The Annals of Statistics
1997, Vol. 25, No. 5, 2018–2023

PROOF OF THE CONJECTURES OF H. UHLIG ON THE SINGULAR MULTIVARIATE BETA AND THE JACOBIAN OF A CERTAIN MATRIX TRANSFORMATION

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Uhlig proposes two conjectures. The first concerns the Jacobian of the transformation $Y = B \times B'$ where B is the matrix $m \times m$ and m and X, Y belong to the class of positive semidefinite matrices of the order of $m \times m$ of rank $n < m$, $S^+_{m,n}$. The second is concerned with the singular multivariate Beta distribution. This article seeks to prove the two conjectures. The latter result is then extended to the case of the singular multivariate F distribution, and the respective density functions are located for the nonzero positive eigenvalues of the singular Beta and F matrices.

1. Introduction. Let

$$Y = \begin{pmatrix} Y_1' \\ Y_2' \\ \vdots \\ Y_n' \end{pmatrix}$$

be a matrix of the order $n \times m (n \geq m)$, in which the rows are independent and identically distributed $\mathscr{N}_m(0, \Sigma)$ where Σ is $m \times m$ positive definite. Then, $X = \sum_{i=1}^n Y_i Y_i'$ has a Wishart distribution with n degrees of freedom, denoted as $\mathscr{W}_m(n, \Sigma)$. If $n < m$, this distribution is called pseudo-Wishart or singular Wishart ([4], [3], page 72). Now let $B \sim \mathscr{W}_m(\rho, \Sigma)$, $A \sim \mathscr{W}_m(n, \Sigma)$ and $A + B = T'T$, where T is an upper-triangular matrix with positive diagonal elements. Then the matrix $U = T'^{-1} A T^{-1}$ of the order $m \times m$ and rank $n < m$ has a singular multivariate Beta distribution.

Uhlig [4] found the density function of U on the manifold of positive semidefinite matrices of the order $m \times m$ with n nonzero eigenvalues, $\mathscr{S}^+_{m,n}$, when $n = 1$. Furthermore, he presented as an open problem the case in which $1 < n < m$, giving as a conjecture an expression for the density function in this case. Additionally, given the matrices $X, Y \in \mathscr{S}^+_{m,n}$ and the nonsingular B matrix of the order $m \times m$, Uhlig [4] calculated the Jacobian

Received December 1995; revised February 1997.

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AMS 1991 *subject classifications*. Primary 62H10; secondary 62E15.

Key words and phrases. Wishart distribution, beta distribution, singular matrix distributions.

of the transformation $X = BYB'$ when $n = 1$, and conjectured about the result when $1 < n < m$.

This article proves both conjectures in the general case, that is, when $1 \leq n \leq m$. The proof of the singular multivariate Beta density is given as a consequence of the proof of the conjecture of the Jacobian (see Theorem 2). The proof presented in the calculation of the Jacobian of the transformation $X = BYB'$ is given indirectly: two densities, $f_X(X)$ and $g_Y(Y)$, are proposed for $\mathscr{S}_{m,n}^+$, such that these are related by the transformation $X = BYB'$, and thus, by using the variable change theorem, the Jacobian of the transformation is determined (see Theorem 1). Finally, these results are applied to the definition of the multivariate F singular distribution, called by some authors Type II Beta distribution (see Theorem 3), and to the determination of the densities of the positive nonzero eigenvalues of the singular Beta and F matrices (see Theorem 4).

2. Results. Let $\mathscr{S}$ be a matrix of the order $m \times m$ such that $\mathscr{S} \sim \mathscr{W}_m(n, \Sigma)$, $\Sigma > 0$ positive definite and $n \leq m$. Then if B is a nonsingular matrix of the order $m \times m$, $BSB' \sim \mathscr{W}_m(n, B\Sigma B')$; note that this is still valid whether B is rectangular or singular, ([1] page 303). In this light, let us consider Uhlig's first conjecture.

