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EDGEWORTH EXPANSION FOR U-STATISTICS UNDER MINIMAL CONDITIONS

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Berry–Esseen bounds for U-statistics under the optimal moment conditions were derived by Koroljuk and Borovskich and Friedrich. Under the same optimal moment assumptions with an additional nonlattice condition, we establish a one-term Edgeworth expansion with remainder $o(n^{-1/2})$ for U-statistics.

1. Introduction and main results. Let $X_1, X_2, \ldots, X_n, n \geq 2$, be independent and identically distributed (i.i.d.) random variables with common distribution function $F(x)$. Let $h(x, y)$ be a real-valued Borel measurable function, symmetric in its arguments with $Eh(X_1, X_2) = \theta$. Define a U-statistic by

$$U_n = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} h(X_i, X_j)$$

and

$$g(X_1) = E\,[h(X_1, X_2) \mid X_1] - \theta, \qquad \sigma_g^2 = \operatorname{Var}\bigl(g(X_1)\bigr).$$

Throughout this paper, we shall assume that $\sigma_g^2 > 0$. Our primary goal is to investigate the asymptotic distribution of the standardized U-statistic defined by

$$G_n(x) = P\biggl(\frac{\sqrt{n}(U_n - \theta)}{2\sigma_g} \leq x\biggr).$$

It is well known that $G_n(x)$ converges to the standard normal distribution function, $\Phi(x)$, provided $Eh^2(X_1, X_2) < \infty$ [see Hoeffding (1948)]. In fact, this moment condition can further be reduced to $Eg^2(X_1) < \infty$ and $E|h(X_1, X_2)|^{4/3} < \infty$; see Remark 4.2.4 of Koroljuk and Borovskich [(1994), page 131].

In recent years, there has been considerable interest in obtaining rates of convergence in the asymptotic normality for U-statistics, for instance, by Grams and Serfling (1973), Bickel (1974) and Chan and Wierman (1977). A sharper Berry–Esseen bound was given by Callaert and Janssen (1978), which states that

$$\sup_{t \in R} |G_n(x) - \Phi(x)| \leq A\sigma_g^{-3} E|h(X_1, X_2)|^3 n^{-1/2}$$

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under the assumption that $E|h(X_1, X_2)|^3 < \infty$, where A is an absolute positive constant. However, we note that the sharpest Berry–Esseen bound of order $O(n^{-1/2})$ comes from Koroljuk and Borovskich (1981) and Friedrich (1989), who established the ideal bound

$$\sup_x |G_n(x) - \Phi(x)| \le \{A_1 \sigma_g^{-3} E|g(X_1)|^3 + A_2 \sigma_g^{-5/3} E|h(X_1, X_2)|^{5/3}\} n^{-1/2},$$

under the condition that

$$E|g(X_1)|^3 < \infty, \qquad E|h(X_1, X_2)|^{5/3} < \infty, \tag{1.1}$$

where A_1 and A_2 are absolute positive constants. Indeed, Bentkus, Götze and Zitikis (1994) showed that the moment conditions in (1.1) are the weakest possible in the Berry–Esseen bounds of order $O(n^{-1/2})$ for U-statistics. It is worth mentioning that Alberink and Bentkus (2001) considered another type of bound for U-statistics under optimal conditions.

One way to improve the normal approximation is to use the Edgeworth expansion. For instance, Bickel, Götze and van Zwet (1986) derived the following Edgeworth expansion of order $o(n^{-1/2})$:

$$\sup_x |G_n(x) - F_n(x)| = o(n^{-1/2}),$$

[where $F_n(x)$ is given in Theorem 1.1 below] under the nonlattice condition of $g(X_1)$ and the moment conditions

$$E|g(X_1)|^3 < \infty, \qquad E|h(X_1, X_2)|^{2+\varepsilon} < \infty \qquad \text{for some } \varepsilon > 0. \tag{1.2}$$

Clearly, these moment conditions are stronger than those in (1.1), the optimal conditions for Berry–Esseen bounds. The natural question is whether the moment conditions in (1.2) can be reduced to those in (1.1), which would then serve as the optimal moment conditions for one-term Edgeworth expansions as well as for Berry–Esseen bounds. The answer to this question is affirmative, as can be seen below.

THEOREM 1.1. *Assume that*:

(i) $\sigma_g^2 > 0$, $E|g(X_1)|^3 < \infty$, $E|h(X_1, X_2)|^{5/3} < \infty$;
(ii) *the d.f. of* $g(X_1)$ *is nonlattice*.

Then, as $n \to \infty$,

$$\sup_x |G_n(x) - F_n(x)| = o(n^{-1/2}), \tag{1.3}$$

where

$$F_n(x) = \Phi(x) - \frac{\Phi^{(3)}(x)}{6\sqrt{n}\sigma_g^3}\{Eg^3(X_1) + 3Eg(X_1)g(X_2)h(X_1, X_2)\}.$$

2. Proofs. Throughout the proofs, we use C to denote a generic positive constant, which may differ at each occurrence.

The proof of our main theorem relies critically on the next lemma.

