2$, we obtain (4.2). Hence Theorem 4.1. \square

5. Asymptotic minimaxity of the sequential procedure. In this section we shall show that the estimate of the mean μ of the process (1.1) defined by (2.10), (2.33) is optimal in a minimax sense in some class of sequential and nonsequential estimation procedures. In addition to the conditions imposed on the distribution of the noise in (1.1), (2.2) and (3.1), we assume that it has a piecewise continuously differentiable density $f(x)$ such that

$$(5.1) \quad J(f) = \int_{-\infty}^{\infty} \frac{(f')^2}{f(x)} dx < \infty.$$

Let \mathcal{P} denote the class of all noise distributions satisfying these conditions. First by making use of the sequential estimate $(T(H), \tilde{\mu}(H))$ defined by (2.9), (2.10) and (2.33), we construct a sequential procedure $\tilde{\mu}$. Let $\{H_n\}$ be some nondecreasing sequences of positive numbers H_n with $\lim H_n = +\infty$. Define the sequential procedure as

$$(5.2) \quad \tilde{\mu} = \{ (T(H_n), \tilde{\mu}(H_n)), n \geq 1 \}.$$

The performance of this procedures depends both on the unknown parameter μ and the nuisance parameters λ, d . For each number $\mu_1 \in (\mu_0 - q, \mu_0 + q)$ we introduce a special class of estimation procedures \mathcal{M} . Let $\hat{\mu} = \{ (t_n, \hat{\mu}_n), n \geq 1 \}$ be an arbitrary estimation procedure, where t_n is a Markovian moment with respect to the process $\{x_k\}$, and $\tilde{\mu}_n$ is a Borelian function of the observations $x_0, \dots, x_{t_n}, n \geq 1$. Denote by \mathcal{M} the class of all procedures $\hat{\mu}$ satisfying the following condition: there exists an interval $U(\mu_1) \subset (\mu_0 - q, \mu_0 + q)$ such that

$$(5.3) \quad E_\theta t_n \leq E_\theta T(n)$$

for all $\theta \in \{ \theta = (\mu, \lambda', d): \mu \in U(\mu_1), \Lambda, d \neq 0 \}$ under any noise distribution from the class \mathcal{P} . For each procedure $\hat{\mu} \in \mathcal{M}$ we define the risk as

$$(5.4) \quad R_{\mu_1}(\hat{\mu}) = \sup_{d \neq 0} \sup_{\lambda \in \Lambda} \sup_{f \in \mathcal{P}} \limsup_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{|\mu - \mu_1| < \delta} E_\theta |\phi(n)(\tilde{\mu}_n - \mu)|^\gamma,$$

where $0 < \gamma \leq 2$; $\phi(n) = w(E_{\theta_1} T(n))$,

$$(5.5) \quad w(x) = \sqrt{x}(1 - 1'_p \lambda)/d, \quad \theta_1 = (\mu_1, \lambda', d).$$

Further we shall need some definitions and results from the theory of local asymptotic normality (LAN).

Let the parametric family of finite-dimensional distributions $\{Q_{\mu, n}: \mu \in S, n \geq 1\}$ be given, where S is an open parametric set on the line $(-\infty, +\infty)$.

We denote $dQ_{\mu_2, n}/dQ_{\mu_1, n}$ the Radon–Nikodim derivative of the absolutely continuous component of the measure $Q_{\mu_2, n}$ with respect to the measure $Q_{\mu_1, n}$.

DEFINITION. The family $\{Q_{\mu, n}: \mu \in S, n \geq 1\}$ is said to have the property LAN at the point $\mu_1 \in S$ if for some deterministic function $v(k)$ with $\lim v(k) = +\infty$ and for each $-\infty < u < \infty$,

$$\ln \frac{dQ_{\mu_1 + uv_n^{-1}}}{dQ_{\mu_1, n}} = u\Delta_n - \frac{u^2}{2} + \psi_n(\mu_1, u),$$

where Δ_n is asymptotically (as $n \rightarrow \infty$) normal with parameters (0,1) with respect to the distribution $Q_{\mu_1, n}$, that is,

$$\lim_{n \rightarrow \infty} Q_{\mu_1, n}\{\Delta_n \leq z\} = \Phi(z), \quad -\infty < z < \infty,$$

and $\psi_n(\mu_1, u)$ converges to zero by distribution $Q_{\mu_1, n}$, that is, for $\delta > 0$,

$$\lim_{n \rightarrow \infty} Q_{\mu_1, n}\{|\psi(\mu_1, u)| > \delta\} = 0.$$

PROPOSITION 5.1. Let a family of distributions $\{Q_{\mu_1, n}\}$ have the property LAN at point $\mu_1 \in S$ and $\{(t_n, \hat{\mu}_n), n \geq 1\}$ be a sequential estimation procedure for μ . Then

$$(5.6) \quad \kappa = \liminf_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \sup_{|\mu - \mu_1| < \delta} E_\mu |v(n)(\hat{\mu}_n - \mu)|^\gamma \geq I(\mu_1),$$

$$(5.7) \quad I(\mu_1) = \liminf_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \left| \frac{v(n)}{\sup_{|\mu - \mu_1| < \delta} E_\mu v(t_n)} \right|^\gamma E|\xi|^\gamma;$$

ξ is a Gaussian random variable with parameters (0,1).

This result follows directly from Theorem 1 of Efroimovich (1980).

Let $p(x_0, \dots, x_n; \mu, \lambda, d)$ be the probability density of a vector (x_0, \dots, x_n) consisting of the values of the autoregressive process (1.1).

PROPOSITION 5.2. The family of probability densities

$$(5.8) \quad \{p(x_0, \dots, x_n; \mu, \lambda, d) : \mu \in (\mu_0 - q, \mu_0 + q), n \geq 1\}$$

has the LAN property for $\lambda \in \Lambda, d \neq 0$ with $v(n) = \sqrt{J(f)}w(n)$.

[For proof we refer the reader to Beran (1976) and Akritas and Johnson (1982).] Now we can state the basic result of this section.

THEOREM 5.1. For each $\mu_1 \in (\mu_0 - q, \mu_0 + q)$ the sequential procedure (5.2) is optimal in the minimax sense in the class \mathcal{M} , that is, the risk (5.4) satisfies the equality

$$(5.9) \quad R_{\mu_1}(\tilde{\mu}) = \inf_{\hat{\mu} \in \mathcal{M}} R_{\mu_1}(\hat{\mu}).$$

PROOF. First we derive the lower bound for the risk (5.4) of Proposition 5.1. for the family (5.8). By Theorem 3.1 we have

$$(5.10) \quad \lim_{n \rightarrow \infty} \frac{E_{\theta_1} T(n)}{n} = \frac{d_1}{\text{tr } F},$$

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{|\mu - \mu_1| < \delta} \left| \frac{E_{\theta} T(n)}{E_{\theta_1} T(n)} - 1 \right| = 0.$$

By (5.4), (5.10) the risk $R_{\mu_1}(\hat{\mu})$ and the left-hand side of (5.6) satisfy inequality $f \in \mathcal{P}$ by the formula

$$(5.11) \quad R_{\mu_1}(\hat{\mu}) \geq \kappa \left(\frac{d_1}{J(f) \text{tr } F(\theta_1)} \right)^{\gamma/2}.$$

Further, in view of (5.2), (5.10) we obtain

$$(5.12) \quad \begin{aligned} I(\mu_1) &\geq \liminf_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \left(\frac{\sqrt{n}}{\sup_{|\mu - \mu_1| < \delta} E_{\theta} \sqrt{t_n}} \right)^{\gamma} E|\xi|^{\gamma} \\ &\geq E|\xi|^{\gamma} \liminf_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \inf_{|\mu - \mu_1| < \delta} \left(\frac{n}{E_{\theta} T(n)} \right)^{\gamma/2} \\ &= E|\xi|^{\gamma} \liminf_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \inf_{|\mu - \mu_1| < \delta} \left(\frac{E_{\theta_1} T(n)}{E_{\theta} T(n)} \frac{n}{E_{\theta_1} T(n)} \right)^{\gamma/2} \\ &= E|\xi|^{\gamma} \left(\frac{d_1}{\text{tr } F(\theta_1)} \right)^{-\gamma/2}. \end{aligned}$$

From here and (5.6), (5.11) it follows that

$$(5.13) \quad R_{\mu_1}(\hat{\mu}) \geq \sup_{f \in \mathcal{P}} \frac{E|\xi|^{\gamma}}{(J(f))^{\gamma/2}} = E|\xi|^{\gamma}$$

for each $\hat{\mu} \in \mathcal{M}$. The last equality holds because $J(f) \geq 1$ for all $f \in \mathcal{P}$ and $J(f) = 1$ for the Gaussian density $f \in \mathcal{P}$. It remains to verify that

$$(5.14) \quad R_{\mu_1}(\hat{\mu}) = E|\xi|^{\gamma}, \quad \mu_1 \in (\mu_0 - q, \mu_0 + q).$$

For this we need the following assertion.

