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ASYMPTOTICALLY EFFICIENT ESTIMATION OF THE INDEX OF REGULAR VARIATION[1]

BY XIAOYING WEI

Merck Research Laboratories

We propose a conditional MLE of the index of regular variation when the functional form of a slowly varying function is assumed known in the tail, and we study its asymptotic properties. We prove asymptotic normality of $P_\theta^{k_n}$, a probability measure whose density is the product of the joint conditional density of the k_n largest order statistics from $F_\theta(x)$ given Z_{n-k}, the $(n-k)$th order statistic, and a density of Z_{n-k} with parameter θ. Based on this result, we show that this conditional MLE is asymptotically normal and asymptotically efficient in many senses whenever k_n is $o(n)$. We also propose an iterative estimator of θ given only partial knowledge of $L_\theta(x)$. This estimator is asymptotically normal, asymptotically unbiased and asymptotically efficient.

1. Introduction. The problem of estimating the index of regular variation has attracted much attention recently, and various estimators have been proposed. For related work, see, for example, Hill (1975), Pickands (1975), de Haan and Resnick (1980), Hall (1982), Mason (1982), Davis and Resnick (1984), Haeusler and Teugels (1985), Csörgő and Mason (1985), Csörgő, Deheuvels and Mason (1985), Welsh (1986), Csörgő, Horváth and Révész (1987) and Smith (1987). In this paper, we propose asymptotically efficient estimators of the index of regular variation under both full and partial knowledge of the slowly varying function $L_\theta(x)$ in the tail, and we explore their various asymptotic properties. In practice, the functional form of $L_\theta(x)$ is usually unknown or only partially known. Estimates of θ, however, depend on $L_\theta(x)$ in one way or another. For example, Hill's estimator appears not to depend on $L_\theta(x)$, but the optimal number of order statistics used in estimation and its asymptotic properties do depend on the functional form of $L_\theta(x)$. It is, therefore, sensible and interesting to investigate the problem of estimating θ assuming that $L_\theta(x)$ is completely known in the tail. The result will be a very useful guide to the investigation of asymptotic properties of other estimators of θ under more realistic assumptions on $L_\theta(x)$.

The paper is organized as follows. In Section 2, a conditional MLE of the index of regular variation under the assumption that the functional form of a slowly varying function is fully known in the tail is proposed, and it is proved that when k_n, the number of order statistics used in estimation, is $o(n)$, this

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conditional MLE is asymptotically normally distributed and asymptotically efficient. It will become clear in Section 2 why we choose a conditional maximum likelihood estimator instead of a maximum likelihood estimator. In Section 3, an iterative estimator is proposed and its asymptotic properties derived when only partial knowledge of $L_\theta(x)$ is assumed. Applying the results obtained in Section 2, it is easy to see that this iterative estimator is asymptotically efficient only when k_n tends to infinity appropriately with the sample size n. For simplicity, we shall denote k_n by k from now on.

2. Conditional maximum likelihood estimator of θ and its asymptotic properties.

2.1. *Introduction and motivation.* Let $F_\theta(x)$ be regularly varying at ∞, that is, $1 - F_\theta(x) = x^{-\theta}L_\theta(x)$. In this section we assume that the functional form of $L_\theta(x)$ is completely known in the tail and that a density function $f_\theta(x)$ of $F_\theta(x)$ exists. We also assume that $Z_1 \le Z_2 \le \cdots \le Z_n$ are the order statistics corresponding to n i.i.d. random variables from $F_\theta(x)$. Our estimator is based on the $(k+1)$ largest order statistics $Z_{n-k}, \dots, Z_n$, where k is such that $k = o(n) \to \infty$ when $n \to \infty$, since only tail behavior of $f_\theta(x)$ is assumed and $Z_{n-k} \to \infty$ in probability as $k = o(n) \to \infty$. A natural estimator in this case seems to be a maximum likelihood estimator with the likelihood function being a joint density of $Z_{n-k}, \dots, Z_n$. However, a simple example shows that the local likelihood ratio does not behave properly; that is, the local asymptotic normality (LAN) does not hold. Let us consider the simplest case of $L_\theta(x) = 1$. A joint density of $Z_{n-k}, \dots, Z_n$ is the following:

$$g_\theta(z_{n-k}, \dots, z_n) = \frac{n!}{(n-k+1)!} F_\theta^{n-k-1}(z_{n-k}) f_\theta(z_{n-k}) \prod_{j=1}^{k} f_\theta(z_{n-k+j}).$$

Suppose LAN holds for the probability measure with this joint density. Then the normalized first derivative of the log-likelihood must weakly converge to a normal distribution. A simple calculation shows

$$\begin{aligned}\frac{1}{\sqrt{k}} & \frac{\partial \log g_\theta(Z_{n-k}, \dots, Z_n)}{\partial\theta} \\ &= \frac{1}{\sqrt{k}} \sum_{j=1}^{k} \left(\theta^{-1} - \log \frac{Z_{n-k+j}}{Z_{n-k}} \right) \\ &\quad + \sqrt{k} \log Z_{n-k} \left(\frac{(n-k-1)(1-F_\theta(Z_{n-k}))}{kF_\theta(Z_{n-k})} - 1 - \frac{1}{k} \right) + \frac{\theta^{-1}}{\sqrt{k}}.\end{aligned}$$

The first term is the normalized Hill's estimator, and the third term goes to zero as $k = o(n) \to \infty$. In view of the results on Hill's estimator this will converge only when the second term converges. However, the second term is of the form $S_k \log Z_{n-k}$, with S_k converging to a normal random variable [Csörgő and Mason (1985)]. This is a contradiction since $\log Z_{n-k}$ is unbounded in probability. Hence, instead of a maximum likelihood estimator, we consider a conditional likelihood estimator with the conditional likelihood function being the conditional density of $Z_{n-k+1}, \dots, Z_n$ given Z_{n-k}.

2.2. *Local and uniform asymptotic normality.* To simplify notation we will let $f_k(x, \theta)$ denote $f_\theta(x)/(1 - F_\theta(Z_{n-k}))$ from now on. Let $P^k_{\theta+u/\sqrt{k}}$ be the probability measure on $\mathscr{R}^{k+1}$ with the following Lebesgue density: $\prod_{j=1}^k f_k(z_{n-k+j}, \theta + u/\sqrt{k})h_\theta(z_{n-k})$, where $\theta \in \mathscr{R}^+$ is fixed throughout, $u \in \mathscr{R}$ and $h_\theta(z_{n-k})$ is the density function of Z_{n-k} with parameter θ. Note that $dP^k_{\theta+u/\sqrt{k}}/dP^k_\theta$ is the conditional likelihood ratio. In order to investigate asymptotic properties of our conditional MLE, we need to obtain the following local and uniform asymptotic normality results for P^k_θ. We first prove the local asymptotic normality for P^k_θ.

THEOREM 1. *Suppose $F_\theta(x)$ is regularly varying at ∞, that is, $F_\theta(x) = 1 - x^{-\theta}L_\theta(x)$ as $x \to \infty$, and admits a density function $f_\theta(x) = \theta x^{-\theta-1}L^*_\theta(x)$ as $x \to \infty$, where $L^*_\theta(x) = L_\theta(x) - \theta^{-1}xL'_\theta(x)$ is a slowly varying function. Consider the following conditions*:

(C1) *$L_\theta(x)$ is normalized, that is, if we define $\varepsilon_\theta(x) = xL'_\theta(x)/L_\theta(x)$, then $\log L_\theta(x) = \eta_\theta + \int_1^x \varepsilon_\theta(y)/y\,dy$, where η_θ is a real constant and $\varepsilon_\theta(x) \to 0$ as $x \to \infty$; in this case, we have $L^*_\theta(x) = L_\theta(x)(1 + \varepsilon_\theta(x)/\theta)$;*

(C2) *$\partial \log L_\theta(x)/\partial\theta \to 0$ as $x \to \infty$;*

(C3) *$\partial \varepsilon_\theta(x)/\partial\theta \to 0$ as $x \to \infty$;*

(C4) *$L^*_\theta(tx)/L^*_\theta(x) - 1 = O(g_\theta(x))$ as $x \to \infty$, where $g_\theta(x) \to 0$ as $x \to \infty$ and $g_\theta(tx)/g_\theta(x) \le Ct^\tau$, for $\tau \le 0$, $t > 1$, $C < \infty$ and $x \ge x_0 > 0$; or $L^*_\theta(tx)/L^*_\theta(x) - 1 = k(t)g_\theta(x) + o(g_\theta(x))$ as $x \to \infty$, for each $t > 0$, where $g_\theta(x)$ is regularly varying with some index $\rho \le 0$ which satisfies $g_\theta(x) \to 0$ as $x \to \infty$ and $k(t) = c\int_1^t u^{\rho-1}\,du$, with c being a constant.*