LEMMA 2.1. *For $n \geq 2$, let $V_n(x)$ and $W_n(x, y)$ be real functions and $W_n(x, y)$ be symmetric in its arguments. Assume that*:

(a) $EV_n(X_1) = 0$, $EV_n^2(X_1) = 1$, $|V_n(X_1)|^3$ *are uniformly integrable and the d.f. of $V_n(X_1)$ is nonlattice for all sufficiently large n.*

(b) $E(W_n(X_1, X_2)|X_1) = 0$, $|W_n(X_1, X_2)|^{5/3}$ *are uniformly integrable and $|W_n(X_i, X_j)| \leq n^{3/2}$ for all $n \geq 2$ and $i \neq j$.*

Then, we have, as $n \to \infty$,

$$\sup_x \left| P\left(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} V_n(X_j) + \frac{1}{n^{3/2}} \sum_{1 \leq i < j \leq n} W_n(X_i, X_j) \leq x\right) - F_n^{(1)}(x) \right| = o(n^{-1/2}), \tag{2.1}$$

where

$$F_n^{(1)}(x) = \Phi(x) - \frac{\Phi^{(3)}(x)}{6\sqrt{n}}\big(EV_n^3(X_1) + 3EV_n(X_1)V_n(X_2)W_n(X_1, X_2)\big).$$

PROOF. For convenience, we write

$$\rho = \sup_{n \geq 2} E|V_n(X_1)|^3,$$

$$A_0 = \sup_{n \geq 2} E|W_n(X_1, X_2)|^{5/3},$$

$$\gamma_n(t) = Ee^{itV_n(X_1)/\sqrt{n}},$$

$$f_n(t) = \{1 + n[\gamma_n(t) - 1] + t^2/2\}e^{-t^2/2},$$

$$B_n = n^{-5/2} \sum_{1 \leq i < j \leq n} EV_n(X_i)V_n(X_j)W_n(X_i, X_j),$$

$$\varphi_n(t) = -t^2 B_n e^{-t^2/2},$$

$$\mathcal{L}_n(x) = nE\{\Phi(x - V_n(X_1)/\sqrt{n}) - \Phi(x)\} - \tfrac{1}{2}\Phi^{(2)}(x),$$

$$S_k = n^{-1/2} \sum_{j=1}^{k} V_n(X_j) \qquad \text{for } 1 \leq k \leq n,$$

$$\Delta_m = n^{-3/2} \sum_{j=m+1}^{n} W_n(X_m, X_j) \qquad \text{for } 1 \leq m \leq n-1,$$

and

$$\Delta_{n,m} = \begin{cases} \sum_{k=m}^{n-1} \Delta_k, & \text{if } 0 < m < n, \\ 0, & \text{if } m \geq n, \end{cases}$$

$$F_n^*(x) = \Phi(x) + \mathcal{L}_n(x) - B_n \Phi^{(3)}(x).$$

Simple calculation shows that $\rho < \infty$, $A_0 < \infty$,

$$\int_{-\infty}^{\infty} e^{itx} \, d\big(\Phi(x) + \mathcal{L}_n(x) - B_n \Phi^{(3)}(x)\big) = f_n(t) + it\varphi_n(t).$$

Applying Esseen's smoothing lemma with $T = a\sqrt{n}$, where a is chosen to be so large that $|dF_n^*(x)/dx| \leq \varepsilon a$, we have

$$\begin{aligned} &\sup_x \big| P(S_n + \Delta_{n,1} \leq x) - F_n^*(x) \big| \\ &\quad \leq \frac{1}{\pi} \int_{|t| \leq a\sqrt{n}} \frac{1}{|t|} \big| Ee^{it(S_n + \Delta_{n,1})} - f_n(t) - it\varphi_n(t) \big| \, dt + \frac{\varepsilon}{\sqrt{n}} \\ &\quad \leq \sum_{j=1}^{5} \beta_{jn} + \frac{\varepsilon}{\sqrt{n}}, \end{aligned} \tag{2.2}$$

where

$$\beta_{1n} = \int_{|t| \leq \sqrt{n}/(4\rho)} \frac{1}{|t|} \big| Ee^{it(S_n + \Delta_{n,1})} - Ee^{itS_n} - itE\Delta_{n,1}e^{itS_n} \big| \, dt,$$

$$\beta_{2n} = \int_{|t| \leq \sqrt{n}/(4\rho)} \frac{1}{|t|} \big| Ee^{itS_n} - f_n(t) \big| \, dt,$$

$$\beta_{3n} = \int_{|t| \leq \sqrt{n}/(4\rho)} \big| E\Delta_{n,1}e^{itS_n} - \varphi_n(t) \big| \, dt,$$

$$\beta_{4n} = \int_{\sqrt{n}/(4\rho) \leq |t| \leq a\sqrt{n}} \frac{1}{|t|} \big| Ee^{it(S_n + \Delta_{n,1})} \big| \, dt,$$

$$\beta_{5n} = \int_{\sqrt{n}/(4\rho) \leq |t| \leq a\sqrt{n}} \frac{1}{|t|} \big| f_n(t) + it\varphi_n(t) \big| \, dt.$$

Since ε is arbitrary, (2.1) follows easily from

$$\beta_{jn} = o(n^{-1/2}), \qquad j = 1, 2, 3, 4, 5, \tag{2.3}$$

and

$$\left| \mathcal{L}_n(x) + \frac{\Phi^{(3)}(x)}{6\sqrt{n}} EV_n^{(3)}(X_1) \right| = o(n^{-1/2}). \tag{2.4}$$

The proofs of $\beta_{2n} = o(n^{-1/2})$ and (2.4) follow easily from the classical results; see Hall (1982), for example. The proof of $\beta_{5n} = o(n^{-1/2})$ is simple and hence omitted. It then remains to show $\beta_{jn} = o(n^{-1/2})$ for $j = 1, 3$ and 4.