LEMMA 5.1. *Under the conditions of Theorem 5.1, for some $r > 2$,*

$$\limsup_{H \rightarrow \infty} \sup_{|\mu - \mu_1| < \delta} E_{\theta} |\sqrt{H}(\tilde{\mu}(H) - \mu)|^r < \infty,$$

where $\mu_1 \in (\mu_0 - q, \mu_0 + q)$, $0 < \delta < 1$, $\lambda \in \Lambda$.

The proof of Lemma 5.1 is given in the Appendix. By applying Lemma 5.1 and Theorem 4.1 to the risk (5.3) we obtain (5.14). Hence Theorem 5.1.

APPENDIX

In the sequel we shall need the Burkholder inequality in the following form.

LEMMA A.1 [Shiryaev (1984), Liptser and Shiryaev (1986)]. *If $X=(X_n, \mathcal{F}_n)$ is a martingale and T is a Markovian moment, then for each $p > 1$ there exists the universal constant C_p (independent of X and T) such that*

$$E|X_T|^p \leq C_p E \left(\sum_{k=1}^T (X_k - X_{k-1})^2 \right)^{p/2}, \quad X_0 = 0.$$

A.1. Proof of Lemma 3.1. By (2.17)

$$(A.1) \quad \begin{aligned} E_\theta z^2 (b(z) - b_0)^4 &= z^2 E_\theta (b(z) - b_0)^4 \chi_{\{\det G(z) > 0\}} \\ &\quad + z^2 b_0^4 P_\theta \{\det G(z) = 0\}, \end{aligned}$$

where χ_A denotes the indicator of a set A . Since $b(z) \leq 1$ and $b_0 \leq 1$, then on the set $\{\det G(z) > 0\}$,

$$(A.2) \quad |b(z) - b_0| \leq 2 \left| \frac{1}{zR(z)\|G^{-1}(z)\|} - \frac{1}{\text{tr } F\|F^{-1}\|} \right| \leq 2 \left\| \frac{G(z)}{zR(z)} - \frac{F}{\text{tr } F} \right\|.$$

Now we show that

$$(A.3) \quad \limsup_{z \rightarrow \infty} \sup_{\theta \in \Theta} z^2 E_\theta \left\| \frac{G(z)}{zR(z)} - \frac{F}{\text{tr } F} \right\|^4 < \infty.$$

From the definition of $G(z)$ in (2.15), it follows that

$$(A.4) \quad \begin{aligned} &\left\| \frac{G(z)}{zR(z)} - \frac{1}{\text{tr } F} \right\| \\ &= \left\| \frac{D(z)}{zR(z)} - \frac{(1 - \alpha(z))Y_{\tau(z)}Y'_{\tau(z)}}{zR(z)} + F \left(\frac{\tau(z) - r(z)}{zR(z)} - \frac{F}{\text{tr } F} \right) \right\|, \end{aligned}$$

where

$$(A.5) \quad D(z) = \sum_{k=r(z)+1}^{\tau(z)} (Y_k Y'_k - F).$$

By (2.13), (2.16) we have

$$(A.6) \quad \left| \frac{\tau(z) - r(z)}{zR(z)} - \frac{F}{\text{tr } F} \right| \leq \frac{\sqrt{p+1}\|D(z)\| + \|Y_{\tau(z)}\|^2}{zR(z)\text{tr } F}.$$

By (A.4) and (A.6),

$$\left\| \frac{G(z)}{zR(z)} - \frac{F}{\text{tr } F} \right\| \leq \frac{(\sqrt{p+1} + 1)\|D(z)\| + 2\|Y_{\tau(z)}\|^2}{zR(z)}$$

and, hence, taking into account that $R(z) \geq 1$,

$$(A.7) \quad \left\| \frac{G(z)}{zR(z)} - \frac{F}{\text{tr } F} \right\|^4 \leq 8 \frac{(\sqrt{p+1} + 1)^4 \|D(z)\|^4 + 16 \|Y_{\tau(z)}\|^8}{(zR(z))^4} \leq 8 \frac{(\sqrt{p+1} + 1)^4 \|D(z)\|^4 + 16 \|Y_{\tau(z)}\|^8}{z^4}.$$

Now we show that

$$(A.8) \quad \sup_{\theta \in \mathcal{K}} E_\theta \|Y_{\tau(z)}\|^8 \leq L(1+z),$$

$$(A.9) \quad \sup_{\theta \in \mathcal{K}} E_\theta \|D(z)\|^4 \leq L(1+z)^2.$$

Write (2.3) in vector form:

$$(A.10) \quad X_n = (\mu - \mu_0) \mathbf{1}_p + \xi_n, \quad \xi_n = A \xi_{n-1} + d \eta_n, \quad \xi_0 = \zeta - (\mu - \mu_0) \mathbf{1}_p,$$

where $X_n = (y_n, \dots, y_{n-p+1})'$, $\eta_n = (\varepsilon_n, 0, \dots, 0)'$ and the matrix A is defined in (2.27).

Since

$$(A.11) \quad \xi_k = A^k \xi_0 + d \sum_{j=1}^k A^{k-j} \eta_j,$$

then there exists some constant L for the compact set \mathcal{K} that for all $\theta \in \mathcal{K}$,

$$(A.12) \quad \|Y_k\| \leq L(1 + \|\xi_{k-1}\|).$$

Therefore

$$(A.13) \quad \sup_{k \geq 1} \sup_{\theta \in \mathcal{K}} E_\theta \|Y_k\|^8 < \infty.$$

Further, by the definition of $\tau(z)$ we have

$$E_\theta \tau(z) \leq z E_\theta R(z) + r(z) + 1 \leq \frac{z \sum_{k=1}^{r(z)+1} E_\theta \|Y_k\|^2}{r(z)} + r(z) + 1.$$

From here it follows that

$$(A.14) \quad E_\theta \tau(z) \leq L(1+z), \quad z > 0.$$

By (A.11) and Hölder's inequality,

$$\|\xi_k\|^8 \leq 2^7 \left(\|A^k\|^8 \|\xi_0\|^8 + \left(\sum_{j=1}^k \|A^j\|^{8/7} \right)^7 \sum_{j=1}^k |\varepsilon_j|^8 \right).$$

Taking into account (A.14) and applying the Wald identity, we obtain

$$(A.15) \quad \sup_{\theta \in \mathcal{K}} E_\theta \|\xi_{\tau(z)}\|^8 \leq L(1+z).$$

This inequality and (A.12) imply (A.8). Now we verify (A.9). By (A.5) and (A.10)

$$\begin{aligned} \|D(z)\|^2 &= 2 \left\| \sum_{k=r+1}^{\tau(z)} \xi_{k-1} \right\|^2 + \left\| \sum_{k=r+1}^{\tau(z)} (X_{k-1}X'_{k-1} - (\mu - \mu_0)^2 \mathbf{1}_p \mathbf{1}'_p - F_0) \right\|^2 \\ &\leq 2(1 + 4(\mu - \mu_0)^2 p) \left\| \sum_{k=r+1}^{\tau(z)} \xi_{k-1} \right\|^2 + 2\|S(z)\|^2, \end{aligned}$$

where

$$(A.16) \quad S(z) = \sum_{k=r+1}^{\tau(z)} (\xi_{k-1} \xi'_{k-1} - F_0).$$

Hence,

$$(A.17) \quad \|D(z)\|^4 \leq 8(1 + 4q^2 p)^2 \left\| \sum_{k=r+1}^{\tau(z)} \xi_{k-1} \right\|^4 + 2\|S(z)\|^4.$$