THEOREM 1. *Let $X, Y \in \mathscr{S}_{m,n}^+$ be related by $X = BYB'$, where B is $m \times m$ of full rank. Form the representations $X = G_1KG_1'$, $Y = H_1LH_1'$, where $G_1H_1 \in V_{n,m}$ the Stiefel manifold of $m \times n$ matrices H_1 with orthogonal columns, $H_1'H_1 = I_n$ and K, L are $n \times n$ diagonals, $L = \operatorname{diag}(l_1, \dots, l_n)$ and $K = \operatorname{diag}(k_1, \dots, k_n)$ with $l_1 > \cdots > l_n > 0$ and $k_1 > \cdots > k_n > 0$. Then*

$$\begin{aligned}(dX) &= |G_1'BH_1|^{m+1-n}|B|^n(dY) = |H_1'B'G_1|^{m+1-n}|B|^n(dY)\\ &= |K|^{(m+1-n)/2}|L|^{(m+1-n)/2}|B|^n(dY).\end{aligned}$$

PROOF. Suppose that $Y \sim \mathscr{W}_m(n, \Sigma)$. Then $X \sim \mathscr{W}_m(n, \Xi)$ with $\Xi = B\Sigma B'$ and thus the density functions are given, respectively, by

$$f_X(X) = \frac{\pi^{-(mn-n^2)/2}|K|^{(n-m-1)/2}}{2^{mn/2}\Gamma_n\left[\frac{1}{2}n\right]|\Xi|^{n/2}}\operatorname{etr}\left(-\frac{1}{2}\Xi^{-1}X\right)$$

and

$$g_Y(Y) = \frac{\pi^{-(mn-n^2)/2}|L|^{(n-m-1)/2}}{2^{mn/2}\Gamma_n\left[\frac{1}{2}n\right]|\Sigma|^{n/2}}\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}Y\right)$$

with $X = G_1KG_1'$, $Y = H_1LH_1'$ [4] and $\operatorname{etr}(\cdot)$ denotes $\exp(\operatorname{tr}(\cdot))$. The Jacobian of the transformation $X = BYB'$ is written as $(dX) = |J(X \to Y)|(dY)$. From the variable change theorem, it is known ([1], page 166) that

$$\begin{aligned} g_Y(Y) &= f_X(BYB')|J(X \to Y)| \\ &= \frac{\pi^{-(mn-n^2)/2}|K|^{(n-m-1)/2}}{2^{mn/2}\Gamma_n\left[\frac{1}{2}n\right]|\Xi|^{n/2}} \\ &\quad \times \operatorname{etr}\left(-\frac{1}{2}\Xi^{-1}X\right)|_{X=BYB'}|J(X \to Y)| \\ &= \frac{\pi^{-(mn-n^2)/2}|G_1'BYB'G_1|^{(n-m-1)/2}}{2^{mn/2}\Gamma_n\left[\frac{1}{2}n\right]|\Xi|^{n/2}} \\ &\quad \times \operatorname{etr}\left(-\frac{1}{2}\Xi^{-1}BYB'\right)|J(X \to Y)|. \end{aligned} \tag{1}$$

This follows because

$$|G_1'BYB'G_1| = |G_1'XG_1| = |K|. \tag{2}$$

We also have

$$|G_1'BYB'G_1| = |G_1'BH_1|^2|L|. \tag{3}$$

Furthermore, given that $\Xi = B\Sigma B'$,

$$\operatorname{tr}\Xi^{-1}BYB' = \operatorname{tr}\Sigma^{-1}Y \tag{4}$$

and

$$|\Xi| = |B\Sigma B'| = |B|^2|\Sigma|. \tag{5}$$

Substituting (3), (4) and (5) in (1), we have

$$g_Y(Y) = g_Y(Y)\frac{|G_1'BH_1|^{n-m-1}|J(X \to Y)|}{|B|^n}$$

and therefore

$$(dX) = |G_1'BH_1|^{m-n+1}|B|^n(dY).$$

Finally, the other two expressions for (dX) are obtained from (2) and (3) and from the fact that $|A| = |A'|$. □

An alternative proof to that given in Theorem 1 may be developed by assuming that the conjecture is true; from the density of X, and employing this Jacobian, we then find the density of Y, and the desired result is then found by the variable change theorem

The following theorem proves the second conjective proposed by Uhlig [4] in his Theorem 7.