First, we prove $\beta_{3n} = o(n^{-1/2})$. Similar to the proof of Bickel, Götze and van Zwet (1986), we have

$$\begin{aligned} &EW_n(X_i, X_j)e^{it(V_n(X_i)+V_n(X_j))/\sqrt{n}} \\ &\qquad = -\frac{t^2}{n}EV_n(X_i)V_n(X_j)W_n(X_i, X_j) + \theta_{ij}^{(1)}(t), \end{aligned} \tag{2.5}$$

where, by using $|e^{ix} - 1 - ix| \le 2|x|^{6/5}$ and Holder's inequality, we have

$$\begin{aligned} \theta_{ij}^{(1)}(t) &\le \left|EW_n(X_i, X_j)\left(e^{itV_n(X_i)/\sqrt{n}} - 1 - \frac{itV_n(X_i)}{\sqrt{n}}\right)\left(e^{itV_n(X_j)/\sqrt{n}} - 1\right)\right| \\ &\quad + \frac{|t|}{\sqrt{n}}\left|EW_n(X_i, X_j)V_n(X_i)\left(e^{itV_n(X_j)/\sqrt{n}} - 1 - \frac{itV_n(X_j)}{\sqrt{n}}\right)\right| \\ &\le 2\left(\frac{|t|}{\sqrt{n}}\right)^{11/5} E|W_n(X_1, X_2)| \\ &\quad \times \left(|V_n(X_1)|^{6/5}|V_n(X_2)| + |V_n(X_1)||V_n(X_2)|^{6/5}\right) \\ &\le 4\left(\frac{|t|}{\sqrt{n}}\right)^{11/5} \left\{E|W_n(X_1, X_2)|^{5/3}\right\}^{3/5}\left\{E|V_n(X_1)|^3 E|V_n(X_2)|^{5/2}\right\}^{2/5} \\ &\le 4\left(\frac{|t|}{\sqrt{n}}\right)^{11/5} A_0^{3/5}\rho^{11/15}. \end{aligned}$$

By combining this with the relations [see Petrov (1975), pages 109–111]

$$|\gamma_n^n(t)| \le 2e^{-t^2/3} \quad \text{and} \quad \left|\gamma_n^n(t) - e^{-t^2/2}\right| \le 16|t|^3 e^{-t^2/3}\frac{E|V_n(X_1)|^3}{\sqrt{n}} \tag{2.6}$$

for $|t| \le \sqrt{n}/(4\rho)$, we obtain

$$\begin{aligned} &EW_n(X_i, X_j)e^{itS_n} \\ &\qquad = \gamma_n^{n-2}(t)EW_n(X_i, X_j)e^{it(V_n(X_i)+V_n(X_j))/\sqrt{n}} \\ &\qquad = -\frac{1}{n}t^2 e^{-t^2/2}EV_n(X_i)V_n(X_j)W_n(X_i, X_j) + \theta_{ij}^{(2)}(t), \end{aligned} \tag{2.7}$$

where

$$\left|\theta_{ij}^{(2)}(t)\right| \le C(n^{-3/2}\rho^{5/3} + n^{-11/10}\rho^{11/15})A_0^{3/5}(t^2 + t^6)e^{-t^2/3}.$$

It follows from (2.7) that

$$\beta_{3n} = \int_{|t|\le\sqrt{n}/(4\rho)} n^{-3/2}\left|\sum_{1\le i<j\le n}\theta_{ij}^{(2)}(t)\right|dt$$
$$\le C(n^{-1}\rho^{5/3} + n^{-3/5}\rho^{11/15})A_0^{3/5} = o(n^{-1/2}).$$

Secondly, we prove $\beta_{4n} = o(n^{-1/2})$. Note that for any fixed $1 \le m < k \le n$ and $1 \le p \le 2$, we have

$$E|\Delta_{n,m} - \Delta_{n,k}|^p \le 8n^{-3p/2+1}(k-m)E|W_n(X_1, X_2)|^p; \tag{2.8}$$

see Theorem 2.1.3 in Koroljuk and Borovskich (1994). In view of (2.8), by using the inequality $|e^{ix} - 1 - ix| \le 2|x|^{5/3}$, we have

$$\begin{aligned}&\left|Ee^{it(S_n+\Delta_{n,1})} - Ee^{it(S_n+\Delta_{n,m})} - itE(\Delta_{n,1} - \Delta_{n,m})e^{it(S_n+\Delta_{n,m})}\right|\\ &\quad\le 16|t|^{5/3}n^{-3/2}mA_0.\end{aligned} \tag{2.9}$$

Noting that $\Delta_{n,m}$ depends only on $X_m, \dots, X_n$, it follows from (2.9) that, for any $1 \le m \le n$,

$$\begin{aligned}&\left|Ee^{it(S_n+\Delta_{n,1})}\right|\\ &\quad\le |\gamma_n^{m-1}(t)| + n^{-1/2}mA_0^{3/5}|t||\gamma_n^{m-3}(t)| + 16|t|^{5/3}n^{-3/2}mA_0.\end{aligned} \tag{2.10}$$

Since the d.f. of $V_n(X_1)$ is nonlattice for sufficiently large n, by using Lemma 4 given in Feller [(1971), page 501], there exists a $b > 0$ such that when $1/(4\rho) \le |t|/\sqrt{n} \le a$,

$$|\gamma_n(t)| \le 1 - b < e^{-b}.$$

Therefore, by choosing $m = [\frac{4\log n}{b}] + 3$ in (2.10), simple calculation shows that

$$\beta_{4n} = \int_{\sqrt{n}/(4\rho)\le|t|\le a\sqrt{n}} \frac{1}{|t|}\left|Ee^{it(S_n+\Delta_{n,1})}\right|dt = o(n^{-1/2}).$$

Finally, we shall prove $\beta_{1n} = o(n^{-1/2})$. Noting that $\Delta_{n,1} = \sum_{m=1}^{n-1}\Delta_m$, $\Delta_m = \Delta_{n,m} - \Delta_{n,m+1}$ and $\Delta_{n,n} = 0$, we may write

$$\begin{aligned}&Ee^{it(S_n+\Delta_{n,1})} - Ee^{itS_n} - itE\Delta_{n,1}e^{itS_n}\\ &\quad= \sum_{m=1}^{n-1} Ee^{it(S_n+\Delta_{n,m+1})}\left(e^{it\Delta_m} - 1 - it\Delta_m\right)\\ &\qquad + it\sum_{m=1}^{n-1} E\Delta_m\left(e^{it(S_n+\Delta_{n,m+1})} - e^{itS_n}\right)\\ &\quad= Z_{n1}(t) + itZ_{n2}(t) \qquad \text{say.}\end{aligned} \tag{2.11}$$