Since by (A.10),

$$\sum_{k=r+1}^{\tau(z)} \xi_{k-1} = (I - A)^{-1}(\xi_r - \xi_{\tau(z)}) + d(I - A)^{-1} \sum_{k=r+1}^{\tau(z)} \eta_k,$$

then for all $\theta \in \mathcal{H}$,

$$\begin{aligned} E_\theta \left\| \sum_{k=r+1}^{\tau(z)} \xi_{k-1} \right\|^4 &\leq L \left(E_\theta \|\xi_r\|^4 + E_\theta \|\xi_{\tau(z)}\|^4 + E_\theta \left(\sum_{k=r+1}^{\tau(z)} \varepsilon_k^2 \right)^2 \right) \\ &\leq L \left(E_\theta \|\xi_r\|^4 + E_\theta \|\xi_{\tau(z)}\|^4 + E_\theta \left(\sum_{k=r+1}^{\tau(z)} (\varepsilon_k^2 - 1) \right)^2 \right. \\ &\quad \left. + E_\theta (\tau(z) - r(z))^2 \right) \\ &\leq L \left(E_\theta \|\xi_r\|^4 + E_\theta \|\xi_{\tau(z)}\|^4 + E_\theta (\tau(z) - r(z)) + z^2 E_\theta R^2(z) + 1 \right) \\ &\leq L \left(E_\theta \|\xi_r\|^4 + E_\theta \|\xi_{\tau(z)}\|^4 + z + z^2(r+1) \sum_{j=1}^{r+1} E_\theta \|Y_j\|^4 / r^2 + 1 \right). \end{aligned}$$

From here, (A.13) and (A.15) it follows that

$$(A.18) \quad \sup_{\theta \in \mathcal{H}} E_\theta \left\| \sum_{k=r+1}^{\tau(z)} \xi_{k-1} \right\|^4 \leq L(1 + z^2).$$

Further we note that the matrix (A.16) satisfies the equations

$$S(z) - AS(z)A' = \xi_r \xi'_r - \xi_{\tau(z)} \xi'_{\tau(z)} + dAM_1(z) + dM'_1(z)A' + M_2(z),$$

where

$$M_1(z) = \sum_{k=r+1}^{\tau(z)} \xi_{k-1} \eta'_k, \quad M_2(z) = \sum_{k=r+1}^{\tau(z)} (d^2 \eta_k \eta'_k - B).$$

Therefore for $\theta \in \mathcal{X}$,

$$(A.19) \quad E_\theta \|S(z)\|^4 \leq L(E_\theta \|\xi_r\|^8 + E_\theta \|\xi_{\tau(z)}\|^8 + E_\theta \|M_1(z)\|^4 + E_\theta \|M_2(z)\|^4).$$

From here, (A.15), (A.17) and (A.18) we have

$$(A.20) \quad E_\theta \|D(z)\|^4 \leq L(1 + z^2 + E_\theta \|M_1(z)\|^4 + E_\theta \|M_2(z)\|^4).$$

Show that

$$(A.21) \quad \sup_{\theta \in \mathcal{X}} E_\theta \|M_1(z)\|^4 \leq (1 + z)^2, \quad z > 0,$$

$$(A.22) \quad \sup_{\theta \in \mathcal{X}} E_\theta \|M_2(z)\|^4 \leq (1 + z)^2, \quad z > 0.$$

By applying Lemma A.1 we obtain

$$E_\theta \|M_1(z)\|^4 \leq L E_\theta \left(\sum_{k=r+1}^{\tau(z)} \|\xi_{k-1}\|^2 \varepsilon_k^2 \right)^2.$$

By (A.10),

$$(A.23) \quad \begin{aligned} \|\xi_{k-1}\|^2 &\leq (|\mu - \mu_0| \sqrt{p} + \|X_{k-1}\|)^2 \\ &\leq 2q^2 p(1 + \|X_{k-1}\|^2) = 2q^2 p \|Y_k\|^2. \end{aligned}$$

Therefore, taking into account (2.13), we have inequality

$$\begin{aligned} E_\theta \|M_1(z)\|^4 &\leq L E_\theta \left(\sum_{k=r+1}^{\tau(z)} \|Y_k\|^2 \varepsilon_k^2 \right)^2 \\ &\leq L \left(E_\theta \left(\sum_{k=r+1}^{\tau(z)-1} \|Y_k\|^2 \varepsilon_k^2 \right)^2 + E_\theta \|Y_{\tau(z)}\|^4 \varepsilon_{\tau(z)}^4 \right) \\ &\leq L \left(E_\theta \sum_{k=r+1}^{\tau(z)-1} \|Y_k\|^2 \sum_{k=r+1}^{\tau(z)-1} \|Y_k\|^2 \varepsilon_k^4 + E_\theta \|Y_{\tau(z)}\|^4 \varepsilon_{\tau(z)}^4 \right) \\ &\leq L \left(z^2 E_\theta R(z) E_\theta \left(\sum_{k=r+1}^{\tau(z)-1} \|Y_k\|^2 \varepsilon_k^4 \middle| Y_1, \dots, Y_{r+1} \right) + E_\theta \|Y_\tau\|^4 \right) \\ &\leq L(z^2 E_\theta R^2(z) + E_\theta \|Y_\tau\|^4) \leq L \left(z^2 \sum_{j=1}^{r+1} E_\theta \|Y_j\|^4 / r + E_\theta \|Y_\tau\|^4 \right). \end{aligned}$$

From here and (A.8), (A.13), (A.21) follows. Inequality (A.22) can be proved similarly to (A.18). Combining (A.20)–(A.22) yields (A.9). From (A.7)–(A.9) inequality (A.3) follows.

It remains to prove that for the second addend in (A.1),

$$(A.24) \quad \limsup_{z \rightarrow \infty} \sup_{\theta \in \mathcal{X}} z^2 P_\theta \{ \det G(z) = 0 \} < \infty.$$

Note that

$$\begin{aligned} P_\theta \{ \det G(z) = 0 \} &= P_\theta \left\{ \inf_{\|v\|=1} v' G(z) v / z R(z) = 0 \right\} \\ &= P_\theta \left\{ \inf_{\|v\|=1} (v' F v / \text{tr } F + v' (G(z) / z R(z) - F / \text{tr } F) v) = 0 \right\}. \end{aligned}$$

Let

$$\phi_* = \inf_{\theta \in \mathcal{X}} \inf_{\|v\|=1} v' F(\theta) v / \text{tr } F(\theta),$$

where the matrix $F(\theta)$ is defined by (2.27). By making use of the Chebyshev inequality we have

$$\begin{aligned} P_\theta \{ \det G(z) = 0 \} &\leq P_\theta \{ \|G(z) / z R(z) - F / \text{tr } F\| > \phi_* \} \\ &\leq \frac{E_\theta \|G(z) / z R(z) - F / \text{tr } F\|^4}{\phi_*^4}. \end{aligned}$$

From here by applying (A.3), we obtain (A.24). Hence Lemma 3.1. \square

A.2. Proof of Lemma 3.2. By (2.17), (2.20) and (2.3), taking into account that $b(z) = 0$ if $\det G(z) = 0$, we have

$$\begin{aligned} a^*(h) - a &= \frac{\sum_{k=1}^{\sigma(h)} b(c_k)(a(c_k) - a)}{\sum_{k=1}^{\sigma(h)} b(c_k)} \\ &= d \frac{\sum_{k=1}^{\sigma(h)} b(c_k) G^{-1}(c_k) M(c_k) \chi_{\{\det G(c_k) > 0\}}}{\sum_{k=1}^{\sigma(h)} b(c_k)}, \end{aligned}$$

where $M(z)$ is defined in (2.30). By the Cauchy–Bunyakovskii inequality,

$$\begin{aligned} (A.25) \quad E_\theta \|a^*(h) - a\|^4 &\leq d^4 E_\theta \left(\frac{\sum_{k=1}^{\sigma(h)} \sqrt{b(c_k)} \|M(c_k)\| / c_k R(c_k)}{\sum_{k=1}^{\sigma(h)} b(c_k)} \right)^4 \\ &\leq d^4 E_\theta \frac{(\sum_{k=1}^{\sigma(h)} b(c_k))^2 (\sum_{k=1}^{\sigma(h)} \|M(c_k)\|^2 / c_k^2 R^2(c_k))^2}{(\sum_{k=1}^{\sigma(h)} b(c_k))^4} \\ &\leq d^4 \frac{\left(\sum_{k \geq 1} \sqrt{E_\theta \|M(c_k)\|^4 / c_k^4 R^4(c_k)} \right)^2}{h^2} \\ &\leq d^4 \frac{\left(\sum_{k \geq 1} \sqrt{E_\theta \|M(c_k)\|^4 / c_k^4} \right)^2}{h^2}. \end{aligned}$$

(The last inequality holds because $R(c_j) \geq 1$.)

