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ON THE CONSTRUCTION AND EXISTENCE OF ORTHOGONAL ARRAYS WITH THREE LEVELS AND INDEXES 1 AND 2

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We study the construction of orthogonal arrays with three levels and index 1 and the existence of orthogonal arrays with three levels and index 2. For strength greater than or equal to 2, we show that orthogonal arrays with three levels and index 1 are unique, and we establish the maximum number of factors for orthogonal arrays with three levels and index 2.

1. Introduction. An $N \times k$ matrix A with entries from a set S of cardinality s is called an orthogonal array of strength t, $1 \leq t \leq k$, denoted by $OA(N, k, s, t)$, if every $N \times t$ submatrix of A contains each t-tuple based on S equally often as a row. The common frequency with which each of the t-tuples appears as a row in a submatrix must be equal to N/s^t, which is referred to as the index of the array and is denoted by λ. The number of rows N and columns k are also called the number of runs and factors of the array, respectively, while s is called the number of levels of the array. We will always take $S = \{0, 1, \dots, s-1\}$.

Orthogonal arrays were introduced by Rao (1946, 1947) under the name of "hypercubes." Besides being used for the construction of various other combinatorial configurations, they are popular among statisticians for their properties in fractional factorial experiments. It is well known that an orthogonal array of strength t is a fractional factorial design of resolution $t + 1$ (see, e.g., Raktoe, Hedayat and Federer (1981)). Cheng (1980) and Mukerjee (1982) showed that orthogonal arrays have desirable optimality properties as fractional factorials. The often-used regular fractional factorial designs, that is, the fractions that are obtained from a defining relation, are examples of orthogonal arrays. The class of orthogonal arrays, however, contains also nonregular fractions. For example, for a regular fractional factorial design, the number of runs must be a power of s, or equivalently, the index must be a power of s; for an orthogonal array this restriction does not apply.

Orthogonal arrays with two or three levels are especially of interest for statistical applications. Orthogonal arrays with two levels are often used at

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the exploratory stage of an investigation when not much is known about how the factors affect a response of interest. Orthogonal arrays with three levels can be useful for detecting and testing of linear and quadratic effects of a factor.

Important research problems in the area of orthogonal arrays include the construction of orthogonal arrays of various strengths and indexes, and the identification of the maximum possible number of factors in an orthogonal array for given strength, index and number of levels. See Hedayat, Sloane and Stufken (1997) for further details.

We will study the construction of orthogonal arrays with three levels and index 1 and the existence of orthogonal arrays with three levels and index 2. In Section 2, we will show that, for any strength $t \geq 2$, there is, up to isomorphism, only one orthogonal array with three levels, index 1 and the maximum possible number of factors. In Section 3, we will, again for any strength $t \geq 2$, establish the maximum number of factors for orthogonal arrays with three levels and index 2. These arrays can be constructed easily, as we will see.

Some of the techniques used in this paper are also potentially useful for studying the existence and construction of other orthogonal arrays.

2. On the construction and uniqueness of orthogonal arrays with three levels and index 1. We will consider a general method of construction for orthogonal arrays with three levels and index 1. The method provides an orthogonal array $OA(3^t, t+1, 3, t)$ and we will show that, up to isomorphism, there is a unique orthogonal array with these parameters. It should be noted that for $t \geq 3$ such an orthogonal array possesses the maximum possible number of factors [Bush (1952)].

For the method of construction, let $c_1, \dots, c_t \in \{1, 2\}$, and $c \in \{0, 1, 2\}$. Construct a $3^t \times (t+1)$ array as follows:

1. Use each of the 3^t-tuples based on 0, 1 and 2 exactly once as a row to form the first t columns of the array.
2. With $(x_1, \dots, x_t)$ as a row restricted to the first t columns, take $\sum_{i=1}^{t} c_i x_i + c \pmod 3$ as the entry for that row in the $(t+1)$th column.