If $L_\theta(x)$ satisfies (C1)–(C4), *then we have*

$$\frac{dP^k_{\theta+u/\sqrt{k}}}{dP^k_\theta} = \exp\left\{\Delta_{k,\theta}u - \frac{\theta^{-2}u^2}{2} + \psi_{n,k}(\theta, u)\right\}, \tag{1}$$

where for $k = o(n) \to \infty$,

$$P^k_\theta(|\psi_{n,k}(\theta, u)| > \varepsilon) \to 0 \quad \forall \varepsilon > 0 \tag{2}$$

and

$$\Delta_{k,\theta} = \frac{1}{\sqrt{k}} \sum_{j=1}^k \frac{\partial \log f_k(Z_{n-k+j}, \theta)}{\partial\theta} \to_D N(0, \theta^{-2}). \tag{3}$$

NOTATION. Here L'_θ denotes the derivative with respect to x, and $\partial L_\theta / \partial\theta$ denotes the derivative with respect to θ.

NOTE. Condition (C4) is the same as SR1 and SR2 defined by Goldie and Smith (1987).

PROOF OF THEOREM 1. Applying Theorem 3.14 from Fabian and Hannan (1987) to our case, we only have to verify the following to prove the theorem:

$$\text{(a)} \qquad U = \left\{ \frac{1}{\sqrt{k}} \frac{\partial \log f_k(Z_{n-k+j}, \theta)}{\partial\theta},\ j = 1, 2, \dots, k \right\}$$

is a Lindeberg asymptotic martingale difference array;

$$\text{(b)} \qquad \sum_{j=1}^{k} \int \left(\sqrt{f_k\left(z_{n-k+j}, \theta + \frac{u}{\sqrt{k}}\right)} - \sqrt{f_k(z_{n-k+j}, \theta)} - \frac{u}{\sqrt{k}} \frac{\partial \sqrt{f_k(z_{n-k+j}, \theta)}}{\partial\theta} \right)^2 dz_{n-k+j}$$

$\to 0$ in P_θ^k-probability; and

$$\text{(c)} \qquad \frac{1}{k} \sum_{j=1}^{k} E\left[\left(\frac{\partial \log f_k(Z_{n-k+j}, \theta)}{\partial\theta} \right)^2 \middle| Z_{n-k} \right] \to \theta^{-2} \quad \text{in } P_\theta^k\text{-probability.}$$

We shall verify conditions (a), (b) and (c) through a sequence of lemmas, which will be proved below.

Condition (a) is true by Lemma 3(b) and the fact that

$$E\left[\frac{\partial \log f_k(Z_{n-k+j}, \theta)}{\partial\theta} \right] = 0, \qquad j = 1, 2, \dots, k;$$

that is, U is actually a martingale difference array with respect to $\{\mathscr{F}_{k,j}\}$, where $\mathscr{F}_{k,0} = \sigma(Z_{n-k})$ and $\mathscr{F}_{k,j} = \sigma(Z_{n-k}, \dots, Z_{n-k+j})$, $j = 1, 2, \dots, k$. Condition (b) follows from Lemma 3(c) and the fact that $(Z_{n-k+j},\ j = 1, 2, \dots, k)$ can be assumed without loss of generality being conditional i.i.d. given Z_{n-k}. Condition (c) is true because of Lemma 2. □

We now prove the lemmas mentioned above.

LEMMA 1. *Let Z_{n-k} be the $(n-k)$th order statistic from $F_\theta(x) = 1 - x^{-\theta}L_\theta(x)$, where $\theta \in (0, \infty)$ and $L_\theta(x)$ is a slowly varying function. Then, $\forall\, M > 0$, $P(Z_{n-k} < M) \to 0$ if $k = o(n) \to \infty$.*

PROOF. The proof is immediate. □

LEMMA 2. *Suppose the assumptions in Theorem 1 are satisfied. Denote the conditional Fisher information based on the conditional density function $f_k(x, \theta)$ as $I(\theta, Z_{n-k})$, where Z_{n-k} is the $(n-k)$th order statistic from $F_\theta(x)$. Let $\theta_k = \theta + u/\sqrt{k}$. Then, $\forall\, 0 < L < \infty$,*

$$\sup_{|u| \le L} I(\theta_k, Z_{n-k}) \to_P \theta^{-2}$$

as $k = o(n) \to \infty$ for all $\theta \in (0, \infty)$.

PROOF. Let

$$\begin{aligned} r(\theta_k, X, Z_{n-k}) &= \frac{\partial \log f_k(X, \theta_k)}{\partial \theta_k} \\ &= \theta_k^{-1} - \log \frac{X}{Z_{n-k}} + \frac{\partial \log L^*_{\theta_k}(X)}{\partial \theta_k} - \frac{\partial \log L_{\theta_k}(Z_{n-k})}{\partial \theta_k}, \end{aligned}$$

where $X \sim f_k(x, \theta_k)$ given Z_{n-k}. Then

$$I(\theta_k, Z_{n-k}) = E_{\theta_k}\big[r^2(\theta_k, X, Z_{n-k}) | Z_{n-k}\big].$$

Let $r_*(\theta_k, X, Z_{n-k}) = \theta_k^{-1} - \log(X/Z_{n-k})$. We shall first prove that

$$\sup_{|u| \le L} E\big[r_*^2(\theta_k, X, Z_{n-k}) | Z_{n-k}\big] \to_P \theta^{-2}, \tag{4}$$

as $k = o(n) \to \infty$, $\forall\, \theta \in (0, \infty)$. We then prove that

$$\sup_{|u| \le L} E\big[|r^2(\theta_k, X, Z_{n-k}) - r_*^2(\theta_k, X, Z_{n-k})|\, | Z_{n-k}\big] \to_P 0.$$

We now prove (4). Since $L^*_\theta(x) = L_\theta(x)(1 + \varepsilon_\theta(x)/\theta)$, the conditional expectation in (4) is

$$\begin{aligned} &\int_{Z_{n-k}}^\infty r_*^2(\theta_k, X, Z_{n-k}) f_k(x, \theta_k)\, dx \\ &\qquad = \int_{Z_{n-k}}^\infty r_*^2(\theta_k, Z_{n-k}) \frac{\theta_k x^{-\theta_k - 1}}{Z_{n-k}^{-\theta_k}} \frac{L^*_{\theta_k}(x)}{L_{\theta_k}(Z_{n-k})}\, dx \\ &\qquad = \left(1 + \frac{1}{\theta} \varepsilon_\theta(Z_{n-k})\right) \int_{Z_{n-k}}^\infty r_*^2(\theta_k, X, Z_{n-k}) \frac{\theta_k x^{-\theta_k - 1}}{Z_{n-k}^{-\theta_k}} \frac{L^*_{\theta_k}(x)}{L^*_{\theta_k}(Z_{n-k})}\, dx \end{aligned}$$

$$= \left(1 + \frac{1}{\theta}\varepsilon_\theta(Z_{n-k})\right)\int_1^\infty \left(\theta_k^{-2} - 2\theta_k^{-1}\log y + (\log y)^2\right) \times \theta_k y^{-\theta_k - 1}\frac{L^*_{\theta_k}(yZ_{n-k})}{L^*_{\theta_k}(Z_{n-k})}\,dy.$$

Applying Proposition 2.5.1 of Goldie and Smith (1987), since $v(y)y^\varepsilon = (\theta_k^{-2} - 2\theta_k^{-1}\log y + (\log y)^2)\theta_k y^{-\theta_k - 1 + \varepsilon}$ is integrable for sufficiently small ε and sufficiently large k $\forall\ \theta \in (0, \varepsilon)$ and condition (C4) is satisfied, the above integral is equal to

$$\left(1 + \frac{1}{\theta}\varepsilon_\theta(Z_{n-k})\right)\left(\int_1^\infty \left(\theta_k^{-2} - 2\theta_k^{-1}\log y + (\log y)^2\right)\theta_k y^{-\theta_k - 1}\,dy + O(g_\theta(Z_{n-k}))\right)$$

$$= \left(1 + \frac{1}{\theta}\varepsilon_\theta(Z_{n-k})\right)\left(\left(\theta + \frac{u}{\sqrt{k}}\right)^{-2} + O_p(g_\theta(Z_{n-k}))\right).$$

Hence we have (4), by conditions (C1) and (C4).