Recalling (2.6), (2.8) and that Δ_m depends only on $X_m, \dots, X_n$, it is clear that

$$
\begin{aligned}
|Z_{n1}(t)| &\le 2\sum_{m=1}^{n-1} |t|^{5/3+\delta} e^{-(m-1)t^2/(3n)} E|\Delta_m|^{5/3+\delta} \\
&\le 16 n^{-3(\delta+1)/2} E|W_n(X_1, X_2)|^{5/3+\delta} \sum_{m=1}^{n-1} |t|^{5/3+\delta} e^{-(m-1)t^2/(3n)},
\end{aligned}
\tag{2.12}
$$

where δ is chosen so that $0 < \delta < 1/3$. By noting that $|W_n(X_i, X_j)| \le n^{3/2}$ and that $|W_n(X_1, X_2)|^{5/3}$ are uniformly integrable, we have

$$
\begin{aligned}
&E|W_n(X_1, X_2)|^{5/3+\delta} \\
&\quad = E|W_n(X_1, X_2)|^{5/3+\delta}\big(I_{(|W_n(X_1,X_2)|\le n)} + I_{(n<|W_n(X_1,X_2)|\le n^{3/2})}\big) \\
&\quad \le n^{\delta} A_0 + n^{3\delta/2} E|W_n(X_1, X_2)|^{5/3} I_{(n<|W_n(X_1,X_2)|\le n^{3/2})} \\
&\quad = o(n^{3\delta/2}) \qquad \text{as } n \to \infty.
\end{aligned}
$$

Therefore, it follows from (2.12) that

$$
\begin{aligned}
&\int_{|t|\le\sqrt{n}/(4\rho)} \frac{1}{|t|} |Z_{n1}(t)|\, dt \\
&\quad \le o(n^{-3/2})\Bigg(\int_{|t|\le\sqrt{n}/(4\rho)} |t|^{2/3+\delta}\, dt \\
&\qquad\qquad + \sum_{m=2}^{n-1} \int_{|t|\le\sqrt{n}/(4\rho)} |t|^{2/3+\delta} e^{-(m-1)t^2/(3n)}\, dt\Bigg) \\
&\quad \le o(n^{-3/2})\Bigg(2\bigg(\frac{n}{4\rho}\bigg)^{(5/3+\delta)/2} \\
&\qquad\qquad + \sum_{m=2}^{n-1} \bigg(\frac{n}{m-1}\bigg)^{(5/3+\delta)/2} \int_{|s|\le\sqrt{m-1}/(4\rho)} |s|^{2/3+\delta} e^{-s^2/3}\, ds\Bigg) \\
&\quad = o(n^{-1/2}).
\end{aligned}
$$

In view of this and (2.11), in order to prove $\beta_{1n} = o(n^{-1/2})$, it suffices to show that

$$
\int_{|t|\le\sqrt{n}/(4\rho)} |Z_{n2}(t)|\, dt = o(n^{-1/2}).
\tag{2.13}
$$

To prove this, let us define

$$
l_{m,k} = \frac{1}{n^{3/2}} \sum_{j=k+1}^{n} W_n(X_m, X_j) \qquad \text{if } 0 \le k < n,
$$

and $l_{m,k} = 0$ if $k \geq n$. Further define $j(m)$ to be the largest integer such that $mj(m) < n$. Clearly, $\Delta_m = l_{m,m}$. Also recall that $\Delta_{n,m} = 0$ for $m \geq n$. Thus we can write

$$\begin{aligned}
Z_{n2}(t) &= \sum_{m=1}^{n-1} E\Delta_m e^{it(S_n+\Delta_{n,m+1})} - \sum_{m=1}^{n-1} E\Delta_m e^{itS_n} \\
&= \sum_{m=1}^{n-1}\sum_{j=1}^{j(m)} E\big(l_{m,jm}e^{it(S_n+\Delta_{n,jm+1})} - l_{m,(j+1)m}e^{it(S_n+\Delta_{n,(j+1)m+1})}\big) \\
&\quad - \sum_{m=1}^{n-1}\sum_{j=1}^{j(m)} E\big(l_{m,jm} - l_{m,(j+1)m}\big)e^{itS_n} \\
&= \sum_{m=1}^{n-1}\sum_{j=1}^{j(m)} El_{m,jm}e^{itS_n}\big(e^{it\Delta_{n,jm+1}} - e^{it\Delta_{n,(j+1)m+1}}\big) \\
&\quad + \sum_{m=1}^{n-1}\sum_{j=1}^{j(m)} E\big(l_{m,jm} - l_{m,(j+1)m}\big)e^{itS_n}\big(e^{it\Delta_{n,(j+1)m+1}} - 1\big) \\
&= Z_{n2}^{(1)}(t) + Z_{n2}^{(2)}(t), \qquad \text{say.}
\end{aligned} \tag{2.14}$$