The array constructed in this way is an orthogonal array $OA(3^t, t+1, 3, t)$ and will be denoted by $A(c_1, \dots, c_t, c)$. As fractional factorial designs, the arrays $A(c_1, \dots, c_t, c)$ are precisely the regular fractional factorial designs. We will show that any orthogonal array $OA(3^t, t+1, 3, t)$ can be obtained via this method of construction, and thus also that any such array is a regular fractional factorial design. This result will also imply that any two arrays $OA(3^t, t+1, 3, t)$ are isomorphic, meaning that one can be obtained from the other by a sequence of permutations of rows, columns, and levels of one or more of the factors.

LEMMA. *For $t \geq 2$, if A is an orthogonal array $OA(3^t, t+1, 3, t)$, then A is a regular fractional factorial design.*

PROOF. We need to show that there are $c_1, \dots, c_t \in \{1,2\}$ and $c \in \{0,1,2\}$ such that the runs of A are, up to a permutation, precisely those of $A(c_1, \dots, c_t, c)$. In the first t columns of A, every t-tuple based on 0, 1 and 2 appears exactly once. Hence, the runs of A are of the form $(x_1, \dots, x_t, g(x_1, \dots, x_t))$, where $g(x_1, \dots, x_t) \in \{0,1,2\}$ and $x_i = 0$, 1 or 2 for $i = 1, \dots, t$. The goal is to show that there are $c_i \in \{1,2\}$, $i = 1, \dots, t$ and $c \in \{0,1,2\}$ such that $g(x_1, \dots, x_t) \equiv \sum_{i=1}^{t} c_i x_i + c \pmod 3$ for all x_i's. This means that $g(0, \dots, 0) = c$. Subtracting c from each entry in the last column of A leads to an orthogonal array $OA(3^t, t+1, 3, t)$ with the property that $g(0, \dots, 0) = 0$. Hence, we may assume without loss of generality that $g(0, \dots, 0) = 0$, that is, that the all-zero run is part of the array. With e_i, $i = 1, \dots, t$, denoting the t-tuple with a 1 in position i and a 0 elsewhere, it follows now also that $c_1 = g(e_1), \dots, c_t = g(e_t)$, and that all of the c_i's are nonzero. (Since the array has strength t and index 1, the difference of any two runs in the array, computed modulo 3, must have at least two nonzero elements. Considering the all-zero run and the runs that start with the e_i's gives the desired conclusion. This type of reasoning is also used repeatedly in what follows.) We will show that

$$g(x_1, \dots, x_t) \equiv \sum_{i=1}^{t} c_i x_i \pmod 3, \tag{1}$$

for all $x_i = 0$, 1 or 2, $i = 1, \dots, t$, by using induction on the weight of $(x_1, \dots, x_t)$, which is defined as the number of nonzero entries in $(x_1, \dots, x_t)$ and which is denoted by $\mathrm{wt}(x_1, \dots, x_t)$. If $\mathrm{wt}(x_1, \dots, x_t) = 1$, then, using that $g(0, \dots, 0)$, $g(e_i)$ and $g(2e_i)$ are different for every $i = 1, \dots, t$ since the array has strength t and index 1, it follows that $g(2e_i) = 2c_i$ and that (1) is true.

Assume now that (1) is true for all $(x_1, \dots, x_t)$ with $\mathrm{wt}(x_1, \dots, x_t) \le l$, where $l \le t-1$. We need to show that (1) is true for all $y = (x_1, \dots, x_t)$ with $\mathrm{wt}(y) = l+1$. Without loss of generality we can assume that $y = (x_1, \dots, x_{l+1}, 0, \dots, 0)$, where each of the first $l+1$ entries is nonzero. We need to show that $g(y) \equiv \sum_{i=1}^{l+1} c_i x_i \pmod 3$. To see this, suppose that $g(y) \not\equiv \sum_{i=1}^{l+1} c_i x_i \pmod 3$. Now define the following other t-tuples.

For $j = 1, \dots, l+1$, let y_{1j} be obtained from y by changing its jth entry to 0.

Let y_2 be obtained from y by changing its $(l+1)$th entry to $2x_{l+1}$.

Let y_3 be obtained from y_2 by changing its 1st entry to 0.