Now let us consider

$$\int_{Z_{n-k}}^\infty |r^2(\theta_k, X, Z_{n-k}) - r_*^2(\theta_k, X, Z_{n-k})| f_k(x, \theta_k)\,dx.$$

Using the inequality $|a^2 - b^2| \le (1 + \alpha)|a - b|^2 + b^2/\alpha$, $\alpha > 0$, $a, b \in \mathscr{R}$, and letting $a = r^2(\theta_k, X, Z_{n-k})$ and $b = r_*^2(\theta_k, X, Z_{n-k})$ gives

$$\int_{Z_{n-k}}^\infty |r^2(\theta_k, X, Z_{n-k}) - r_*^2(\theta_k, X, Z_{n-k})| f_k(x, \theta_k)\,dx$$

$$\le (1 + \alpha)\int_{Z_{n-k}}^\infty |r(\theta_k, X, Z_{n-k}) - r_*(\theta_k, X, Z_{n-k})|^2 f_k(x, \theta_k)\,dx + \frac{1}{\alpha}\int_{Z_{n-k}}^\infty r_*^2(\theta_k, X, Z_{n-k}) f_k(x, \theta_k)\,dx$$

$$= (1 + \alpha)\mathrm{I} + \frac{1}{\alpha}\mathrm{II},$$

where

$$\mathrm{I} = \int_{Z_{n-k}}^\infty \left(\frac{\partial \log L^*_{\theta_k}(x)}{\partial\theta} - \frac{\partial \log L_{\theta_k}(Z_{n-k})}{\partial\theta}\right)^2 f_k(x, \theta_k)\,dx;$$

I $\to_P$ 0 as $k = o(n) \to \infty$ by Lemma 2 and conditions (C2) and (C3) on $L_\theta(x)$, since condition (C3) implies that $\partial \log L^*_\theta(x)/\partial\theta \to 0$ as $x \to \infty$. The term II is asymptotically bounded in probability from the first part of the proof. Hence, if we let $n \to \infty$ first and then $\alpha \to \infty$, we have

$$\sup_{|u|\le L} \int_{Z_{n-k}}^{\infty} |r^2(\theta_k, X, Z_{n-k}) - r_*^2(\theta_k, X, Z_{n-k})| f_k(x, \theta_k)\, dx \to_P 0. \qquad \square$$

LEMMA 3. *Under the same assumptions as in Theorem* 1 *we have, for any* $L > 0$,

$$\text{(a)} \qquad \sup_{|u|\le L} \int \left(\frac{\partial \sqrt{f_k(x, \theta + u/\sqrt{k})}}{\partial\theta} - \frac{\partial \sqrt{f_k(x, \theta)}}{\partial\theta} \right)^2 dx \to_P 0,$$

$$\text{(b)} \qquad E\left[\left| \frac{f_k'(x, \theta)}{f_k(x, \theta)} \right|^2 I\left\{ \left| \frac{f_k'(x, \theta)}{f_k(x, \theta)} \right| > \varepsilon\sqrt{k} \right\} \middle| Z_{n-k} \right] \to_P 0 \quad \forall\, \varepsilon > 0$$

and

$$\text{(c)} \quad k \int \left(\sqrt{f_k\left(x, \theta + \frac{u}{\sqrt{k}}\right)} - \sqrt{f_k(x, \theta)} - \frac{u}{2\sqrt{k}} \frac{f_k'(x, \theta)}{\sqrt{f_k(x, \theta)}} \right)^2 dx \to_P 0,$$

where $f_k'(x, \theta) = \partial f_k(x, \theta)/\partial\theta$.

PROOF. Let

$$\begin{aligned} \varphi_k(x, \theta) &= \frac{\partial \sqrt{f_k(x, \theta)}}{\partial\theta} = \frac{f_k'(x, \theta)}{2\sqrt{f_k(x, \theta)}} \\ &= \frac{1}{2}\left(\theta^{-1} - \log \frac{x}{Z_{n-k}} + \frac{\partial \log L^*_\theta(x)}{\partial\theta} - \frac{\partial \log L_\theta(Z_{n-k})}{\partial\theta} \right) \sqrt{f_k(x, \theta)} \end{aligned}$$

and

$$\varphi_k^*(x, \theta) = \frac{1}{2}\left(\theta^{-1} - \log \frac{x}{Z_{n-k}} \right) \sqrt{\frac{\theta x^{-\theta-1}}{Z_{n-k}^{-\theta}}}\,.$$

If we do a variable transformation $y = x/Z_{n-k}$ in $\varphi_k(x, \theta)$ and $\varphi_k^*(x, \theta)$ and write the functions after transformation as $\psi_k(y, \theta)$ and $\psi^*(y, \theta)$, respectively, then

$$\begin{aligned} \psi_k(y, \theta) = \frac{1}{2}\Bigg(\theta^{-1} - \log y &+ \frac{\partial \log L^*_\theta(yZ_{n-k})}{\partial\theta} \\ &- \frac{\partial \log L_\theta(Z_{n-k})}{\partial\theta} \Bigg) \sqrt{g_k(y, \theta)} \end{aligned}$$

and

$$\psi^*(y,\theta) = \frac{1}{2}(\theta^{-1} - \log y)\sqrt{\theta y^{-\theta-1}} = \frac{1}{2}(\theta^{-1} - \log y)\sqrt{g^*(y,\theta)},$$

where $g_k(y,\theta) = \theta y^{-\theta-1} L^*_\theta(yZ_{n-k})/L_\theta(Z_{n-k})$ and $g^*(y,\theta) = \theta y^{-\theta-1}$. Hence the integral on the left-hand side of (a) can be written as

$$\begin{aligned}
&\int_{Z_{n-k}}^\infty \left(\frac{\partial\sqrt{f_k(x,\theta+u/\sqrt{k})}}{\partial\theta} - \frac{\partial\sqrt{f_k(x,\theta)}}{\partial\theta} \right)^2 dx \\
&\quad = \int_{Z_{n-k}}^\infty \left[\left(\varphi_k\left(x,\theta+\frac{u}{\sqrt{k}}\right) - \varphi_k^*(x,\theta) \right) - \left(\varphi_k(x,\theta) - \varphi_k^*(x,\theta)\right) \right]^2 dx \\
&\quad \le C\left[\int_{Z_{n-k}}^\infty \left(\varphi_k\left(x,\theta+\frac{u}{\sqrt{k}}\right) - \varphi_k^*(x,\theta) \right)^2 dx \right. \\
&\qquad\qquad \left. + \int_{Z_{n-k}}^\infty \left(\varphi_k(x,\theta) - \varphi_k^*(x,\theta)\right)^2 dx \right] \\
&\quad = C\left[\int_1^\infty \left(\psi_k\left(y,\theta+\frac{u}{\sqrt{k}}\right) - \psi^*(y,\theta) \right)^2 dy \right. \\
&\qquad\qquad \left. + \int_1^\infty \left(\psi_k(y,\theta) - \psi^*(y,\theta)\right)^2 dy \right] \\
&\quad = C[\mathrm{I} + \mathrm{II}],
\end{aligned}$$

where $C > 0$. The first term I is

$$\begin{aligned}
\mathrm{I} &= \int_1^\infty \frac{1}{4}\left(\frac{\partial \log g_k(y,\theta+u/\sqrt{k})}{\partial\theta} \frac{\sqrt{g_k(y,\theta+u/\sqrt{k})}}{\sqrt{g^*(y,\theta)}} - (\theta^{-1} - \log y) \right)^2 \\
&\quad \times g^*(y,\theta)\,dy \\
&= E^*\left[\frac{1}{4}\left(\frac{\partial \log g_k(Y,\theta+u/\sqrt{k})}{\partial\theta} \frac{\sqrt{g_k(Y,\theta+u/\sqrt{k})}}{\sqrt{g^*(Y,\theta)}} - (\theta^{-1} - \log Y) \right)^2 \right] \\
&= E^*\left[I\{\mathscr{B}_k\} \frac{1}{4}\left(\frac{\partial \log g_k(Y,\theta+u/\sqrt{k})}{\partial\theta} \frac{\sqrt{g_k(Y,\theta+u/\sqrt{k})}}{\sqrt{g^*(Y,\theta)}} \right.\right. \\
&\qquad\qquad\qquad\qquad \left.\left. -(\theta^{-1} - \log Y) \right)^2 \right]
\end{aligned}$$