By noting that $\Delta_{n,k}$ depends only on $X_k, \dots, X_n$, that $l_{m,jm}$ is independent of $X_1, \dots, X_{m-1}, X_{m+1}, \dots, X_{jm}$ and that $E(l_{m,jm}|X_m) = 0$, it follows from (2.6), the inequality $|e^{it} - 1| \leq 2|t|^{2/3}$, and Holder's inequality that, when $|t| \leq \sqrt{n}/(4\rho)$,

$$\begin{aligned}
&\big|El_{m,jm}e^{itS_n}\big(e^{it\Delta_{n,jm+1}} - e^{it\Delta_{n,(j+1)m+1}}\big)\big| \\
&\quad \leq |\gamma_n(t)|^{jm-1}\, E\big|l_{m,jm}\big(e^{itV_n(X_m)/\sqrt{n}} - 1\big)\big(e^{it(\Delta_{n,jm+1}-\Delta_{n,(j+1)m+1})} - 1\big)\big| \\
&\quad \leq 2e^{-(jm-1)t^2/(3n)} \\
&\qquad \times \Big[E\big\{\big|e^{itV_n(X_m)/\sqrt{n}} - 1\big|^{5/2}\big|e^{it(\Delta_{n,jm+1}-\Delta_{n,(j+1)m+1})} - 1\big|^{5/2}\big\}\Big]^{2/5} \\
&\qquad \times \big\{E|l_{m,jm}|^{5/3}\big\}^{3/5} \\
&\quad \leq 4n^{-1/2}|t|^{5/3}e^{-(jm-1)t^2/(3n)}\big\{E\,|V_n(X_1)|^{5/2}\big\}^{2/5} \\
&\qquad \times \big\{E|\Delta_{n,jm+1} - \Delta_{n,(j+1)m+1}|^{5/3}\big\}^{2/5} \\
&\qquad \times \big\{E|l_{m,jm}|^{5/3}\big\}^{3/5}.
\end{aligned} \tag{2.15}$$

Recalling (2.8), it can be easily shown that

$$\begin{aligned}\{E|V_n(X_1)|^{5/2}\}^{2/5} &\le \{E|V_n(X_1)|^3\}^{1/3} \le \rho^{1/3},\\ \{E|l_{m,jm}|^{5/3}\}^{3/5} &\le 8n^{3/5-3/2}\{E|W_n(X_1,X_2)|^{5/3}\}^{3/5},\\ \{E|\Delta_{n,jm+1}-\Delta_{n,(j+1)m+1}|^{5/3}\}^{2/5} &\le \left\{2\sum_{l=jm+1}^{(j+1)m} E|\Delta_l|^{5/3}\right\}^{2/5}\\ &\le 2m^{2/5}n^{-3/5}\{E|W_n(X_1,X_2)|^{5/3}\}^{2/5}.\end{aligned}$$

Combining all these, we see that when $|t| \le \sqrt{n}/(4\rho)$,

$$Z_{n2}^{(1)}(t) \le 64\rho^{1/3}A_0 n^{-2}\sum_{m=1}^{n-1}\sum_{j=1}^{j(m)} m^{2/5}|t|^{5/3}e^{-(jm-1)t^2/(3n)}.$$

It then follows that

$$\begin{aligned}\int_{|t|\le\sqrt{n}/(4\rho)} &|Z_{n2}^{(1)}(t)|\,dt\\ &\le \frac{C}{n^2}\sum_{m=1}^{n-1}\sum_{j=1}^{j(m)} m^{2/5}\int_{|t|\le\sqrt{n}/(4\rho)}|t|^{5/3}e^{-(jm-1)t^2/(3n)}\,dt\\ &\le \frac{C}{n^2}\Biggl(\int_{|t|\le\sqrt{n}/(4\rho)}|t|^{5/3}\,dt \qquad (2.16)\\ &\qquad+\sum_{m=1}^{n-1}\sum_{j=1}^{j(m)} m^{2/5}\left(\frac{n}{jm}\right)^{4/3}\int_{-\infty}^{\infty}|t|^{5/3}e^{-t^2/3}\,dt\Biggr)\\ &\le Cn^{-3/5}\\ &= o(n^{-1/2}).\end{aligned}$$

Similar to the proof of (2.15), we have

$$\begin{aligned}&\bigl|E(l_{m,jm}-l_{m,(j+1)m})e^{itS_n}(e^{it\Delta_{n,(j+1)m+1}}-1)\bigr|\\ &\quad\le n^{-3/2}\sum_{l=jm+1}^{(j+1)m}\bigl|EW_n(X_m,X_l)e^{itS_n}(e^{it\Delta_{n,(j+1)m+1}}-1)\bigr|\\ &\quad= n^{-3/2}\sum_{l=jm+1}^{(j+1)m}\left|EW_n(X_m,X_l)\exp\left\{it\sum_{k=1}^{(j+1)m}V_n(X_k)/\sqrt{n}\right\}\right| \qquad (2.17)\\ &\qquad\times\left|E\exp\left\{it\sum_{(j+1)m+1}^{n}V_n(X_k)/\sqrt{n}\right\}(e^{it\Delta_{n,(j+1)m+1}}-1)\right|.\end{aligned}$$

By Holder's inequality and (2.6), it can be easily shown that, when $|t| \le \sqrt{n}/(4\rho)$,

$$\begin{aligned}
&\left| E W_n(X_m, X_l) \exp\left\{ it \sum_{k=1}^{(j+1)m} V_n(X_k)/\sqrt{n} \right\} \right| \\
&\quad = |\gamma_n(t)|^{(j+1)m-2} \left| E W_n(X_1, X_2)\big(e^{itV_n(X_1)/\sqrt{n}} - 1\big)\big(e^{itV_n(X_2)/\sqrt{n}} - 1\big) \right| \\
&\quad \le e^{-[(j+1)m-2]t^2/(3n)} \big\{ E|W_n(X_1, X_2)|^{3/2} \big\}^{2/3} \\
&\qquad \times \big\{ E\big|\big(e^{itV_n(X_1)/\sqrt{n}} - 1\big)\big(e^{itV_n(X_2)/\sqrt{n}} - 1\big)\big|^3 \big\}^{1/3} \\
&\quad \le e^{-[(j+1)m-2]t^2/(3n)} \big\{ E|W_n(X_1, X_2)|^{5/3} \big\}^{3/5} \\
&\qquad \times t^2 n^{-1} \big\{ E\,|V_n(X_1)V_n(X_2)|^3 \big\}^{1/3} \\
&\quad \le n^{-1} t^2 e^{-[(j+1)m-2]t^2/(3n)} A_0^{3/5} \rho^{2/3}.
\end{aligned} \tag{2.18}$$