Then, since y and y_{1j} differ for only one entry, it follows that

$$g(y) \ne g(y_{1j}). \tag{2}$$

It follows similarly that

$$g(y) \ne g(y_2), \tag{3}$$

$$g(y_2) \ne g(y_{1,l+1}) \tag{4}$$

and

$$g(y_2) \ne g(y_3). \tag{5}$$

From the induction hypothesis and (2) we see that, for every $j = 1, \dots, l+1$, $g(y) \not\equiv \sum_{i=1}^{l+1} c_i x_i + 2c_j x_j \pmod 3$. With the above assumption that $g(y) \not\equiv \sum_{i=1}^{l+1} c_i x_i \pmod 3$, this implies that $g(y) \equiv \sum_{i=1}^{l+1} c_i x_i + c_j x_j \pmod 3$ for every $j = 1, \dots, l+1$. In particular, this means that $c_1 x_1 \equiv c_2 x_2 \equiv \cdots \equiv c_{l+1} x_{l+1} \pmod 3$, and that $g(y) \equiv (l+2) c_1 x_1 \pmod 3$ and $g(y_{1,l+1}) \equiv l c_1 x_1 \pmod 3$. By (3) and (4) this means that $g(y_2) \equiv (l+1) c_1 x_1 \pmod 3$. But by the induction hypothesis we have now also that $g(y_3) \equiv \sum_{i=2}^{l} c_i x_i + 2c_{l+1} x_{l+1} \equiv (l+1) c_1 x_1 \pmod 3$. This equality of $g(y_2)$ and $g(y_3)$ contradicts (5), and concludes the proof. □

THEOREM 1. *For $t \geq 2$, up to isomorphism, there is a unique orthogonal array $OA(3^t, t+1, 3, t)$.*

PROOF. Clearly, the orthogonal array $A(c_1, \dots, c_t, c)$ $(c \neq 0)$ is isomorphic to the orthogonal array $A(c_1, \dots, c_t, 0)$. Permuting the levels 1 and 2 in those columns of the orthogonal array $A(c_1, \dots, c_t, 0)$ where $c_i = 2$, leads, up to a permutation of the runs, to the orthogonal array $A(1, \dots, 1, 0)$. Therefore, all orthogonal arrays $OA(3^t, t+1, 3, t)$ are isomorphic to each other. □

Only for $t = 2$ do the arrays in the lemma not have the maximum possible number of factors. The maximum is in that case not 3 but 4. It is easy to show that an $OA(9, 4, 3, 2)$ is also unique up to isomorphism.

3. On the existence of orthogonal arrays with three levels and index 2. In this section we will determine the maximum number of factors in an orthogonal array of strength t with three levels and index 2, a maximum that we will denote by $f(2 \cdot 3^t, 3, t)$.

For $t = 2$, applying Theorem 1B of Bose and Bush (1952) leads to $f(2 \cdot 3^2, 3, 2) \leq 7$. An orthogonal array $OA(18, 7, 3, 2)$ can be constructed by the method given in Addelman and Kempthorne (1961). Hence, $f(2 \cdot 3^2, 3, 2) = 7$.

For $t = 3$, Hedayat, Seiden and Stufken (1997) proved that $f(2 \cdot 3^3, 3, 3) = 5$.

For general t, combining the runs of any two orthogonal arrays $OA(3^t, t+1, 3, t)$ as in Section 2 leads to an orthogonal array $OA(2 \cdot 3^t, t+1, 3, t)$. Thus, $f(2 \cdot 3^t, 3, t) \geq t+1$. Further, for $t \geq 4$ it is not hard to see that $f(2 \cdot 3^t, 3, t) \leq f(2 \cdot 3^4, 3, 4) + t - 4$. We will show that $f(2 \cdot 3^4, 3, 4) = 5$, which will then also establish that $f(2 \cdot 3^t, 3, t) = t + 1$ if $t \geq 4$.