$$+ E^*\left[I\{\mathscr{B}_k^c\} \frac{1}{4}\left(\frac{\partial \log g_k(Y, \theta + u/\sqrt{k})}{\partial \theta} \frac{\sqrt{g_k(Y, \theta + u/\sqrt{k})}}{\sqrt{g^*(Y, \theta)}} - (\theta^{-1} - \log Y)\right)^2 \right]$$

$$= \mathrm{I}' + \mathrm{II}',$$

where E^* is the expectation with respect to the density function g^* and

$$\mathscr{B}_k = \left\{ \omega: \left| \frac{1}{4}\left(\frac{\partial \log g_k(Y, \theta + u/\sqrt{k})}{\partial \theta} \frac{\sqrt{g_k(Y, \theta + u/\sqrt{k})}}{\sqrt{g^*(Y, \theta)}} - (\theta^{-1} - \log Y)\right) \right| < \varepsilon \right\}.$$

Obviously, $\mathrm{I}' < \varepsilon$. Using a well-known inequality, the term II' is

$$\begin{aligned} \mathrm{II}' &\le E^*\left[2I\{\mathscr{B}_k^c\} \frac{1}{4}\left(\frac{\partial \log g_k(Y, \theta + u/\sqrt{k})}{\partial \theta} \frac{\sqrt{g_k(Y, \theta + u/\sqrt{k})}}{\sqrt{g^*(Y, \theta)}} \right)^2 + \frac{1}{4}(\theta^{-1} - \log Y)^2 \right] \\ &< 2I\{Z_{n-k} < M\}\left(E^*\left[\frac{1}{4}\left(\frac{\partial \log g_k(Y, \theta + u/\sqrt{k})}{\partial \theta} \frac{\sqrt{g_k(Y, \theta + u/\sqrt{k})}}{\sqrt{g^*(Y, \theta)}} \right)^2 \right] + E^*\left[\frac{1}{4}(\theta^{-1} - \log Y)^2 \right] \right), \end{aligned}$$

where $M > 0$ is such that if $Z_{n-k} > M$, then $I\{\mathscr{B}_k\} = 1$.

The term $\mathrm{II}' \to_P 0$ as $k = o(n) \to \infty$, since, by Lemma 1, $I\{Z_{n-k} < M\} \to_P 0$ and, by Lemma 2,

$$E^*\left[\frac{1}{4}\left(\frac{\partial \log g_k(Y, \theta + u/\sqrt{k})}{\partial \theta} \frac{\sqrt{g_k(Y, \theta + u/\sqrt{k})}}{\sqrt{g^*(Y, \theta)}} \right)^2 \right]$$

$$+ E^*\left[\frac{1}{4}(\theta^{-1} - \log Y)^2\right] \to_P \frac{\theta^{-2}}{2},$$

for all u such that $|u| \le L$, as $k = o(n) \to \infty$. Combining the above results we have $\mathrm{I} \to_P 0$ as $k = o(n) \to \infty$. Similarly it can be shown that $\mathrm{II} \to_P 0$ as $k = o(n) \to \infty$.

Condition (b) is in fact the conditional Lindeberg condition. Let

$$\mathscr{A}_k = \left\{x: \left|\frac{f_k'(x,\theta)}{f_k(x,\theta)}\right| > \varepsilon\sqrt{k}\right\}$$

and

$$\mathscr{A}_k^* = \left\{y: \left|\frac{g_k'(y,\theta)}{g_k(y,\theta)}\right| > \varepsilon\sqrt{k}\right\}.$$

Using the notation introduced above, (b) is

$$\begin{aligned}
E&\left[\left|\frac{f_k'(x,\theta)}{f_k(x,\theta)}\right|^2 I\left\{\left|\frac{f_k'(x,\theta)}{f_k(x,\theta)}\right| > \varepsilon\sqrt{k}\right\}\middle| Z_{n-k}\right] \\
&= 4\int_{\mathscr{A}_k} \varphi_k(x,\theta)^2\,dx \\
&= 4\int_{\mathscr{A}_k^*} (\psi_k(y,\theta) - \psi^*(y,\theta) + \psi^*(y,\theta))^2\,dy \\
&\le C\left[\int_{\mathscr{A}_k^*} (\psi_k(y,\theta) - \psi^*(y,\theta))^2\,dy + \int_{\mathscr{A}_k^*} \psi^*(y,\theta)^2\,dy\right] \\
&\le C\left[\int_1^\infty (\psi_k(y,\theta) - \psi^*(y,\theta))^2\,dy + \int_{\mathscr{A}_k^*} \psi^*(y,\theta)^2\,dy\right],
\end{aligned}$$

where $C > 0$. The first term tends to 0 in probability as $k = o(n) \to \infty$, which follows from the proof of condition (a). The second term

$$\int_{\mathscr{A}_k^*} \psi^*(y,\theta)^2\,dy = \frac{1}{4}\int_{\mathscr{A}_k^*} (\theta^{-1} - \log y)^2 \theta y^{-\theta-1}\,dy \to_P 0,$$

as $k = o(n) \to \infty$ follows easily. Part (c) follows from (a) and a usual argument. □

The uniform asymptotic normality for P_θ^k can be similarly proved.

PROPOSITION 1. *Let the conditions* (C1)–(C4) *on* $L_\theta(x)$ *be satisfied uniformly over compact sets of* $\Theta = (0,\infty)$. *Then* P_θ^k *is uniformly asymptotically normal in any compact set* $K \subset \Theta$; *that is, for any sequences* $\theta_k \in K$ *and*

$u_k \to u$ such that $\theta_k + u_k\theta_k/\sqrt{k} \in K$, the representation

$$\frac{dP^k_{\theta_k + u_k\theta_k/\sqrt{k}}}{dP^k_{\theta_k}} = \exp\left\{u\theta_k\Delta_{k,\theta_k} - \frac{u^2}{2} + \psi_{n,k}(\theta_k, u_k)\right\}$$

is valid; here $P^k_{\theta_k}(|\psi_{n,k}(\theta_k, u_k)| > \varepsilon) \to 0\ \forall\, \varepsilon > 0$ and $\theta_k\Delta_{k,\theta_k} \to_D N(0,1)$.

2.3. *Asymptotic properties of conditional MLE.* In this section we assume the existence and uniqueness of a conditional MLE and investigate its asymptotic properties.

THEOREM 2. *Let $F_\theta(x) = 1 - x^{-\theta}L_\theta(x)$ with $L_\theta(x)$ satisfying conditions* (C1)–(C4). *Let $\hat{\theta}_k$ be a conditional MLE of θ based on the $(k+1)$ largest order statistics pertaining to i.i.d. random variables from $F_\theta(x)$, and let K be any compact set in $\Theta = (a, b)$, with $0 < a < b < \infty$. Then, $\forall\, \delta > 0$,*

$$\sup_{\theta \in K} P^k_\theta\left(|\sqrt{k}\,\theta^{-2}\left(\hat{\theta}_k - \theta\right) - \Delta_{k,\theta}| > \delta\right) \to 0 \tag{5}$$

as $k = o(n) \to \infty$, where $\Delta_{k,\theta}$ is the same as defined in Theorem 1. *Hence we have*

$$\sqrt{k}\left(\hat{\theta}_k - \theta\right) \to_D N(0, \theta^2)$$

uniformly in $\theta \in K$ as $k = o(n) \to \infty$.

NOTE. Result (5) may be called asymptotic sufficiency, which also implies certain asymptotic efficiency.

PROOF OF THEOREM 2. Let

$$Z_{k,\theta}(u, Z_{n-k}) = \prod_{j=1}^{k} \frac{f_k\left(Z_{n-k+j}, \theta + u/\sqrt{k}\right)}{f_k(Z_{n-k+j}, \theta)},$$

where, as before, we may assume without loss of generality that Z_{n-k+j}, $j = 1, 2, \ldots, k$, are i.i.d. from $f_k(x, \theta)$ given Z_{n-k}.