By using (2.8) and methods similar to those used in (2.18), we have, when $|t| \le \sqrt{n}/(4\rho)$,

$$\begin{aligned}
&\left| E \exp\left\{ it \sum_{(j+1)m+1}^{n} V_n(X_k)/\sqrt{n} \right\} \big(e^{it\Delta_{n,(j+1)m+1}} - 1\big) \right| \\
&\quad \le E\big| e^{it\Delta_{n,(j+1)m+1}} - 1 - it\Delta_{n,(j+1)m+1} \big| \\
&\qquad + |t| \left| E\Delta_{n,(j+1)m+1} \exp\left\{ it \sum_{k=(j+1)m+1}^{n} V_n(X_k)/\sqrt{n} \right\} \right| \\
&\quad \le 2|t|^{5/3} E|\Delta_{n,(j+1)m+1}|^{5/3} \\
&\qquad + n^{1/2}|t||\gamma_n(t)|^{n-(j+1)m-3} \\
&\qquad\quad \times \big| E W_n(X_1, X_2)\big(e^{itV_n(X_1)/\sqrt{n}} - 1\big)\big(e^{itV_n(X_2)/\sqrt{n}} - 1\big) \big| \\
&\quad \le 16 n^{-1/2}|t|^{5/3} A_0 + 2n^{-1/2}|t|^3 e^{-[n-(j+1)m-3]t^2/(3n)} A_0^{3/5} \rho^{2/3}.
\end{aligned} \tag{2.19}$$

In terms of (2.17)–(2.19), we obtain, when $|t| \le \sqrt{n}/(4\rho)$,

$$\begin{aligned}
&\big| E(l_{m,jm} - l_{m,(j+1)m}) e^{itS_n} \big(e^{it\Delta_{n,(j+1)m+1}} - 1\big) \big| \\
&\quad \le Cmn^{-3} \big(|t|^{11/3} e^{-[(j+1)m-2]t^2/(3n)} + |t|^5 e^{-(n-5)t^2/(3n)} \big).
\end{aligned} \tag{2.20}$$

By noting $mj(m) < n$, (2.20) implies that

$$\begin{aligned}
&\int_{|t| \le \sqrt{n}/(4\rho)} |Z_{n2}^{(2)}(t)|\, dt \\
&\quad \le \frac{C}{n^3} \sum_{m=1}^{n-1} \sum_{j=1}^{j(m)} m \bigg(\int_{|t| \le \sqrt{n}/(4\rho)} |t|^{11/3} e^{-[(j+1)m-2]t^2/(3n)}\, dt
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \int_{|t|\le\sqrt{n}/(4\rho)} |t|^5 e^{-(n-5)t^2/(3n)}\Big)\,dt \\
&\le \frac{C}{n^3}\int_{|t|\le\sqrt{n}/(4\rho)} |t|^{11/3}\,dt \\
&\quad + \frac{C}{n^3}\sum_{m=1}^{n-1}\sum_{j=1}^{j(m)} m\left(\frac{n}{jm}\right)^{7/3}\int_{-\infty}^{\infty}|t|^{11/3}e^{-t^2/3}\,dt + \frac{C}{n} \\
&\le Cn^{-2/3} \\
&= o(n^{-1/2}).
\end{aligned}
$$

Combining this and (2.16) yields

$$
\int_{|t|\le\sqrt{n}/(4\rho)} |Z_{n2}(t)|\,dt \le \int_{|t|\le\sqrt{n}/(4\rho)} \big(|Z_{n2}^{(1)}(t)| + |Z_{n2}^{(2)}(t)|\big)\,dt = o(n^{-1/2}).
$$

This proves (2.13). Hence, we have shown that $\beta_{1n} = o(n^{-1/2})$.

The proof of Lemma 2.1 thus follows from (2.2)–(2.4). □

PROOF OF THEOREM 1.1. Without loss of generality, we assume that $\theta = 0$. Write

$$
\begin{aligned}
\psi(X_i, X_j) &= h(X_i, X_j) - g(X_i) - g(X_j), \\
\psi^*(X_i, X_j) &= \psi(X_i, X_j) I_{(|\psi(X_i,X_j)|\le n^{3/2}/12)}, \\
g^*(X_i) &= E(\psi^*(X_i, X_j)|X_i), \\
g^{**}(X_i) &= g^*(X_i) I_{(|g^*(X_i)|\le\sqrt{n})}, \\
U_n^* &= \frac{1}{\sqrt{n}\sigma_g}\sum_{j=1}^n g(X_j) + \frac{1}{\sqrt{n}(n-1)\sigma_g}\sum_{1\le i<j\le n}\psi^*(X_i, X_j) \\
&= \frac{1}{\sqrt{n}\sigma_g}\sum_{j=1}^n \big(g(X_j) + g^*(X_j)\big) \\
&\quad + \frac{1}{\sqrt{n}(n-1)\sigma_g}\sum_{1\le i<j\le n}\big(\psi^*(X_i, X_j) - g^*(X_i) - g^*(X_j)\big), \\
U_n^{**} &= \frac{1}{\sqrt{n}\sigma_g}\sum_{j=1}^n \big(g(X_j) + g^{**}(X_j)\big) \\
&\quad + \frac{1}{\sqrt{n}(n-1)\sigma_g}\sum_{1\le i<j\le n}\big(\psi^*(X_i, X_j) - g^*(X_i) - g^*(X_j)\big).
\end{aligned}
$$