Since $f(2 \cdot 3^4, 3, 4) \geq 5$, the result follows if we can show that an orthogonal array $OA(162, 6, 3, 4)$ does not exist. Let A be a 162×6 array based on the symbols 0, 1 and 2. Further, let r be a particular run in A and let $n_i(r)$ denote the number of other runs in A which have exactly i coincidences with r. If A is an orthogonal array $OA(162, 6, 3, 4)$, then it holds that

$$\sum_{i=j}^{6} \binom{i}{j} n_i(r) = \binom{6}{j} (2 \cdot 3^{4-j} - 1), \qquad j = 0, 1, 2, 3, 4. \tag{6}$$

[Bose and Bush (1952)]. Since the strength and index of the array are 4 and 2, respectively, it follows that $n_5(r) + n_6(r) \le 1$. This implies that the only nonnegative integral solutions to (6) are

I. $n_0(r) = 8$, $n_1(r) = 72$, $n_2(r) = 0$, $n_3(r) = 80$, $n_4(r) = 0$, $n_5(r) = 0$ and $n_6(r) = 1$.

II. $n_0(r) = 12$, $n_1(r) = 53$, $n_2(r) = 35$, $n_3(r) = 50$, $n_4(r) = 10$, $n_5(r) = 1$ and $n_6(r) = 0$.

III. $n_0(r) = 13$, $n_1(r) = 48$, $n_2(r) = 45$, $n_3(r) = 40$, $n_4(r) = 15$, $n_5(r) = 0$ and $n_6(r) = 0$.

The desired result, formulated in Theorem 2, is now an immediate consequence of the following three propositions. The proofs of these propositions are given in the appendices.

PROPOSITION 1. *An $OA(162, 6, 3, 4)$, if it exists, cannot contain a run such that the $n_i(r)$'s are as in* I.

PROPOSITION 2. *An $OA(162, 6, 3, 4)$, if it exists, cannot contain a run such that the $n_i(r)$'s are as in* II.

PROPOSITION 3. *Not every run in an $OA(162, 6, 3, 4)$, if it exists, can yield $n_i(r)$'s as in* III.

THEOREM 2. *For $t \ge 4$, the maximum number of factors in an orthogonal array of strength t with three levels and index 2 is $t + 1$.*

In closing, as noted previously an $OA(2 \cdot 3^t, t + 1, 3, t)$ can be constructed by combining the runs of two orthogonal arrays $OA(3^t, t + 1, 3, t)$. This construction can lead to an $OA(2 \cdot 3^t, t + 1, 3, t)$ with no repeated runs (a so-called simple orthogonal array), to one in which each of 3^t runs is repeated twice, or to one in which precisely 3^{t-1} runs are repeated twice. In general, for an array with index λ, we can combine the runs of λ arrays with index unity. A characterization of all possible arrays $OA(\lambda \cdot 3^t, t + 1, 3, t)$ that is similar to that in Seiden and Zemach (1966) for all possible arrays $OA(\lambda \cdot 2^t, t + 1, 2, t)$ is, however, not readily apparent, making it difficult, and possibly not worthwhile, to identify the number of nonisomorphic orthogonal arrays $OA(\lambda \cdot 3^t, t + 1, 3, t)$.

APPENDIX 1

Proof of Proposition 1. If possible, let A be an $OA(162, 6, 3, 4)$ and let r be a run in A such that the nonzero $n_i(r)$'s are as in I: $n_0(r) = 8$, $n_1(r) = 72$,

$n_3(r) = 80$, $n_6(r) = 1$. Without loss of generality, we may take r to be the all-zero run. We will proceed by establishing two properties of the eight runs with none of the factors at level 0, and then argue that these properties are not compatible.

To establish the first property, choose any three factors and any 3-tuple based on levels 1 and 2 only. If this 3-tuple appears α times, say, as a level combination for the selected factors in the 80 runs with precisely three factors at level 0, then, in order to form the right number of 4-tuples in which one of the other factors is at level 0, it must appear $6 - 3\alpha$ times for the three selected factors in the 72 runs with precisely one factor at level 0. Since for any three factors, every 3-tuple must appear six times in the entire array, this means that the selected 3-tuple must appear 2α times for the selected factors in the eight runs with none of the factors at level 0. That establishes the first property: For any three factors we have that every 3-tuple based on 1 and 2 appears an even number of times (possibly zero times) in the eight runs with none of the six factors at level 0.