THEOREM 2. *Let $m > 1$ be an integer and let $p \geq m$ and $n < m$. Let A and B be independent, where $A \sim \mathscr{W}_m(n, \Sigma)$ and $B \sim \mathscr{W}_m(p, \Sigma)$. Put $A + B = T'T$,* *where T is an upper-triangular $m \times m$ matrix with positive diagonal elements. Let U be the $m \times m$ symmetric matrix defined by*

$$U = T'^{-1} A T^{-1}$$

Then $A + B$ and U are independent; $A + B \sim \mathscr{W}_m(p + n, \Sigma)$ and the density of U on the space $\mathscr{S}_{m,p}^{+}$ with respect to the volume element (dU) on this space is

$$f_U(U) = \pi^{(-mn+n^2)/2} \frac{\Gamma_m\left[\frac{1}{2}(n + p)\right]}{\Gamma_n\left[\frac{1}{2}n\right]\Gamma_m\left[\frac{1}{2}p\right]} |L|^{(n-m-1)/2} |I_m - U|^{(p-m-1)/2},$$

where $U = H_1 L H_1'$, $H_1 \in V_{n,m}$, $L = \operatorname{diag}(l_1, \dots, l_n)$, $1 > l_1 > l_2 > \cdots > l_n > 0$ and

$$(dU) = 2^{-n} \prod_{i=1}^{n} l_i^{m-n} \prod_{i<j}^{n} (l_i - l_j)(H_1'\, dH_1) \wedge \bigwedge_{i=1}^{n} dl_i;$$

see [4], *Theorem* 2. *This will be denoted as $U \sim \mathscr{B}_m(n/2, p/2)$.*

Taking into account Theorem 1, the proof is identical to that given for Theorem 7 [4].

One consequence of the above result is the F singular density function, also called Beta Type II; [3] page 92.

THEOREM 3. *Let $m > 1$ be an integer and let $p \leq m$ and $n < m$. Let A and B be independent, where $A \sim \mathscr{W}_m(n, I_m)$ and $B \sim \mathscr{W}_m(p, I_m)$. Put $B = T'T$, where T is an upper-triangular $m \times m$ matrix with positive diagonal elements. Let F be the $m \times m$ symmetric matrix defined by*

$$F = T^{-1} A T'^{-1}.$$

Then the density of F on the $\mathscr{S}_{m,p}^{+}$ with respect to the volume element (dF) on this space defined above is

$$h_F(F) = \pi^{(-mn+n^2)/2} \frac{\Gamma_m\left[\frac{1}{2}(n + p)\right]}{\Gamma_n\left[\frac{1}{2}n\right]\Gamma_m\left[\frac{1}{2}p\right]} |Q|^{(n-m-1)/2} |I_m + F|^{-(p+n)/2},$$

where $F = H_1 Q H_1'$, $H_1 \in V_{n,m}$, $Q = \operatorname{diag}(q_1, \dots, q_n)$, $q_1 > q_2 > \cdots q_n > 0$. This will be denoted as $F \sim \mathscr{F}_m(n/2, p/2)$.

PROOF. Find the factorization $A = G_1 K G_1'$, where $G_1 \in V_{n,m}$ and $K = \operatorname{diag}(k_1 k_2, \dots, k_n)$, $k_1 > k_2 > \cdots > k_n > 0$. Then the joint density of A and B is

$$d(n, m, p) |B|^{(p-m-1)/2} |K|^{(n-m-1)/2} \operatorname{etr}\left(-\tfrac{1}{2}(A + B)\right)(dA) \wedge (dB),$$

where

$$d(n, m, p) = \frac{\pi^{(-mn+n^2)/2}}{2^{m(p+n)/2} \Gamma_n\left[\frac{1}{2}n\right]\Gamma_m\left[\frac{1}{2}p\right]}.$$