Now we estimate $E_\theta \|M(c_k)\|^4$. Denoting $\langle Y \rangle_i$ the i th coordinate of vector Y , we have

$$\begin{aligned} E_\theta \|M(c_k)\|^4 &= E_\theta \left(\sum_{j=1}^{p+1} \langle M(c_k) \rangle_j^2 \right)^2 \\ &\leq (p+1) \sum_{j=1}^{p+1} E_\theta \langle M(c_k) \rangle_j^4 = (p+1) \sum_{j=1}^{p+1} E_\theta (X_{\tau_k}^{(j)})^4, \end{aligned}$$

where

$$X_n^{(j)} = \sum_{l=1}^n v_l \langle Y_l \rangle_j \varepsilon_l, \quad v_l = \chi_{\{r_{k+1} \leq l < \tau_k\}} + \alpha(c_k) \chi_{\{\tau_k=l\}}.$$

Note that v_l is a function depending on y_1, \dots, y_{l-1} . By applying Lemma A.1 to martingales $\{X_n^{(j)}\}$ with Markovian moment (2.19) and the Cauchy–Bunyakovskii inequality we obtain (A.26)

$$\begin{aligned} E_\theta \|M(c_k)\|^4 &\leq LE_\theta \left(\sum_{l=r_k+1}^{\tau_k-1} \|Y_l\|^2 \varepsilon_l^2 + \alpha^2(c_k) \|Y_{\tau_k}\|^2 \varepsilon_{\tau_k}^2 \right)^2 \\ &\leq LE_\theta \left(\sum_{l=r_k+1}^{\tau_k-1} \|Y_l\|^2 + \alpha^2(c_k) \|Y_{\tau_k}\|^2 \right) \\ &\quad \times \left(\sum_{l=r_k+1}^{\tau_k-1} \|Y_l\|^2 \varepsilon_l^4 + \alpha^2(c_k) \|Y_{\tau_k}\|^2 \varepsilon_{\tau_k}^4 \right) \\ &\leq Lc_k E_\theta R(c_k) \\ &\quad \times E_\theta \left(\sum_{l=r_k+1}^{\tau_k-1} \|Y_l\|^2 \varepsilon_l^4 + \alpha^2(c_k) \|Y_{\tau_k}\|^2 \varepsilon_{\tau_k}^4 \mid Y_1, \dots, Y_{r_k+1} \right) \\ &\leq Lc_k^2 E_\theta R^2(c_k) E \varepsilon_1^4 \leq LE \varepsilon_1^4 c_k^2 (r_k + 1) \sum_{j=1}^{r_k+1} E_\theta \|Y_j\|^4 / r_k^2. \end{aligned}$$

By (A.13),

$$E_\theta \|M(c_k)\|^4 \leq Lc_k^2.$$

From here, (A.25) and (3.2),

$$\sup_{\theta \in \mathcal{X}} E_\theta \|a^*(h) - \alpha\|^4 \leq L \frac{(\sum_{k \geq 1} 1/c_k)^2}{h^2} = \frac{L}{h^2} \left(\frac{n_0(H)}{H} + \sum_{k > n_0(H)} \left(\frac{1}{k} \right)^{1+\delta} \right)^2$$

By virtue of (2.14) we establish (3.8) for sufficiently large H . Hence Lemma 3.2. \square

A.3. Proof of inequality (3.14). In view of (3.2) we have

$$\begin{aligned}
 E_\theta(c_{\sigma(h_{\nu(H)})})^{1+\gamma} &= (1 + \gamma) \int_0^\infty t^\gamma P_\theta\{c_{\sigma(h_{\nu(H)})} > t\} dt \\
 &\leq (1 + \gamma) \int_0^H t^\gamma dt + (1 + \gamma) \\
 &\quad \times \int_0^\infty t^\gamma P_\theta\{\sigma(h_{\nu(H)}) > t^{1/(1+\delta)}, \sigma(h_{\nu(H)}) > n_0(H)\} dt \\
 (A.27) \quad &= H^{1+\gamma} + (1 + \gamma)(1 + \delta) \\
 &\quad \times \int_0^\infty s^{\delta+\gamma(1+\delta)} P_\theta\{\sigma(h_{\nu(H)}) > s, \sigma(h_{\nu(H)}) > n_0(H)\} ds \\
 &= H^{1+\gamma} + I_1(H, \theta) + (1 + \gamma)(1 + \delta)(I_2(H, \theta) + I_3(H, \theta)),
 \end{aligned}$$

where

$$\begin{aligned}
 I_1(H, \theta) &= (n_0)^{(1+\gamma)(1+\delta)} P_\theta\{\sigma(h_{\nu(H)}) > n_0(H)\}, \\
 (A.28) \quad I_2(H, \theta) &= \int_{n_0(H)}^\infty s^{\delta+\gamma(1+\delta)} P_\theta\{\sigma(H) > s\} ds, \\
 I_3(H, \theta) &= \int_{n_0(H)}^\infty s^{\delta+\gamma(1+\delta)} P_\theta\{\sigma(h_{\nu(H)}) > s, \nu(H) > n_0(H)\} ds.
 \end{aligned}$$

By applying Propositions (3.1) and (3.2), we obtain the inequality

$$\begin{aligned}
 P_\theta\{\sigma(h_{\nu(H)}) > n_0(H)\} &\leq P_\theta\{\sigma(H) > n_0(H)\} + P_\theta\{\nu(H) > n_0(H)\} \\
 &\leq Ll^2(H)/H^2
 \end{aligned}$$

for sufficiently large H . Therefore

$$(A.29) \quad \lim_{H \rightarrow \infty} \sup_{\theta \in \mathcal{K}} I_1(H, \theta) = 0$$

for $0 < \gamma < (1 - \delta)/(1 + \delta)$.

Further, by Proposition 3.1,

$$\begin{aligned}
 I_2(H, \theta) &\leq L \frac{l(H)n^3(H)}{H} \int_{n_0(H)}^\infty \frac{ds}{s^{4-\delta-\gamma(1+\delta)}} + Ll(H) \int_{n_0(H)}^\infty \frac{ds}{s^{2-\gamma(1+\delta)}} \\
 &= L \frac{l(H)(n_0(H))^{\delta+\gamma(1+\delta)}}{(3 - \delta - \gamma(1 + \delta))H} + L \frac{l(H)}{(1 - \gamma(1 + \delta))(n_0(H))^{1-\gamma(1+\delta)}}.
 \end{aligned}$$

From here and (2.14) it follows that

$$(A.30) \quad \lim_{H \rightarrow \infty} \sup_{\theta \in \mathcal{K}} I_2(H, \theta) = 0$$

for $0 < \gamma < (1 - \delta)/(1 + \delta)$.