The second property is established by considering 2-tuples based on 1 and 2 only. Choose any two factors and any 2-tuple based on 1 and 2. All 4-tuples with these two factors at the level combination given by this 2-tuple and with two other factors at level 0 must appear twice. This implies that the 2-tuple must appear four times for the two selected factors in the runs with precisely three factors at level 0. Considering 3-tuples with these two factors at the level combination given by this 2-tuple and with only one other factor at level 0, this now also means that the 2-tuple must appear 12 times for the two selected factors in the runs with precisely one factor at level 0. Since every 2-tuple must appear 18 times in the entire array for every two factors, this means that the selected 2-tuple must appear twice for the selected factors in the eight runs with none of the factors at level 0. This establishes the second property: the eight runs with none of the factors at level 0 must form an orthogonal array $OA(8, 6, 2, 2)$ based on levels 1 and 2.

By the second property the eight runs with none of the factors at level 0 must, up to a permutation of the runs, be of the following form, where $*$ indicates that an entry is 1 or 2.

$$\begin{array}{cccccc}
1 & 1 & * & * & * & * \\
1 & 1 & * & * & * & * \\
1 & 2 & * & * & * & * \\
1 & 2 & * & * & * & * \\
2 & 1 & * & * & * & * \\
2 & 1 & * & * & * & * \\
2 & 2 & * & * & * & * \\
2 & 2 & * & * & * & *
\end{array}$$

By the first property the third entry in the two runs of the form $1\,1 * * * *$ must be the same. The same conclusion holds for the fourth, fifth and sixth entries. In other words, the two runs that start with 1 1 must be identical. The same is true for the two runs that start with 1 2, for those that start with 2 1 and for those that start with 2 2. Hence, the eight runs can be formed by repeating each run of a four-run array twice. In order for the eight-run array to be an $OA(8, 6, 2, 2)$, as required by the second property, the four-run array must be an orthogonal array $OA(4, 6, 2, 2)$. However, the maximum possible value for k in an $OA(4, k, 2, 2)$ is 3, so that we have shown that there cannot be an array that possesses both of the above properties. □

APPENDIX 2

Proof of Proposition 2. If possible, let A be an $OA(162, 6, 3, 4)$ and let r be a row in A such that the nonzero $n_i(r)$'s are as in II: $n_0(r) = 12$, $n_1(r) = 53$, $n_2(r) = 35$, $n_3(r) = 50$, $n_4(r) = 10$, $n_5(r) = 1$. Without loss of generality, we can take r and the run with five coincidences with r as 000001 and 000002. To describe the remaining 160 runs, we let $R(0^j, i)$ denote the number of runs of the array with precisely j of the first five factors at level 0 and with factor 6 at level i. It can be shown that the values of the $R(0^j, i)$'s that are not zero are

$$R(0^3,0) = 20, \quad R(0^3,1) = R(0^3,2) = 10, \quad R(0^2,1) = R(0^2,2) = 20,$$
$$R(0^1,0) = 30, \quad R(0^1,1) = R(0^1,2) = 15, \quad R(0^0,0) = 4,$$
$$R(0^1,1) = R(0^0,2) = 8.$$

Moreover, for each $i \in \{0, 1, 2\}$ and $j \in \{0, 1, 2, 3\}$, each possible pattern of $\binom{5}{j}$ zeros for the first five factors appears equally often in the $R(0^j, i)$ runs.