Now let $B = T'T$, there T is an upper-triangular matrix with positive diagonal elements and $A = TFT'$. Find the factorization $F = H_1 Q H_1'$, $H_1 \in V_{n,m}$, $Q = \operatorname{diag}(q_1 q_2, \dots, q_n)$, $q_1 > q_2 > \cdots > q_n > 0$. Then, from Theorem 1,

$$(dA) \wedge (dB) = |K|^{(m+1-n)/2} |Q|^{-(m+1-n)/2} |T|^n (dF) \wedge (dB);$$

remembering that T is a function of B, the joint density of B and F is given by

$$d(n,m,p) |Q|^{(n-m-1)/2} |B|^{(n+p-m-1)/2} \operatorname{etr}\left(-\tfrac{1}{2}(I+F)B\right)(dF) \wedge (dB).$$

The result follows from integrating with respect to $B > 0$ (see [2], page 61, Theorem 2.1.11).

Finally, we find the density functions of the nonzero eigenvalues of the Beta and F matrices.

THEOREM 4. (i) *Suppose that* $U \sim \mathscr{B}_m(n/2, p/2)$, $n < m$. *Then*

$$f(\lambda_1, \dots, \lambda_n) = \frac{\pi^{n^2/2} \Gamma_m\left[\frac{1}{2}(n+p)\right]}{\Gamma_m\left[\frac{1}{2}p\right] \Gamma_n\left[\frac{1}{2}n\right] \Gamma_m\left[\frac{1}{2}n\right]} \times \prod_{i=1}^{n} \lambda_1^{(m-n-1)/2} \prod_{i=1}^{n} (1-\lambda_i)^{(p-m-1)/2} \prod_{i<j}^{n} (\lambda_i - \lambda_j).$$

(ii) *Let* $F \sim \mathscr{F}_m(n/2, p/2)$, $n < m$. *Then*

$$f(\delta_1, \dots, \delta_n) = \frac{\pi^{n^2/2} \Gamma_m\left[\frac{1}{2}(n+p)\right]}{\Gamma_m\left[\frac{1}{2}p\right] \Gamma_n\left[\frac{1}{2}n\right] \Gamma_m\left[\frac{1}{2}n\right]} \times \prod_{i=1}^{n} \delta_i^{(m-n-1)/2} \prod_{i=1}^{n} (1+\delta_i)^{-(n+p)/2} \prod_{i<j}^{n} (\delta_i - \delta_j),$$

where λ_1, δ_i, $i = 1, \dots, n$, *are the nonzero eigenvalues of* U *and* F, *respectively.*

PROOF. (i) From Theorem 2, note that L is a diagonal matrix; moreover, $l_i = \lambda_i$, λ_i the ith nonzero eigenvalue of U, $i = 1, \dots, n$ from which

$$|I_m - U| = |I_m - H_1 L H_1'| = |I_n - L| = \prod_{i=1}^{n} (1 - \lambda_i),$$

given that $U = H_1 L H_1'$, with $H_1 \in V_{n,m}$. Furthermore $|L| = \prod_{i=1}^{n} \lambda_i$ and from Theorem 2,

$$(dU) = 2^{-n} \prod_{i=1}^{n} l_i^{m-n} \prod_{i<j}^{n} (l_i - l_j)(H_1' \, dH_1) \wedge \bigwedge_{i=1}^{n} dl_i.$$

Substituting these results in the density of U and given that

$$\int_{H_1 \in V_{n,m}} (H_1' \, dH_1) = 2^n \pi^{mn/2} / \Gamma_m\left[\tfrac{1}{2}n\right],$$

(see [2], page 70, Theorem 2.1.15) the joint density of $\lambda_1, \dots, \lambda_m$ is obtained.

(ii) The proof is analagous to that of the above part (i). □

Acknowledgments. The authors are indebted to Professor Lawrence D. Brown, an associate editor and the referees for constructive comments on the preliminary version of this paper.

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