In order to estimate $I_3(H, \theta)$, we examine the probability in the integrand. By (2.21) for sufficiently large H and $s \geq n_0(H)$,

$$\begin{aligned}
 & P_\theta\{\sigma(h_{\nu(H)}) > s, \nu(H) > n_0(H)\} \\
 &= P_\theta\left\{\sum_{k=1}^{[s]} b(c_k) < h_{\nu(H)}, \nu(H) > n_0(H)\right\} \\
 \text{(A.31)} \quad & \leq P_\theta\left\{\sum_{k=1}^{[s]} b(c_k) < b_0(\theta)[s]/2\right\} \\
 & \quad + P_\theta\{h_{\nu(H)} > b_0(\theta)[s]/2, \nu(H) > n_0(H)\},
 \end{aligned}$$

where $b_0(\theta)$ is defined by (2.28). Note that

$$\begin{aligned}
 P_\theta\left\{\sum_{k=1}^{[s]} b(c_k) < \frac{b_0(\theta)[s]}{2}\right\} & \leq P_\theta\left\{\sum_{k=1}^{[s]} |b(c_k) - b_0(\theta)| > \frac{b_*s}{4}\right\} \\
 & \leq \frac{4^4 E_\theta(\sum_{k=1}^{[s]} |b(c_k) - b_0(\theta)|)^4}{(b_*s)^4} \\
 & \leq \frac{4^4 (\sum_{k=1}^{[s]} 1/c_k^{1/3})^3 \sum_{k=1}^{[s]} c_k E_\theta |b(c_k) - b_0|^4}{b_*^4 s^4};
 \end{aligned}$$

b_* is defined in (3.7). By applying Lemma 3.1 and taking into account (3.2), we obtain

$$\text{(A.32)} \quad P_\theta\left\{\sum_{k=1}^{[s]} b(c_k) < b_0(\theta)[s]/2\right\} \leq Ll(H)\left(\frac{n_0^3(H)}{Hs^4} + \frac{1}{s^{2+\delta}}\right).$$

Now we estimate the second addend in the right-hand side of (A.31). In view of (3.2), we have

$$\begin{aligned}
 & P_\theta\{h_{\nu(H)} > b_0(\theta)[s]/2, \nu(H) > n_0(H)\} \\
 & \leq P_\theta\{\nu(H) > s^{1/(1+\delta)}\tilde{b}, \nu(H) > n_0(H)\},
 \end{aligned}$$

where $\tilde{b} = (b_*/4)^{1/(1+\delta)}$. By making use of this inequality and (A.31), (A.32), we can estimate $I_3(H, \theta)$:

$$\begin{aligned}
 \text{(A.33)} \quad I_3(H, \theta) & \leq Ll(H)\frac{n_0^3(H)}{H} \int_{n_0(H)}^\infty \frac{ds}{s^{4-\delta-\gamma(1+\delta)}} \\
 & \quad + Ll(H) \int_{n_0(H)}^\infty \frac{ds}{s^{2-\gamma(1+\delta)}} + I_4(H, \theta) \\
 & = Ll(H)\frac{(n_0(H))^{\delta+\gamma(1+\delta)}}{(3-\delta-\gamma(1+\delta))H} \\
 & \quad + Ll(H)\frac{1}{(1-\gamma(1+\delta))(n_0(H))^{1-\gamma(1+\delta)}} + I_4(H, \theta),
 \end{aligned}$$

where

$$I_4(H, \theta) = \int_{n_0(H)}^\infty s^{\delta+\gamma(1+\delta)} P_\theta\{\nu(H) > \tilde{b}s^{1/(1+\delta)}, \nu(H) > n_0(H)\} ds.$$

This quantity can be estimated as

$$\begin{aligned} I_4(H, \theta) &\leq \frac{1 + \delta}{\tilde{b}^{1+\delta_1}} \int_0^\infty t^{\delta_1} P_\theta\{\nu(H) > t, \nu(H) > n_0(H)\} dt \\ &= \frac{1 + \delta}{(1 + \delta_1)\tilde{b}^{1+\delta_1}} \left((n_0(H))^{1+\delta_1} P_\theta\{\nu(H) > n_0(H)\} \right. \\ &\quad \left. + \int_{n_0(H)}^\infty t_1^\delta P_\theta\{\nu(H) > t\} dt \right), \end{aligned}$$

$\delta_1 = (1 + \delta)(\delta + \gamma(1 + \delta)) + \delta$. From here and Proposition 3.2 it follows that for sufficiently large H ,

$$I_4(H, \theta) \leq L \left(\frac{l^2(H)(n_0(H))^{1+\delta_1}}{H^2} + \frac{l^3(H)}{(n_0(H))^{1+\delta-\delta_1}} \right).$$

Since $0 < \delta < \sqrt{2} - 1$ then for all $0 < \gamma < (1 - 2\delta - \delta^2)/(1 + \delta)^2$,

$$\limsup_{H \rightarrow \infty} \sup_{\theta \in \mathcal{N}} I_4(H, \theta) = 0$$

and by making use of (A.33) we obtain

$$\limsup_{H \rightarrow \infty} \sup_{\theta \in \mathcal{N}} I_3(H, \theta) = 0.$$

Combining this limiting relationship and (A.27), (A.29) and (A.30) yields inequality (3.14). \square

A.4. Proof of Lemma 3.3. By (2.13) and (A.12),

$$(A.34) \quad \sum_{k=1}^n y_k^2 \leq \sum_{k=1}^{n+1} \|Y_k\|^2 \leq L \sum_{k=1}^n (1 + \|\xi_k\|^2).$$

In view of (A.11),

$$\begin{aligned} (A.35) \quad \|\xi_k\|^2 &\leq 2 \left(\|A^k\|^2 \|\xi_0\|^2 + d^2 \left(\sum_{j=1}^k \|A^{k-j}\| |\varepsilon_j| \right)^2 \right) \\ &\leq 2 \left(\|A^k\|^2 \|\xi_0\|^2 + d^2 \sum_{j=1}^k \|A^{k-j}\| \sum_{j=1}^k \|A^{k-j}\| (\varepsilon_j)^2 \right) \\ &\leq 2 \left(\|A^k\|^2 \|\xi_0\|^2 + d^2 \sum_{j \geq 0} \|A^j\| \sum_{j=0}^{k-1} \|A^j\| (\varepsilon_{k-j})^2 \right). \end{aligned}$$

Since $\sum_{j \geq 0} \|A^j\|$ is bounded for $\theta \in \mathcal{K}$, then

$$\begin{aligned} \sum_{k=1}^n \|\xi_k\|^2 &\leq 2\|\xi_0\|^2 \sum_{k=1}^n \|A^k\|^2 + 2d^2 \sum_{j \geq 1} \|A^j\| \sum_{j=0}^{n-1} \|A^j\| \sum_{k=j+1}^n (\varepsilon_{k-j})^2 \\ &\leq \max_{\theta \in \mathcal{K}} Q(\theta) \left(\|\xi_0\|^2 + \sum_{k=1}^n \varepsilon_k^2 \right), \end{aligned}$$

where

$$Q(\theta) = 2 \max \left(\sum_{j \geq 1} \|A^j\|^2, d^2 \left(\sum_{j \geq 0} \|A^j\| \right)^2 \right).$$

From here and (A.34), taking into account the definition of ξ_0 in (A.10), we obtain (3.10). Hence Lemma 3.3. \square

A.5. Proof of relationship (3.21). By making use of (A.6) we obtain

$$\begin{aligned} \left| \frac{\tau(H)}{H} - \frac{d_1}{\text{tr } F} \right| &\leq R(H) \left| \frac{\tau(H) - r(H)}{HR(H)} - \frac{1}{\text{tr } F} \right| \\ (A.36) \quad &+ \frac{|R(H) - d_1|}{\text{tr } F} + \frac{r(H)}{H} \\ &\leq L \left(\frac{\|D(H)\|}{H} + \frac{\|Y_{\tau(H)}\|^2}{H} + |R(H) - d_1| + \frac{r(H)}{H} \right). \end{aligned}$$