Our attention will focus entirely on the 54 runs with factor 6 at level 0. Restricted to the first five factors, these 54 runs form an orthogonal array $OA(54, 5, 3, 3)$ with each pattern of three zeros appearing twice (20 runs), each pattern of one zero appearing six times (30 runs), and with four runs with none of the factors at level 0. For this $OA(54, 5, 3, 3)$, if it exists, choose two factors and a 2-tuple based on 1 and 2 only. If this 2-tuple appears α times, say, for the two factors in the 20 runs with precisely three zeros, then, in order to form the right number of 3-tuples with one of the other factors at level 0, it must appear $6 - 3\alpha$ times for the selected factors in the 30 runs with precisely one zero. Consequently, it must appear 2α times for the selected factors in the four runs with none of the factors at level 0. Thus, each 2-tuple must appear an even number of times (possibly zero times) for any pair of factors in the four runs with none of the five factors at level 0. It is easily seen that this implies that at least two of these four runs must be identical. That also means that the original $OA(162, 6, 3, 4)$ must have two identical runs, which implies that there is a row r with $n_6(r) = 1$. This possibility, however, was already excluded by Proposition 1, which establishes the result. □

APPENDIX 3

Proof of Proposition 3. If possible, let A be an $OA(162, 6, 3, 4)$ such that for each run r the nonzero $n_i(r)$'s are as in III: $n_0(r) = 13$, $n_1(r) = 48$, $n_2(r) = 45$, $n_3(r) = 40$, $n_4(r) = 15$. Without loss of generality, we may assume that the all-zero run is one of the runs. It can be shown that the remaining 161 runs can be partitioned into the following five parts.

Part A. Fifteen runs with precisely four factors at level 0, each pattern of four zeros appearing once.

Part B. Forty runs with precisely three factors at level 0, each pattern of three zeros appearing twice.

Part C. Forty-five runs with precisely two factors at level 0, each pattern of two zeros appearing thrice.

Part D. Forty-eight runs with precisely one factor at level 0, each pattern of one zero appearing eight times.

Part E. Thirteen runs with none of the factors at level 0.

The 15 rows in Part A are as shown below, where we have named the runs from A1 to A15:

A1	0	0	0	0		
A2	0	0	0		0	
A3	0	0	0			0
A4	0	0		0	0	
A5	0	0		0		0
A6	0	0			0	0
A7	0		0	0	0	
A8	0		0	0		0
A9	0		0		0	0
A10	0			0	0	0
A11		0	0	0	0	
A12		0	0	0		0
A13		0	0		0	0
A14		0		0	0	0
A15			0	0	0	0

In runs A11–A15 we can without loss of generality assume that at least three of the runs have factor 1 at level 1. We can also assume without loss of generality that these three runs are A11, A12 and A13. Further, in the runs A11, A12 and A13 we can choose the nonzero levels of factors 6, 5 and 4, respectively, as level 2. Thus, runs A11–A13 can be taken as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \end{bmatrix}.$$

From these three runs and the fact that any two runs can have at most four coincidences, we find that Part B must contain the following six runs:

$$\begin{bmatrix}
\begin{matrix}2\\2\end{matrix} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \boxed{\begin{matrix}1\\2\end{matrix}} & \begin{matrix}1\\2\end{matrix} \\[2ex]
\begin{matrix}2\\2\end{matrix} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \boxed{\begin{matrix}1\\2\end{matrix}} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}1\\2\end{matrix} \\[2ex]
\begin{matrix}2\\2\end{matrix} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \boxed{\begin{matrix}1\\2\end{matrix}} & \begin{matrix}1\\2\end{matrix} & \begin{matrix}0\\0\end{matrix}
\end{bmatrix}$$

(A box that contains a 1 and a 2 indicates that of the two entries covered by the box exactly one must be equal to 1 and the other must be equal to 2.) These six runs and runs A11–A13 determine runs A1–A3 in Part A as:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Considering runs A1 and A2 and using again that any two runs have at most four coincidences leads to the following two runs in Part B:

$$\begin{bmatrix}
\begin{matrix}0\\0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \boxed{\begin{matrix}1\\2\end{matrix}} & \boxed{\begin{matrix}1\\2\end{matrix}} & \begin{matrix}2\\2\end{matrix}
\end{bmatrix}.$$

These last two runs and runs A2 and A3 contain the 4-tuple 0001 three times for the first four factors. This contradicts that the array is an orthogonal array with strength 4 and index 2. □

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