First, noting that $E(\psi(X_1, X_2)|X_1) = 0$, we have

$$
\begin{aligned}
|g^*(X_j)| &= \big|E\big(\psi(X_i, X_j)I_{(|\psi(X_i,X_j)|\le n^{3/2}/12)}|X_j\big)\big| \\
&\le E\big(|\psi(X_i, X_j)|\, I_{(|\psi(X_i,X_j)|\ge n^{3/2}/12)}|X_j\big) \qquad (2.21)\\
&\le (12)^{2/3}n^{-1}E\big(|\psi(X_i, X_j)|^{5/3}I_{(|\psi(X_i,X_j)|\ge n^{3/2}/12)}|X_j\big).
\end{aligned}
$$

From this and the definitions of U_n^* and U_n^{**}, we get

$$
\begin{aligned}
&\sup_x |P(U_n^* \le x) - P(U_n^{**} \le x)| \\
&\quad\le nP\big(|g^*(X_1)| \ge \sqrt{n}\big) \\
&\quad\le \sqrt{n}E|g^*(X_1)| \qquad (2.22)\\
&\quad= \sqrt{n}E\big|E\big(\psi(X_1, X_2)I_{(|\psi(X_1,X_2)|\ge n^{3/2}/12)}|X_1\big)\big| \\
&\quad\le 12^{2/3}n^{-1/2}E|\psi(X_1, X_2)|^{5/3}I_{(|\psi(X_1,X_2)|\ge n^{3/2}/12)} \\
&\quad= o(n^{-1/2}).
\end{aligned}
$$

Secondly, by noting that

$$
\frac{\sqrt{n}U_n}{2\sigma_g} = \frac{1}{\sqrt{n}\sigma_g}\sum_{j=1}^{n} g(X_j) + \frac{1}{\sqrt{n}(n-1)\sigma_g}\sum_{1\le i<j\le n}\psi(X_i, X_j),
$$

we have

$$
\begin{aligned}
&\sup_x \left|P\left(\frac{\sqrt{n}U_n}{2\sigma_g} \le x\right) - P(U_n^* \le x)\right| \\
&\quad\le \sum_{1\le i<j\le n} P\big(|\psi(X_i, X_j)| \ge n^{3/2}/12\big) \qquad (2.23)\\
&\quad\le 12^{5/3}n^{-1/2}E|\psi(X_1, X_2)|^{5/3}I_{(|\psi(X_i,X_j)|\ge n^{3/2}/12)} \\
&\quad= o(n^{-1/2}).
\end{aligned}
$$

Thus from (2.22) and (2.23), Theorem 1.1 follows if we can prove

$$
\sup_x \big|P(U_n^{**} \le x) - F_n(x)\big| = o(n^{-1/2}). \qquad (2.24)
$$

To prove (2.24), without loss of generality, we assume

$$
\sigma_g^2 = 1
$$

and further let

$$
\begin{aligned}
\sigma_g^{**2} &= E\{g(X_1) + g^{**}(X_1) - Eg^{**}(X_1)\}^2, \\
\theta(x) &= \frac{x - \sqrt{n}Eg^{**}(X_1)}{\sigma_g^{**}},
\end{aligned}
$$

$$V_n(X_j) = \frac{1}{\sigma_g^{**}} \{g(X_j) + g^{**}(X_j) - Eg^{**}(X_j)\},$$

$$W_n(X_i, X_j) = \frac{n}{(n-1)\sigma_g^{**}} \{\psi^*(X_i, X_j) - g^*(X_i) - g^*(X_j)\}.$$

We have

$$P(U_n^{**} \le x) = P\left(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} V_n(X_j) + \frac{1}{n^{3/2}} \sum_{1\le i<j\le n} W_n(X_i, X_j) \le \theta(x)\right).$$

From this, it suffices to show that

$$\sup_y \left| P\left(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} V_n(X_j) + \frac{1}{n^{3/2}} \sum_{1\le i<j\le n} W_n(X_i, X_j) \le y\right) - F_n^{(1)}(y)\right| = o(n^{-1/2}), \tag{2.25}$$

$$\sup_x \left|F_n^{(1)}(\theta(x)) - F_n^{(1)}(x)\right| = o(n^{-1/2}), \tag{2.26}$$

$$\sup_x \left|F_n^{(1)}(x) - F_n(x)\right| = o(n^{-1/2}), \tag{2.27}$$

where

$$F_n^{(1)}(y) = \Phi(y) - \frac{\Phi^{(3)}(y)}{6\sqrt{n}} \big(EV_n^3(X_1) + 3EV_n(X_1)V_n(X_2)W_n(X_1, X_2)\big).$$

In the following, we give the proofs of (2.25)–(2.27).

We first prove (2.25). From Lemma 2.1, it suffices to check that $|V_n(X_1)|^3$, $n \ge 2$, are uniformly integrable and $|W_n(X_1, X_2)|^{5/3}$, $n \ge 2$, are uniformly integrable. By noting that $|g^{**}(X_j)| \le \sqrt{n}$, $|g^{**}(X_j)| \le |g^*(X_j)|$ and (2.21), we obtain

$$E|g^{**}(X_1)| \le (12)^{2/3} n^{-1} E\,|\psi(X_1, X_2)|^{5/3}\, I_{(|\psi(X_1,X_2)|\ge n^{3/2}/12)}, \tag{2.28}$$

$$\begin{aligned}
&\left|\sigma_g^{**2} - 1\right| \\
&\quad \le 2E\left|g(X_1)\big(g^{**}(X_1) - Eg^{**}(X_1)\big)\right| + E\big(g^{**}(X_1) - Eg^{**}(X_1)\big)^2 \\
&\quad \le 8\sqrt{n} E|g^{**}(X_1)| + 4\sqrt{n} E|g(X_1)| I_{(|g(X_1)|\ge\sqrt{n})} \\
&\quad \le 48 n^{-1/2} E\,|\psi(X_1, X_2)|^{5/3}\, I_{(|\psi(X_1,X_2)|\ge n^{3/2}/12)} \\
&\qquad + 4n^{-1/2} E|g(X_1)|^3 I_{(|g(X_1)|\ge\sqrt{n})}.
\end{aligned} \tag{2.29}$$