By the definition of $R(H)$ in (2.13),

$$\begin{aligned} |R(H) - d_1| &= \left| \frac{\sum_{k=1}^r \langle Y_{k+1} Y'_{k+1} \rangle_{22}}{r(H)} - \langle F \rangle_{22} \right| \\ (A.37) \quad &\leq \frac{1}{r(H)} \left\| \sum_{k=2}^{r(H)+1} (Y_k Y'_k - F) \right\|_{22} \\ &\leq \frac{1}{r(H)} \left\| \sum_{k=2}^{r(H)+1} (Y_k Y'_k - F) \right\|. \end{aligned}$$

By an argument similar to the proof of inequality (A.9) for the matrix (A.5), one can show that

$$\sup_{\theta \in \mathcal{K}} E_\theta \left\| \sum_{k=1}^{r(H)+1} (Y_k Y'_k - F) \right\|^4 \leq L(1 + r(H))^2.$$

From here and from (A.36), (A.37), (A.8) and (A.9), it follows that

$$\sup_{\theta \in \mathcal{K}} E_\theta \left| \frac{\tau(H)}{H} - \frac{d_1}{\text{tr } F} \right|^4 \leq L \left(\frac{1}{H^2} + \frac{1}{H^3} + \frac{1}{H^4} + \frac{1}{(r(H))^2} + \frac{1}{(r(H))^4} + \frac{r(H)}{H} \right).$$

This inequality by virtue of (2.13), (2.14), completes the proof of (3.21). \square

A.6. Proof of Proposition 4.2. By substituting (2.3) in (2.15) we obtain on the set $\{\det G(H) > 0\}$,

$$(A.38) \quad \zeta(H) = \sqrt{H}V'(a(H) - a) = d_2(V)\zeta_0(H) + \Delta(H),$$

where

$$\zeta_0(H) = \sum_{j=1}^{\tau_0} g_j \varepsilon_j / \sqrt{H}, \quad g_j = \frac{V'F^{-1}Y_j \sqrt{\text{tr } F}}{\sqrt{d_1 V'F^{-1}V}} \chi_{\{j>r\}},$$

$$\tau_0 = \tau_0(H) = \inf \left\{ k \geq 1: \sum_{j=1}^k g_j^2 \geq H \right\},$$

$$\Delta(H) = \Delta_1(H) + \Delta_2(H) + \Delta_3(H),$$

$$\Delta_1(H) = -d(1 - \alpha(H)) \frac{V'F^{-1}Y_{\tau} \varepsilon_{\tau} \text{tr } F}{d_1 \sqrt{H}},$$

$$\Delta_2(H) = dV'(HG^{-1}(H) - F^{-1} \text{tr } F / d_1)M(H) / \sqrt{H},$$

$$\Delta_3(H) = d_2(V) \left(\sum_{j=1}^{\tau} g_j \varepsilon_j - \sum_{j=1}^{\tau_0} g_j \varepsilon_j \right) / \sqrt{H};$$

$M(H)$ and τ are defined by (2.30) and (2.13).

In view of (A.24), to prove the desired conclusion (4.1) it suffices to show the following.

1. The random variable ζ_0 in (A.38) is asymptotically normal uniformly in parameter $\theta \in \mathcal{X}$, that is,

$$(A.39) \quad \limsup_{H \rightarrow \infty} \sup_{\theta \in \mathcal{X}} \sup_{-\infty < t < \infty} |P_{\theta}\{\zeta_0(H) \leq t\} - \Phi(t)| = 0;$$

2. The random variable $\Delta(H)$ converges to zero in probability uniformly in $\theta \in \mathcal{X}$ as H tends to infinity, that is, for every $\delta > 0$,

$$(A.40) \quad \limsup_{H \rightarrow \infty} \sup_{\theta \in \mathcal{X}} P_{\theta}\{|\Delta(H)| > \delta, \det G(H) > 0\} = 0.$$

To prove (A.39) we apply Proposition 4.1 to the sequences g_k, ε_k and the stopping moment τ_0 . Let us verify its conditions. Conditions (i)–(iii) are evident. Condition (iv) holds because by (2.26),

$$\lim_{N \rightarrow \infty} \frac{\sum_{j=1}^N g_j^2}{N} = \frac{\text{tr } F}{d_1} \quad P_{\theta}\text{-a.s.}$$

for $\theta \in \mathcal{X}$. Condition (v) holds because $E_{\theta} g_n^2$ is bounded for $\theta \in \mathcal{X}$ and $n \geq 0$, due to (A.12), (A.35). In order to verify condition (vi) it suffices to show that for any $0 < \delta < 1$,

$$(A.41) \quad \sum_{n > n_1} \sup_{\theta \in \mathcal{X}} P_{\theta} \left\{ g_n^2 > \delta \sum_{j=1}^{n-1} g_{j-1}^2 \right\} < \infty,$$

where $n_1 = n_1(r, \delta) = [r/(1 - \delta)] + 1$.

We have

$$(A.42) \quad P_\theta \left\{ g_n^2 > \delta \sum_{j=1}^{n-1} g_j^2 \right\} \leq P_\theta \left\{ g_n^2 > \delta^2(n-1) \right\} + P_\theta \left\{ \sum_{j=1}^{n-1} g_j^2 \leq \delta(n-1) \right\}.$$

By the Chebyshev inequality and (A.13),

$$(A.43) \quad \sup_{\theta \in \mathcal{X}} P_\theta \{ g_n^2 > \delta^2(n-1) \} \leq \sup_{\theta \in \mathcal{X}} \frac{E_\theta g_n^2}{\delta^2(n-1)^4} \leq \frac{L}{(n-1)^4}.$$

Further, we make use of the representation

$$\sum_{j=1}^{n-1} g_j^2 = \frac{\text{tr } F(n-r-1)}{d_1} + \frac{V'F^{-1}(\sum_{j=r+1}^{n-1}(Y_jY'_j - F))F^{-1}V \text{tr } F}{V'F^{-1}Vd_1}.$$

From here it follows that for sufficiently small $\delta > 0$,

$$(A.44) \quad \begin{aligned} & P_\theta \left\{ \sum_{j=1}^{n-1} g_j^2 \leq \delta(n-1) \right\} \\ & \leq P_\theta \left\{ \frac{\|V'F^{-1}\|^2}{V'F^{-1}V} \left\| \sum_{j=r+1}^{n-1} (Y_jY'_j - F) \right\| > (n-r-1) - \tilde{\delta}(n-1) \right\} \\ & \leq L \frac{E_\theta \left\| \sum_{j=r+1}^{n-1} (Y_jY'_j - F) \right\|^4}{((1-\tilde{\delta})n-r)^4}, \end{aligned}$$

where $\tilde{\delta} = \delta d_1 / \text{tr } F$. Now we note that

$$(A.45) \quad \sup_{\theta \in \mathcal{X}} E_\theta \left\| \sum_{j=r+1}^{n-1} (Y_jY'_j - F) \right\|^4 \leq Ln^2, \quad n \geq r+2.$$

This inequality can be shown by an argument similar to that used in the proof of (A.9). Combining (A.42)–(A.45) yields (A.41). This completes the proof of (A.39). It remains to verify (A.40).