Given $Z_{n-k} = z_{n-k}$, the conditional Fisher information $I(\theta, z_{n-k}) \to \theta^{-2}$ as $z_{n-k} \to \infty$. Hence there exists a $0 < M < \infty$ such that if $z_{n-k} \ge M$, then

$$0 < \inf_{\{z_{n-k} > M\}} \inf_{\theta \in \Theta} I(\theta, z_{n-k}) \le \sup_{\{z_{n-k} > M\}} \sup_{\theta \in \Theta} I(\theta, z_{n-k}) < \infty.$$

Define $\mathscr{A}_{n,k} = \{Z_{n-k} \ge M\}$ and $\Xi_{k,\theta}(u, Z_{n-k}) = Z_{k,\theta}(u, Z_{n-k}) I\{\mathscr{A}_{n,k}\}$. Let θ^*_k be a MLE based on $\Xi_{k,\theta}(u, Z_{n-k})$. Then $\hat{\theta}_k - \theta^*_k = 0$ and hence $\sqrt{k}\,(\hat{\theta}_k - \theta^*_k) = 0$ if $Z_{n-k} \ge M$, which implies

$$P\left(\sqrt{k}\left(\hat{\theta}_k - \theta^*_k\right) = 0\right) > P(Z_{n-k} \ge M) \to 1$$

as $k = o(n) \to \infty$. Therefore, we only have to prove that

$$\sup_{\theta \in K} P^k_\theta\left(|\sqrt{k}\,\theta^{-2}(\theta^*_k - \theta) - \Delta_{k,\theta}| > \delta\right) \to 0.$$

Applying Theorem III.1.2 from Ibragimov and Has'minskii (1981), we only need to verify the following:

(a) Uniform asymptotic normality for $\Xi_{k,\theta}(u, Z_{n-k})$.
(b) For any compact set $K \subset \Theta$ and $u, v \in U_{k,\theta}$ and $|u|, |v| \le L$, $0 < L < \infty$:

$$\sup_{\theta \in K} E\left[\left|\sqrt{\Xi_{k,\theta}(u, Z_{n-k})} - \sqrt{\Xi_{k,\theta}(v, Z_{n-k})}\right|^2\right] \le H(u-v)^2,$$

where H is a positive constant which does not depend on θ, v and u and $U_{k,\theta} = \sqrt{k}\,\theta^{-2}(\Theta - \theta)$.

(c) For any compact set $K \subset \Theta$ and any $N > 0$, there exists a k_0 such that

$$\sup_{k > k_0} \sup_{\theta \in K} \sup_{u \in U_{k,\theta}} |u|^N E\left[\Xi_{k,\theta}(u, Z_{n-k})^{1/2}\right] \le \sup_{\theta \in K} \sup_{u \in U_{k,\theta}} |u|^N \exp\{-Cu^2\},$$

where C is a positive constant.

Condition (a) follows from uniform asymptotic normality of $Z_{k,\theta}(u, Z_{n-k})$ since

$$\Xi_{k,\theta}(u, Z_{n-k}) = Z_{k,\theta}(u, Z_{n-k}) - Z_{k,\theta}(u, Z_{n-k}) I\{\mathscr{A}_{n,k}^c\}.$$

In fact, for any sequence $\theta_k \in K$, $u_k \to u$ such that $\theta_k + k^{-1/2}\theta_k u_k \in K$, uniform asymptotic normality of $Z_{k,\theta}(u, Z_{n-k})$ implies

$$Z_{k,\theta_k}(u_k, Z_{n-k}) \to_D \exp\{\Delta u - \tfrac{1}{2}u^2\} = Z(u)$$

for any fixed u, where Δ is a standard normal variable. Since $I\{\mathscr{A}_{n,k}^c\} \to_P 0$, $Z_{k,\theta_k}(u_k, Z_{n-k}) I\{\mathscr{A}_{n,k}^c\} \to_D 0$. Therefore, we have $\Xi_{k,\theta_k}(u_k, Z_{n-k}) \to_D Z(u)$.

We now verify condition (b). The left-hand side of (b) is

$$\begin{aligned}
&\sup_{\theta \in K} E\left[\left|\sqrt{\Xi_{k,\theta}(u, Z_{n-k})} - \sqrt{\Xi_{k,\theta}(v, Z_{n-k})}\right|^2\right] \\
&\quad = \sup_{\theta \in K} E\left[I\{\mathscr{A}_{n,k}\} E\left[\left|\sqrt{Z_{k,\theta}(u, Z_{n-k})} - \sqrt{Z_{k,\theta}(v, Z_{n-k})}\right|^2 \Big| Z_{n-k}\right]\right] \\
&\quad = \sup_{\theta \in K} E\Bigg[I\{\mathscr{A}_{n,k}\} \int \Bigg| \sqrt{\prod_{j=1}^{k} f_k\left(z_{n-k+j}, \theta + \frac{u}{\sqrt{k}}\right)} \\
&\qquad\qquad - \sqrt{\prod_{j=1}^{k} f_k\left(z_{n-k+j}, \theta + \frac{v}{\sqrt{k}}\right)}\Bigg|^2 \prod_{j=1}^{k} dz_{n-k+j}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\theta\in K} E\Bigg[I\{\mathscr{A}_{n,k}\} \int \prod_{j=1}^{k} dz_{n-k+j} \\
&\qquad \times \left(\int_{\theta+v\theta/\sqrt{k}}^{\theta+u\theta/\sqrt{k}} \frac{\partial \sqrt{\prod_{j=1}^{k} f_k(z_{n-k+j}, t)}}{\partial t}\, dt \right)^2 \Bigg] \\
&\le \sup_{\theta\in K} \frac{\theta(u-v)}{\sqrt{k}} E\Bigg[I\{\mathscr{A}_{n,k}\} \int \prod_{j=1}^{k} dz_{n-k+j} \\
&\qquad \times \int_{\theta+v\theta/\sqrt{k}}^{\theta+u\theta/\sqrt{k}} \left(\frac{\partial \sqrt{\prod_{j=1}^{k} f_k(z_{n-k+j}, t)}}{\partial t} \right)^2 dt \Bigg] \\
&= \sup_{\theta\in K} \frac{\theta(u-v)}{\sqrt{k}} E\Bigg[I\{\mathscr{A}_{n,k}\} \int_{\theta+v\theta/\sqrt{k}}^{\theta+u\theta/\sqrt{k}} dt \\
&\qquad \times \int \left(\frac{\partial \sqrt{\prod_{j=1}^{k} f_k(z_{n-k+j}, t)}}{\partial t} \right)^2 \prod_{j=1}^{k} dz_{n-k+j} \Bigg] \\
&= \sup_{\theta\in K} \frac{\sqrt{k}\,\theta}{4}(u-v) E\left[I\{\mathscr{A}_{n,k}\} \int_{\theta+v\theta/\sqrt{k}}^{\theta+u\theta/\sqrt{k}} I(t, Z_{n-k})\, dt \right] \\
&\le \frac{1}{4}(u-v)^2 \sup_{z_{n-k}\ge M} \sup_{\theta\in\Theta} I(\theta, z_{n-k})\theta^2 \\
&= H(u-v)^2,
\end{aligned}
$$

where $H < \infty$.