In terms of (2.29) and $|g^{**}(X_j)| \le \sqrt{n}$, it follows that, for sufficiently large n, $\sigma_g^{**} \ge 1/2$ and hence

$$\begin{aligned} |V_n(X_1)|^3 &\le 64\{|g(X_1)|^3 + |g^{**}(X_1) - Eg^{**}(X_1)|^3\} \\ &\le 64|g(X_1)|^3 + 8^3 n|g^{**}(X_1)| + 8^3\{E|g^{**}(X_1)|\}^3. \end{aligned} \tag{2.30}$$

This, together with (2.28), implies that $|V_n(X_1)|^3$, $n \ge 2$, are uniformly integrable. Similarly, it can be shown that $|W_n(X_1, X_2)|^{5/3}$, $n \ge 2$, are uniformly integrable. By using Lemma 2.1, (2.25) follows directly.

The proof of (2.26) is simple because, by using (2.28) and (2.29),

$$|\theta(x) - x| \le \frac{\sqrt{n}E|g^{**}(X_1)|}{\sigma_g^{**}} + \frac{|x||1-\sigma_g^{**2}|}{\sigma_g^{**}(1+\sigma_g^{**})} = o\{(1+|x|)n^{-1/2}\}.$$

Next, we prove (2.27). To do this, it suffices to show that

$$EV_n^3(X_1) - Eg^3(X_1) = o(1) \tag{2.31}$$

and

$$EV_n(X_1)V_n(X_2)W_n(X_1, X_2) - Eg(X_1)g(X_2)h(X_1, X_2) = o(1). \tag{2.32}$$

We only prove (2.32) below since the proof of (2.31) is similar but simpler and hence omitted. Write

$$\begin{aligned} V_n^*(X_j) &= g(X_j) + g^{**}(X_j) - Eg^{**}(X_j), \\ W_n^*(X_i, X_j) &= \psi^*(X_i, X_j) - g^*(X_i) - g^*(X_j). \end{aligned}$$

It can be easily shown that

$$E|W_n^*(X_1, X_2)|^{3/2} \le 8E|\psi(X_1, X_2)|^{3/2}$$

and, [by using (2.21), (2.28) and (2.30)]

$$\begin{aligned} E|g^{**}(X_1) - Eg^{**}(X_1)|^3 &\le (12)^{2/3}E|\psi(X_1, X_2)|^{5/3}I_{(|\psi(X_1,X_2)|\ge n^{3/2}/12)}, \\ E|V_n^*(X_1)|^3 &\le C(E|g(X_1)|^3 + E|\psi(X_1, X_2)|^{5/3}). \end{aligned}$$

Combining all these estimates and Hölder's inequality, we obtain

$$\begin{aligned} &E\{|g^{**}(X_1) - Eg^{**}(X_1)||V_n^*(X_2)||W_n^*(X_1, X_2)|\} \\ &\quad \le \{E|g^{**}(X_1) - Eg^{**}(X_1)|^3\}^{1/3}\{E|V_n^*(X_2)|^3\}^{1/3}\{E|W_n^*(X_1, X_2)|^{3/2}\}^{2/3} \\ &\quad = o(1), \\ &E\{|g(X_1)||g^{**}(X_2) - Eg^{**}(X_2)||W_n^*(X_1, X_2)|\} \\ &\quad \le \{E|g(X_1)|^3\}^{1/3}\{E|g^{**}(X_1) - Eg^{**}(X_1)|^3\}^{1/3}\{E|W_n^*(X_1, X_2)|^{3/2}\}^{2/3} \\ &\quad = o(1). \end{aligned}$$

It then follows that

$$\begin{aligned}
&EV_n^*(X_1)V_n^*(X_2)W_n^*(X_1, X_2)\\
&\quad = Eg(X_1)V_n^*(X_2)W_n^*(X_1, X_2)\\
&\qquad + E\big(g^{**}(X_1) - Eg^{**}(X_1)\big)V_n^*(X_2)W_n^*(X_1, X_2)\\
&\quad = Eg(X_1)g(X_2)W_n^*(X_1, X_2)\\
&\qquad + Eg(X_1)\big(g^{**}(X_2) - Eg^{**}(X_2)\big)W_n^*(X_1, X_2) + o(1)\\
&\quad = Eg(X_1)g(X_2)\psi^*(X_1, X_2) + o(1)\\
&\quad = Eg(X_1)g(X_2)h(X_1, X_2)\\
&\qquad + Eg(X_1)g(X_2)\psi(X_1, X_2)I_{(|\psi(X_1,X_2)|\ge n^{3/2}/12)} + o(1).
\end{aligned}$$

From this, $\sigma_g^{**2} = 1 + o(n^{-1/2})$ and

$$\begin{aligned}
&E|g(X_1)||g(X_2)||\psi(X_1, X_2)|I_{(|\psi(X_1,X_2)|\ge n^{3/2}/12)}\\
&\quad \le \big\{E|g(X_1)|^3\big\}^{2/3}\big\{E|\psi(X_1, X_2)|^{3/2}I_{(|\psi(X_1,X_2)|\ge n^{3/2}/12)}\big\}^{2/3}\\
&\quad = o(1),
\end{aligned}$$

we immediately get (2.32). This completes the proof of Theorem 1.1. □

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