We have

$$\begin{aligned} E_\theta |\Delta_1(H)|^4 & \leq \frac{d^4(\text{tr } F)^4}{(d_1)^4 H^2} E(\varepsilon_1)^4 E_\theta |V'F^{-1}Y_\tau|^4 \\ & \leq \frac{d^4(\text{tr } F)^4 (V'F^{-2}V)^2 E(\varepsilon_1)^4 E_\theta \|Y_\tau\|^4}{(d_1)^4 H^2}. \end{aligned}$$

By (A.12) and (A.15),

$$(A.46) \quad \sup_{\theta \in \mathcal{X}} E_\theta \|Y_\tau\|^4 \leq L(1+H).$$

Thus

$$(A.47) \quad \lim_{H \rightarrow \infty} \sup_{\theta \in \mathcal{X}} E_\theta |\Delta_1|^4 = 0.$$

Next consider $\Delta_2(H)$. On the set $\{\det G(H) > 0\}$, we have

$$(A.48) \quad |\Delta_2(H)| \leq \frac{d}{d_1} \|V\| \|G^{-1}(H)H\| \left\| \frac{G(H)}{H} - \frac{d_1 F}{\text{tr } F} \right\| \frac{\|M(H)\|}{\sqrt{H}} \|F^{-1} \text{tr } F\|.$$

By the definition of τ and $R(H)$ in (2.13) and (A.34), it follows that

$$(A.49) \quad \sup_{\theta \in \mathcal{X}} E_\theta \frac{\|M(H)\|^2}{H} \leq \sup_{\theta \in \mathcal{X}} E_\theta R(H) \leq L < \infty.$$

Further, we show that

$$(A.50) \quad \lim_{H \rightarrow \infty} \sup_{\theta \in \mathcal{X}} E_\theta \left\| \frac{G(H)}{H} - \frac{d_1 F}{\text{tr } F} \right\| = 0.$$

We have

$$(A.51) \quad \left\| \frac{G(H)}{H} - \frac{d_1 F}{\text{tr } F} \right\| \leq R(H) \left\| \frac{G(H)}{HR(H)} - \frac{F}{\text{tr } F} \right\| + \frac{\|F\|}{\text{tr } F} |R(H) - d_1|.$$

By (A.7)–(A.9),

$$(A.52) \quad \sup_{\theta \in \mathcal{X}} E_\theta (R(H))^4 \left\| \frac{G(H)}{HR(H)} - \frac{F}{\text{tr } F} \right\|^4 \leq L \frac{(1+H)^2}{H^4}.$$

By (A.37),

$$(A.53) \quad \sup_{\theta \in \mathcal{X}} E_\theta \frac{\|F\|}{\text{tr } F} |R(H) - d_1| \leq L \frac{1 + \sqrt{r(H)}}{r(H)}.$$

Combining this inequality and (A.51), (A.52) yields (A.50). From (A.48)–(A.50) we obtain

$$(A.54) \quad \lim_{H \rightarrow \infty} \sup_{\theta \in \mathcal{X}} P_\theta \{|\Delta_2| > \delta, \det G(H) > 0\} = 0.$$

To analyze $\Delta_3(H)$ we need the following relationship:

$$(A.55) \quad \limsup_{H \rightarrow \infty} \sup_{\theta \in \mathcal{X}} E_\theta \frac{\tau_0(H)}{H} < \infty.$$

By definition of τ_0 in (A.38),

$$(A.56) \quad \begin{aligned} E_\theta \tau_0(H) &\leq 1 + \sum_{n \geq 1} P_\theta \{ \tau_0(H) > n \} \\ &\leq 1 + \sum_{n \geq 1} P_\theta \left\{ \sum_{j=1}^n g_j^2 < H \right\} \\ &\leq 1 + \sum_{n \geq 1} \chi_{\{n\delta < H\}} + \sum_{n \geq [H/\delta]} P_\theta \left\{ \sum_{j=1}^n g_j^2 < n\delta \right\}, \quad \delta > 0. \end{aligned}$$

Further we shall make use of (A.44). Let $0 < \delta < \inf_{\theta \in \mathcal{X}} \text{tr } F/d_1$. By (2.13) there exists such a number $H_0 > 0$ that for all $n \geq [H_0/\delta]$,

$$(1 - \tilde{\delta})n - r > (1 - \tilde{\delta})n/2.$$

Therefore, by (A.44), (A.45),

$$P_\theta \left\{ \sum_{j=1}^n g_j^2 < n\delta \right\} \leq L \frac{n^2}{((1-\tilde{\delta})n-r)^4} \leq L/n^2, \quad n \geq \left\lceil \frac{H}{\delta} \right\rceil,$$

for all $H \geq H_0$. This estimate and (A.56) imply (A.55). Further, we show that

$$(A.57) \quad \limsup_{H \rightarrow \infty} \sup_{\theta \in \mathcal{X}} E_\theta \frac{(g_{\tau_0})^8}{H^2} = 0.$$

By the definition of g_k in (A.38) and (A.12),

$$|g_{\tau_0}|^8 \leq L \left(1 + \sum_{j=1}^{\tau_0} |\varepsilon_j|^8 \right).$$

In view of (3.1), applying the Wald identity we obtain (A.57). Now we can estimate the second moment of $\Delta_3(H)$. From the definitions of Δ_3 and τ_0 , it follows that

$$\begin{aligned} E_\theta(\Delta_3(H))^2 &= \frac{(d_2(V))^2}{H} E_\theta \left| \sum_{k=1}^{\tau} g_k^2 - \sum_{k=1}^{\tau_0} g_k^2 \right| \\ &\leq (d_2(V))^2 E_\theta \left| \frac{\text{tr } F}{Hd_1 V' F^{-1} V} V' F^{-1} \sum_{k=r+1}^{\tau} Y_k Y_k' F^{-1} V - 1 \right| \\ (A.58) \quad &+ (d_2(V))^2 \frac{E_\theta(g_{\tau_0})^2}{H} \\ &= (d_2(V))^2 E_\theta \left| \frac{(\tau-r) \text{tr } F}{Hd_1} - 1 \right| \\ &+ (d_2(V))^2 \frac{\text{tr } F}{Hd_1} \frac{|V' F^{-1} D(H) F^{-1} V|}{V' F^{-1} V} + (d_2(V))^2 \frac{E_\theta(g_{\tau_0})^2}{H}, \end{aligned}$$

where the matrix $D(H)$ is defined by (A.5). Further, applying (A.6), (A.8), (A.9) and (A.53), we have

$$\begin{aligned} (A.59) \quad &E_\theta \left| \frac{\tau(H) - r(H) \text{tr } F}{H} \frac{1}{d_1} - 1 \right| \\ &\leq \frac{\text{tr } FR(H)}{d_1} E_\theta \left| \frac{\tau(H) - r(H)}{HR(H)} - \frac{1}{\text{tr } F} \right| + \frac{E_\theta |R(H) - d_1|}{d_1} \\ &\leq \frac{\sqrt{p+1}}{d_1 H} E_\theta \|D(H)\| + \frac{E_\theta \|Y_{\tau(H)}\|^2}{d_1} + \frac{E_\theta |R(H) - d_1|}{d_1} \\ &\leq L \left(\frac{1 + \sqrt[4]{H} + \sqrt{H}}{H} + \frac{1 + \sqrt{r(H)}}{r(H)} \right), \quad \theta \in \mathcal{X}. \end{aligned}$$

The second term in the right-hand side of (A.58), in view of (A.9), can be estimated as

$$\sup_{\theta \in \mathcal{X}} E_{\theta}(d_2(V))^2 \frac{\text{tr } F}{Hd_1} \frac{|V'F^{-1}D(H)F^{-1}V|}{V'F^{-1}V} \leq L \frac{1 + \sqrt{H}}{H}.$$

Combining this estimate with (A.57)–(A.59) yields

$$\lim_{h \rightarrow \infty} \sup_{\theta \in \mathcal{X}} E_{\theta}(\Delta_3(H))^2 = 0.$$

The required conclusion (A.40) follows from here and (A.47), (A.54). Hence Proposition 4.2. \square

A.7. Proof of Lemma 5.1. On the event $\Gamma(H)$ defined in (4.3), (2.10) and (2.33) imply

$$\tilde{\mu}(H) - \mu = \frac{\Delta(H)}{1 - 1'_p \lambda(H)}$$

where $\Delta(H) = V'_0(a(H) - a)$, $V_0 = (1, (\mu - \mu_0)1'_p)'$; $a(H)$ is defined by (2.15). Observe that if $|\mu - \mu_1| < \delta$, then

$$(A.60) \quad |\Delta(H)| \leq L \|G^{-1}(H)\| \|M(H)\|,$$

where $M(H)$ is defined in (2.30). Now we estimate each of the quantities

$$(A.61) \quad I_1(H) = E_{\theta} |\tilde{\mu}(H) - \mu|^r \chi_{\Gamma(H)},$$

$$(A.62) \quad I_2(H) = E_{\theta} |\tilde{\mu}(H) - \mu|^r \chi_{\Gamma^c(H)},$$