We now verify condition (c). We shall show first that, $\forall\, \delta > 0$ and $z_{n-k} \ge M$ for M sufficiently large, there exists an $\eta > 0$ such that the following is true:

$$
(6)\quad \inf_{\theta\in K} \inf_{\{h:\, \theta+h\in\Theta,\, |h|>\delta\}} \int_{z_{n-k}}^{\infty} \left(\sqrt{f_k(x, \theta+h)} - \sqrt{f_k(x, \theta)} \right)^2 dx \ge \eta > 0.
$$

This is equivalent to showing that

$$
(7)\quad \sup_{\theta\in K} \sup_{\{h:\, \theta+h\in\Theta,\, |h|>\delta\}} \int_{z_{n-k}}^{\infty} \sqrt{f_k(x, \theta+h)}\,\sqrt{f_k(x, \theta)}\, dx \le 1 - \eta.
$$

Let

$$
\eta_\theta(y, z_{n-k}) = 1 - \frac{L_\theta(yz_{n-k})}{L_\theta(z_{n-k})} + \frac{\varepsilon_\theta(yz_{n-k})}{\theta}
$$

and $\eta^*_{\theta,h}(y, z_{n-k}) = \eta_\theta(y, z_{n-k}) + \eta_{\theta+h}(y, z_{n-k}) - \eta_\theta(y, z_{n-k})\eta_{\theta+h}(y, z_{n-k})$, where $\varepsilon_\theta(x)$ was defined in condition (C1). The left-hand side of (7) can be written as

$$\begin{aligned}
&\sup_{\theta\in K}\ \sup_{\{h:\ \theta+h\in\Theta,\ |h|>\delta\}} \int_{z_{n-k}}^\infty \sqrt{f_k(x,\theta+h)}\sqrt{f_k(x,\theta)}\,dx\\
&\quad= \sup_{\theta\in K}\ \sup_{\{h:\ \theta+h\in\Theta,\ |h|>\delta\}} \int_1^\infty \big((1-\eta_{\theta+h}(y,z_{n-k}))(1-\eta_\theta(y,z_{n-k}))\big)^{1/2}\\
&\qquad\times\sqrt{\theta(\theta+h)}\,y^{-\theta-1-h/2}\,dy\\
&\quad= \sup_{\theta\in K}\ \sup_{\{h:\ \theta+h\in\Theta,\ |h|>\delta\}} \int_1^\infty \big(1-\eta^*_{\theta,h}(y,z_{n-k})\big)^{1/2}\sqrt{\theta(\theta+h)}\,y^{-\theta-1-h/2}\,dy\\
&\quad= \sup_{\theta\in K}\ \sup_{\{h:\ \theta+h\in\Theta,\ |h|>\delta\}} \int_1^\infty \sqrt{\theta(\theta+h)}\,y^{-\theta-1-1/2}\,dy\\
&\qquad+ O\left(\int_1^\infty (\eta^*_{\theta,h}(y,z_{n-k}))\sqrt{\theta(\theta+h)}\,y^{-\theta-1-1/2}\,dy\right).
\end{aligned}$$

An easy integration gives that the first term is, for some $\eta > 0$,

$$\sup_{\theta\in K}\ \sup_{\{h:\ \theta+h\in\Theta,\ |h|>\delta\}} \frac{\sqrt{1+h/\theta}}{1+h/(2\theta)} < 1-\eta.$$

It is easy to see that, by conditions (C1) and (C4), if $z_{n-k} > M$ for M sufficiently large, the second term will be less than ε, where ε can be arbitrarily small. Hence (6) is true.

Applying Theorem I.7.6. from Ibragimov and Has'minskii (1981), one has, if $z_{n-k} > M$,

$$\begin{aligned}
&\inf_{\theta\in K}\liminf_{h\to 0}|h|^{-2}\int_{z_{n-k}}^\infty \left(\sqrt{f_k(x,\theta+h)}-\sqrt{f_k(x,\theta)}\right)^2 dx\\
&\qquad\ge \inf_{\theta\in K}\frac{1}{4}I(\theta,z_{n-k}) \ge \inf_{z_{n-k}\ge M}\ \inf_{\theta\in\Theta}\frac{1}{4}I(\theta,z_{n-k}) > 0.
\end{aligned}\tag{8}$$

Combining (6) and (8) we have the following inequality:

$$\inf_{\theta\in K}\ \inf_{\{h:\ \theta+h\in\Theta\}} \int_{z_{n-k}}^\infty \left(\sqrt{f_k(x,\theta+h)}-\sqrt{f_k(x,\theta)}\right)^2 dx \ge \frac{B|h|^2}{1+|h|^2},\tag{9}$$

if $z_{n-k} > M$, where B does not depend on z_{n-k}. Hence

$$\begin{aligned}
&\sup_{\theta,u} E\left[\sqrt{\Xi_{k,\theta}(u,Z_{n-k})}\right]\\
&\qquad= \sup_{\theta,u} E\left[I\{\mathscr{A}_{n,k}\}E\left[\sqrt{Z_{k,\theta}(u,Z_{n-k})}\,\middle|Z_{n-k}\right]\right]
\end{aligned}$$

$$= \sup_{\theta, u} E\left[I\{\mathscr{A}_{n,k}\} \left(\int_{Z_{n-k}}^{\infty} \sqrt{f_k\left(x, \theta + \frac{u}{\sqrt{k}}\right)} \sqrt{f_k(x,\theta)}\, dx \right)^k \right]$$

$$= \sup_{\theta, u} E\left[I\{\mathscr{A}_{n,k}\} \left(1 - \frac{1}{2} \int_{Z_{n-k}}^{\infty} \left(\sqrt{f_k\left(x, \theta + \frac{u}{\sqrt{k}}\right)} - \sqrt{f_k(x,\theta)} \right)^2 dx \right)^k \right]$$

$$\le \sup_{\theta, u} E\left[I\{\mathscr{A}_{n,k}\} \exp\left\{ -\frac{k}{2} \int_{Z_{n-k}}^{\infty} \left(\sqrt{f_k\left(x, \theta + \frac{u}{\sqrt{k}}\right)} - \sqrt{f_k(x,\theta)} \right)^2 dx \right\} \right]$$

$$\le \sup_{\theta, u} \exp\left\{ -\frac{k}{2} \frac{B|u|^2/k\theta^{-4}}{1 + |u|^2/k\theta^{-4}} \right\}$$

$$\le \sup_{\theta, u} \exp\left\{ -\frac{1}{2} \frac{B|u|^2}{b^{-2} + |u|^2/k} \right\}$$

$$= \sup_{u \in U_{k,\theta}} \exp\{-C|u|^2\},$$

when k is sufficiently large, where the supremum is taken over $\theta \in K$ and $u \in U_{k,\theta}$. □

3. Asymptotic efficient estimators of the index of regular variation under partial knowledge of slowly varying function. In this section we assume that a distribution function which is regularly varying at ∞ has the following form:

$$(10) \quad 1 - F_\theta(x) = Cx^{-\theta}\left(1 + D_1 x^{-\beta} + D_2 x^{-2\beta} + \cdots + D_l x^{-l\beta} - o(x^{-l\beta})\right)$$

as $x \ge x_0 > 0$, where $\theta > 0$, $C > 0$, $\beta > 0$, $l > 1$ and the D_j's are known constants. These special forms of $L_\theta(x)$ have been considered by Hall (1982), Hall and Welsh (1984, 1985), Haeusler and Teugels (1985) and Csörgő, Deheuvels and Mason (1985), among others.

We extend the result by Hall and Welsh (1984) that the optimal convergence rate of any estimator of θ is $n^{-\beta/(2\beta+\theta)}$ when $F_\theta(x)$ has the form (10) but with $l = 1$, and we show that the optimal rate of any estimator when $l > 1$ is $n^{-l\beta/(2l\beta+\theta)}$. We omit the proof here since the proof is very tedious.

We now derive an estimator for θ which achieves this best obtainable rate and is asymptotically efficient. The estimator we propose is obtained by a Newton–Raphson iterative procedure based on $\Delta'_k(\theta)$ but ignoring the term $o(Z_{n-k}^{-l\beta})$, where

$$\Delta'_k(\theta) = -\frac{1}{k} \sum_{j=1}^{k} \left(\log \frac{Z_{n-k+j}}{Z_{n-k}} - E\left[\log \frac{Z_{n-k+j}}{Z_{n-k}} \middle| Z_{n-k} \right] \right).$$

To derive the estimator, we need the following calculation. Let $L(x) =$

$1 + D_1 x^{-\beta} + \cdots + D_l x^{-l\beta} + o(x^{-l\beta})$. Then

$$\begin{aligned}\frac{L(yz)}{L(z)} &= \frac{1 + D_1(yz)^{-\beta} + \cdots + D_l(yz)^{-l\beta} + o(z^{-l\beta})}{1 + D_1 z^{-\beta} + \cdots + D_l z^{-l\beta} + o(z^{-l\beta})} \\ &= \big(1 + D_1(yz)^{-\beta} + \cdots + D_l(yz)^{-l\beta} + o(z^{-l\beta})\big) \\ &\quad \times \big(1 - D_1 z^{-\beta} + \big(D_1^2 - D_2\big) z^{-2\beta} - \big(D_1^3 - 2D_1D_2 + D_3\big) z^{-3\beta} \\ &\qquad + \cdots + h_l(D_1, \dots, D_l) z^{-l\beta} + o(z^{-l\beta})\big) \\ &= \sum_{q=0}^{l} \sum_{m=0}^{q} p_{q,m}(D_1, \dots, D_l) y^{-m\beta} z^{-q\beta} + o(z^{-l\beta}),\end{aligned}$$

where

$$h_l = (-l)^l D_1^l + (-1)^{l-1}(l-1) D_1^{l-2} D_2 + \cdots + 2D_1 D_{l-1} - D_l$$

and the $p_{q,m}(D_1, \dots, D_l)$'s are polynomials of $D_1, \dots, D_l$ with $p_{0,0} = 1$, $p_{1,0} = -p_{1,1} = D_1$ and so on.

Integrating by parts, the conditional expectation of $\log(Z_{n-k+1}/Z_{n-k})$ is

$$\begin{aligned}E&\left[\log \frac{Z_{n-k+1}}{Z_{n-k}} \middle| Z_{n-k}\right] \\ &= \int_1^\infty y^{-\theta-1} \frac{L_\theta(yZ_{n-k})}{L_\theta(Z_{n-k})}\, dy \\ &= \int_1^\infty y^{-\theta-1} \sum_{q=0}^{l} \sum_{m=0}^{q} p_{q,m}(D_1, \dots, D_l) y^{-m\beta} Z_{n-k}^{-q\beta}\, dy + o\big(Z_{n-k}^{-l\beta}\big) \\ &= \sum_{q=0}^{l} \sum_{m=0}^{q} p_{q,m}(D_1, \dots, D_l)(\theta + m\beta)^{-1} Z_{n-k}^{-q\beta} + o\big(Z_{n-k}^{-l\beta}\big).\end{aligned}$$

Formalizing the arguments above, we have the following theorem.

THEOREM 3. *Suppose* $\{Z_{n-k+j},\ j = 0, 1, 2, \dots, k\}$ *are the order statistics corresponding to i.i.d. observations from* (10). *Let* $k_i = \lambda_i n^{2i\beta/(2i\beta+\theta)}$, *where the* λ_i's *are positive constants*, $i = 1, 2, \dots, l$, *and let* $\hat{\theta}_{k_1}$ *be Hill's estimator of* θ *using* k_1 *largest order statistics. Let*

$$\hat{\theta}_{k_{i+1}} = \hat{\theta}_{k_i} - \frac{\Delta^*_{k_{i+1}}\big(\hat{\theta}_{k_i}\big)}{\partial \Delta^*_{k_{i+1}}\big(\hat{\theta}_{k_i}\big)/\partial\theta}, \qquad i = 1, 2, \dots, l-1,$$

where

$$\Delta^*_k(\theta) = -\frac{1}{k} \sum_{j=1}^{k} \left(\log \frac{Z_{n-k+j}}{Z_{n-k}} - \sum_{q=0}^{l} \sum_{m=0}^{q} p_{q,m}(D_1, \dots, D_l)(\theta + m\beta)^{-1} Z_{n-k}^{-q\beta}\right),$$

where the $p_{q,m}(D_1, \dots, D_l)$'s *are polynomials of* $D_1, D_2, \dots, D_l$ *as defined* *before. Then*

$$\sqrt{k_{i+1}}\left(\left(\hat{\theta}_{k_{i+1}}-\theta\right)-\theta^2\Delta_{k_{i+1}}(\theta)\right)\to_P 0.$$

as $n\to\infty$ *for* $i=1,2,\dots,l-1$, *where* $\Delta_k(\theta)=\Delta_{k,\theta}/\sqrt{k}$ *and* $\Delta_{k,\theta}$ *is the same as defined in Theorem* 1.

PROOF. The theorem will be proved by induction on i. Hence we need to verify the theorem for $i=1$ first, and then prove it for the case i under the assumption that the result holds for $i-1$. We combine two steps into one since the proving procedure is similar.

Expanding $\Delta^*_{k_{i+1}}(\hat{\theta}_{k_i})$ at θ, we have

$$\begin{aligned}
&\sqrt{k_{i+1}}\left(\left(\hat{\theta}_{k_{i+1}}-\theta\right)-\theta^2\Delta_{k_{i+1}}(\theta)\right)\\
&\quad=\sqrt{k_{i+1}}\left(\left(\hat{\theta}_{k_i}-\frac{\Delta^*_{k_{i+1}}\left(\hat{\theta}_{k_i}\right)}{\partial\Delta^*_{k_{i+1}}\left(\hat{\theta}_{k_i}\right)/\partial\theta}-\theta\right)-\theta^2\Delta_{k_{i+1}}(\theta)\right)\\
&\quad=\sqrt{k_{i+1}}\left(1-\frac{\partial\Delta^*_{k_{i+1}}\left(\theta^*_{k_i}\right)/\partial\theta}{\partial\Delta^*_{k_{i+1}}\left(\hat{\theta}_{k_i}\right)/\partial\theta}\right)\left(\hat{\theta}_{k_i}-\theta\right)\\
&\qquad-\sqrt{k_{i+1}}\left(\frac{\Delta^*_{k_{i+1}}(\theta)}{\partial\Delta^*_{k_{i+1}}\left(\hat{\theta}_{k_i}\right)/\partial\theta}-\frac{\Delta_{k_{i+1}}(\theta)}{\theta^{-2}}\right)\\
&\quad=\mathrm{I}+\mathrm{II},
\end{aligned}$$

where $\theta^*_{k_i}=\alpha\theta+(1-\alpha)\hat{\theta}_{k_i}$ with $0\le\alpha\le 1$.

We now prove that $\mathrm{I}\to_P 0$. Using the fact that $(\hat{\theta}_{k_i}-\theta)=O_p(1/\sqrt{k_i})$ (this is true for $i=1$ by the asymptotic normality result of Hill's estimator and it is true for $i=2,3,\dots,l-1$ by the assumption that the theorem is true for $i-1$), it is easy to see that

$$\begin{aligned}
&\left(1-\frac{\partial\Delta^*_{k_{i+1}}\left(\theta^*_{k_i}\right)/\partial\theta}{\partial\Delta^*_{k_{i+1}}\left(\hat{\theta}_{k_i}\right)/\partial\theta}\right)\\
&\quad=1-\frac{\theta^{*\,-2}_{k_i}-\sum_{q=1}^{l}\sum_{m=0}^{q}p_{q,m}(D_1,\dots,D_l)\left(\theta^*_{k_i}+m\beta\right)^{-2}Z_{n-k}^{-q\beta}}{\hat{\theta}^{-2}_{k_i}-\sum_{q=1}^{l}\sum_{m=0}^{q}p_{q,m}(D_1,\dots,D_l)\left(\hat{\theta}_{k_i}+m\beta\right)^{-2}Z_{n-k}^{-q\beta}}\\
&\quad=O_p\left(\frac{1}{\sqrt{k_i}}\right).
\end{aligned}$$

Therefore

$$\mathrm{I}=\sqrt{k_{i+1}}\,O_p\left(\frac{1}{k_i}\right)=n^{(i+1)\beta/[2(i+1)\beta+\theta]}O_p\left(n^{-2i\beta/(2i\beta+\theta)}\right)\to_P 0,$$

as $n\to\infty$.

We now prove that II $\to_P 0$. Since $\partial\Delta^*_{k_{i+1}}(\hat\theta_{k_i})/\partial\theta \to_P \theta^{-2}$, it suffices to prove that $\sqrt{k}\,(\Delta_k(\theta) - \Delta^*_k(\theta)) \to_P 0$ for any $k = k_i$ with $i = 2, \dots, l$. Define

$$R(\theta, \beta, Z_{n-k+j}) = \frac{\sum_{i=1}^{l} D_i i\beta\theta^{-2} Z_{n-k+j}^{-i\beta} + o\big(Z_{n-k+j}^{-l\beta}\big)}{1 + \sum_{i=1}^{l} D_i(1 - i\beta\theta^{-1}) Z_{n-k+j}^{-i\beta} + o\big(Z_{n-k+j}^{-l\beta}\big)}.$$

By an elementary calculation and the fact that $E[\Delta_k(\theta) | Z_{n-k}] = 0$, we may write

$$\begin{aligned}\sqrt{k}\,&(\Delta_k(\theta) - \Delta^*_k(\theta))\\ &= \frac{1}{\sqrt{k}} \sum_{j=1}^{k} \Big(R(\theta, \beta, Z_{n-k+j}) - E\big[R(\theta, \beta, Z_{n-k+j}) | Z_{n-k}\big] + o\big(Z_{n-k}^{-l\beta}\big)\Big).\end{aligned}$$