By (2.7), (2.9) we have on the set $\Gamma(H)$ the inequality

$$|1 - 1'_p \lambda(H)| \geq 1/\sqrt{l(H)}$$

which implies for $0 < t < (1 - 1'_p \lambda)/2$ and $\lambda \in \Lambda$ the following estimate:

$$I_1(H) \leq \frac{2^r}{(1 - 1'_p \lambda)} E_{\theta} |\Delta(H)|^r \chi_{\Gamma(H)} + (l(H))^{r/2} E_{\theta} |\Delta(H)|^r \chi_{\{\Gamma(H) \cap \{\|\lambda(H) - \lambda\| > t/\sqrt{p}\}\}}.$$

From here and (A.60) it follows that

$$(A.63) \quad I_1(H) \leq \frac{L}{(1 - 1'_p \lambda)} E_{\theta} \|HG^{-1}(H)\|^r \|M(H)/H\|^r \chi_{\Gamma(H)} + L(l(H))^{r/2} E_{\theta} \|HG^{-1}(H)\|^r \times \|M(H)/H\|^r \chi_{\{\Gamma(H) \cap \{\|\lambda(H) - \lambda\| > t/\sqrt{p}\}\}}.$$

Let us estimate $\|HG^{-1}(H)\|^r$ on the event $\Gamma(H)$. By the definition of σ in (2.21), the function $b(H)$ in (2.17) and due to the choice of the sequence c_k in (3.2), we have the following inclusions:

$$\begin{aligned} \{\sigma(H) \leq n_0(H)\} &= \{n_0(H)b(H) \geq H\} \\ &= \left\{ \frac{[HL(H)]}{H} \geq (R(H)H\|G^{-1}(H)\|)^2 \right\} \\ &\subset \left\{ \sqrt{l(H)} \geq R(H)H\|G^{-1}(H)\| \right\}. \end{aligned}$$

Thus on the event $\Gamma(H)$

$$(A.64) \quad \|HG^{-1}(H)\| \leq \frac{\sqrt{l(H)}}{R(H)} \leq \sqrt{l(H)}.$$

By making use of this, we obtain

$$\begin{aligned} (A.65) \quad \|HG^{-1}(H)\|^r &\leq (\|F^{-1}\| \operatorname{tr} F)^r \\ &\quad \times \left(1 + \|HG^{-1}(H)\| \left\| \frac{G(H)}{HR(H)} - \frac{F}{\operatorname{tr} F} \right\| \right)^r \\ &\leq L \left(1 + (l(H))^{r/2} \left\| \frac{G(H)}{HR(H)} - \frac{F}{\operatorname{tr} F} \right\|^r \right). \end{aligned}$$

The matrix F is defined in (2.27). Combining (A.63)–(A.65) yields

$$\begin{aligned} (A.66) \quad I_1(H) &\leq LE_\theta \|M(H)/H\|^r \\ &\quad + L(l(H))^{r/2} E_\theta \left(\frac{\|M(H)\|}{H} \left\| \frac{G(H)}{HR(H)} - \frac{F}{\operatorname{tr} F} \right\| \right)^r \\ &\quad + L(l(H))^r E_\theta \|M(H)/H\|^r \chi_{\{\|\lambda(H)-\lambda\|>t/\sqrt{p}\}}. \end{aligned}$$

Next we show that for $2 < \gamma \leq 8$,

$$(A.67) \quad \sup_{|\mu-\mu_1|<\delta} E_\theta \|M(H)/H\|^\gamma \leq L \frac{E|\varepsilon_1|^\gamma}{H^{\gamma/2}}.$$

Taking into account the definition of $\tau(H)$ in (2.10) and applying Lemma A.1 and the Hölder inequality, we obtain

$$\begin{aligned} (A.68) \quad E_\theta \|M(H)\|^\gamma &\leq LE_\theta \left(\sum_{j=r(H)+1}^{\tau(H)-1} \|Y_j\|^2 (\varepsilon_j)^2 + (\alpha(H))^2 \|Y_{\tau(H)}\|^2 (\varepsilon_{\tau(H)})^2 \right)^{\gamma/2} \\ &\leq LE_\theta \left(\sum_{j=r(H)+1}^{\tau(H)-1} \|Y_j\|^2 + (\alpha(H))^2 \|Y_{\tau(H)}\|^2 \right)^{\gamma/2-1} \\ &\quad \times \left(\sum_{j=r(H)+1}^{\tau(H)-1} \|Y_j\|^2 |\varepsilon_j|^\gamma + (\alpha(H))^2 \|Y_{\tau(H)}\|^2 |\varepsilon_{\tau(H)}|^\gamma \right) \\ &\leq LE|\varepsilon_1|^\gamma E_\theta (HR(H))^{\gamma/2}. \end{aligned}$$

By (2.13),

$$E_\theta(R(H))^{\gamma/2} \leq E_\theta\left(1 + \sum_{j=1}^{r(H)} \|Y_j\|^2/r(H)\right)^{\gamma/2} \leq 2^{\gamma/2-1}\left(1 + \max_{j \geq 1} E_\theta\|Y_j\|^\gamma\right).$$

This and (A.68) imply (A.67). Next we estimate the second term in the right-hand side of (A.66) with the help of (A.67) and (A.3), assuming $2 < r \leq 8/3$. We have

$$\begin{aligned} (A.69) \quad & E_\theta \left\| \frac{G(H)}{HR(H)} - \frac{F}{\text{tr } F} \right\|^r \|M(H)/H\|^r \\ & \leq \left(E_\theta \left\| \frac{G(H)}{HR(H)} - \frac{F}{\text{tr } F} \right\|^4 \right)^{r/4} (E_\theta \|M(H)/H\|^{4r/(4-r)})^{(4-r)/4} \leq \frac{L}{H^r}. \end{aligned}$$

The last term in the right-hand side of (A.66) can be estimated by applying the Cauchy–Bunyakovsky inequality and (3.8):

$$\begin{aligned} & E_\theta \|M(H)/H\|^r \chi_{\{\|\lambda^*(H) - \lambda\| > t/\sqrt{p}\}} \\ & \leq (E_\theta \|M(H)/H\|^{2r})^{1/2} \sqrt{P_\theta\{\|\lambda^*(H) - \lambda\| > t/\sqrt{p}\}} \leq Ll(H)/H^{1+r/2}, \end{aligned}$$

when H is sufficiently large. This and (A.66), (A.67) and (A.69) imply

$$(A.70) \quad I_1(H) \leq \frac{L}{H^{r/2}} \left(1 + \frac{(l(H))^{r/2}}{H^{r/2}} + \frac{(l(H))^{1+r}}{H} \right)$$

for sufficiently large H . It remains to estimate $I_2(H)$ in (A.62). We have

$$(A.71) \quad I_2(H) \leq (E_\theta(\tilde{\mu}(H) - \mu)^4)^{r/4} (P_\theta\{\Gamma^c(H)\})^{(4-r)/4}.$$

From (2.9), (2.10), (2.33), (3.2) and (3.8) it follows that

$$\begin{aligned} (A.72) \quad & E_\theta(\tilde{\mu}(H) - \mu)^4 \leq \frac{\|V_0\|^4}{H^2} E_\theta \left(\sum_{j=1}^{\nu(H)} \|\alpha^*(h_j) - a\|^2 \right)^2 \\ & \leq \frac{\|V_0\|^4}{H^2} \left(\sum_{j \geq 1} \sqrt{E_\theta \|\alpha^*(h_j) - a\|^4} \right)^2 \leq L \frac{(l(H))^4}{H^2} \end{aligned}$$

for $|\mu - \mu_1| < \delta$, $\lambda \in \Lambda$.

Further, by Propositions 3.1 and 3.2,

$$(A.73) \quad \lim_{H \rightarrow \infty} \sup_{|\mu - \mu_1| < \delta} HP_\theta\{\Gamma^c\} = 0, \quad \lambda \in \Lambda.$$

Combining (A.71)–(A.73) yields

$$I_2(H) \leq L \frac{(l(H))^r}{H^{1+r/2}}.$$

This estimate and (A.70) lead to (5.14) for $2 < r \leq 8/3$. Hence Lemma 5.1. \square

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