Since $\sqrt{k}\, o(Z_{n-k}^{-l\beta}) \to_P 0$, for $k \le k_l$, it only has to be shown that

$$S_k = \frac{1}{\sqrt{k}} \sum_{j=1}^{k} \Big(R(\theta, \beta, Z_{n-k+j}) - E\big[R(\theta, \beta, Z_{n-k+j}) | Z_{n-k}\big]\Big) \to_P 0.$$

Let $\mathscr{F}_{k,0} = \sigma(Z_{n-k})$ and $\mathscr{F}_{k,j} = \sigma(Z_{n-k}, \dots, Z_{n-k+j})$, $j = 1, 2, \dots, k$. It is easy to verify that S_k is a martingale with respect to $\{\mathscr{F}_{k,j}\}$. Therefore we only need to show that the conditional variance of this martingale goes to 0 in probability. It is easy to see that

$$\begin{aligned}\operatorname{Var}(S_k | Z_{n-k}) &\le E\big[R(\theta, \beta, Z_{n-k+1})^2 | Z_{n-k}\big]\\ &= E\left[\left(\frac{\sum_{i=1}^{l} D_i i\beta\theta^{-2} Z_{n-k+1}^{-i\beta} + o\big(Z_{n-k+1}^{-l\beta}\big)}{1 + \sum_{i=1}^{l} D_i(1 - i\beta\theta^{-1}) Z_{n-k+1}^{-i\beta} + o\big(Z_{n-k+1}^{-l\beta}\big)}\right)^2 \middle| Z_{n-k}\right]\\ &\le \big(C_1 Z_{n-k}^{-\beta} + \cdots + C_l Z_{n-k}^{-l\beta} + o\big(Z_{n-k}^{-l\beta}\big)\big)^2\\ &\to_P 0,\end{aligned}$$

where the C_j's may depend on β, θ and $D_1, \dots, D_l$. The last inequality is true since $Z_{n-k+1} \ge Z_{n-k}$ with probability 1. □

NOTE. It is easy to see that in each step of the iterative procedure the modifying term $\Delta^*_{k_{i+1}}(\hat\theta_{k_i})/[\partial\Delta^*_{k_{i+1}}(\hat\theta_{k_i})/\partial\theta]$ only depends on Hill's estimator constructed using the k_{i+1} largest order statistics, $Z_{n-k_{i+1}}$, the previous estimator $\hat\theta_{k_i}$, $D_1, \dots, D_l$ and β.

Since k_i depends on θ, the estimator proposed in Theorem 3 is not really an estimator of θ. In the following theorem, we shall propose an estimator of θ which has the same asymptotic distribution and is a real estimator.

THEOREM 4. *Suppose* $\{Z_{n-k+j},\ j = 0, 1, 2, \dots, k\}$ *are the other statistics corresponding to i.i.d. random variables from* (10) *which are larger than* Z_{n-k}. *Let* $k_i = \lambda_i n^{2i\beta/(2i\beta+\theta)}$ *and let* $\hat k_i$ *be a sequence of positive random variables such that* $\hat k_i / k_i \to_P 1$, $i = 1, 2, \dots, l$. *Let* $\hat\theta_{\hat k_1}$ *be Hill's estimator of* θ

using $\hat{k}_1$ largest order statistics. Let

$$\hat{\theta}_{\hat{k}_{i+1}} = \hat{\theta}_{\hat{k}_i} - \frac{\Delta^*_{\hat{k}_{i+1}}\left(\hat{\theta}_{\hat{k}_i}\right)}{\partial \Delta^*_{\hat{k}_{I+1}}\left(\hat{\theta}_{\hat{k}_i}\right)/\partial\theta}, \qquad i = 1, 2, \ldots, l-1,$$

where $\Delta^*_k(\theta)$ is the same as defined in Theorem 3. Then, for $i = 1, 2, \ldots, l-1$,

$$\sqrt{k_{i+1}}\left(\hat{\theta}_{\hat{k}_{i+1}} - \theta\right) \to_D N(0, \theta^2) \tag{11}$$

as $n \to \infty$.

PROOF. We shall prove the theorem by induction on i. For $i = 1$, we have, by expanding $\Delta^*_{\hat{k}_2}(\hat{\theta}_{\hat{k}_1})$ at θ,

$$\begin{aligned}\sqrt{k_2}\left(\hat{\theta}_{\hat{k}_2} - \theta\right) &= \sqrt{k_2}\left(\hat{\theta}_{\hat{k}_1} - \frac{\Delta^*_{\hat{k}_2}\left(\hat{\theta}_{\hat{k}_1}\right)}{\partial \Delta^*_{\hat{k}_2}\left(\hat{\theta}_{\hat{k}_1}\right)/\partial\theta} - \theta\right) \\ &= \sqrt{k_2}\left(1 - \frac{\partial \Delta^*_{\hat{k}_2}(\theta^*)/\partial\theta}{\partial \Delta^*_{\hat{k}_2}\left(\hat{\theta}_{\hat{k}_1}\right)/\partial\theta}\right)\left(\hat{\theta}_{\hat{k}_1} - \theta\right) - \sqrt{k_2}\,\frac{\Delta^*_{\hat{k}_2}(\theta)}{\partial \Delta^*_{\hat{k}_2}\left(\hat{\theta}_{\hat{k}_1}\right)/\partial\theta} \\ &= \mathrm{I} - \mathrm{II},\end{aligned}$$

where $\hat{\theta} = \alpha\theta + (1-\alpha)\hat{\theta}_{\hat{k}_1}$ with $0 \le \alpha \le 1$.

Using Theorem 4.1 of Hall and Welsh (1985), $\hat{\theta}_{\hat{k}_1} - \theta = O_p(1/\sqrt{k_1})$, we can verify that $I \to_P 0$ as $n \to \infty$. The proof is similar to that of Theorem 3. The numerator of the term II is

$$\begin{aligned}&\sqrt{k_2}\,\Delta^*_{\hat{k}_2}(\theta) \\ &\quad = \sqrt{\frac{\hat{k}_2}{k_2}}\,\frac{1}{\sqrt{\hat{k}_2}}\sum_{j=1}^{\hat{k}_2}\left(\log\frac{Z_{n-\hat{k}2+j}}{Z_{n-\hat{k}_2}} - E\left[\log\frac{Z_{n-\hat{k}_2+j}}{Z_{n-\hat{k}_2}}\middle|Z_{n-\hat{k}_2}\right]\right) + \sqrt{k_2}\,o\left(Z^{-l\beta}_{n-\hat{k}_2}\right) \\ &\quad = \sqrt{\frac{\hat{k}_2}{k_2}}\,\frac{1}{\sqrt{\hat{k}_2}}S_{\hat{k}_2} + \sqrt{k_2}\,o\left(Z^{-l\beta}_{n-\hat{k}_2}\right),\end{aligned}$$

where

$$S_k = \sum_{j=1}^{k}\left(\log\frac{Z_{n-k+j}}{Z_{n-k}} - E\left[\log\frac{Z_{n-k+j}}{Z_{n-k}}\middle|Z_{n-k}\right]\right).$$

We now prove that $S_{\hat{k}_2}/\sqrt{\hat{k}_2} \to_D N(0, \theta^{-2})$. By the random central limit theorem for martingales [Rao (1987), Proposition 1.7.17] and by Theorem 3, we only have to show that

$$\lim_{\varepsilon \to \pm 0}\limsup_{n\to\infty}\frac{E\left[S^2_{k_2}\right] - E\left[S^2_{[k_2(1\pm\varepsilon)]}\right]}{k_2} = 0.$$

Since $\{Z_{n-k+1}, \ldots, Z_n\}$ can be assumed i.i.d. given Z_{n-k},

$$\begin{aligned} E\left[S_{k_2}^2\right] &= \sum_{j=1}^{k_2} E\left[\left(\log\frac{Z_{n-k_2+j}}{Z_{n-k_2}} - E\left[\log\frac{Z_{n-k_2+j}}{Z_{n-k_2}}\middle|Z_{n-k_2}\right]\right)^2\right] \\ &= k_2 E\left[\left(\log\frac{Z_{n-k_2+1}}{Z_{n-k_2}} - E\left[\log\frac{Z_{n-k_2+1}}{Z_{n-k_2}}\middle|Z_{n-k_2}\right]\right)^2\right] \\ &= k_2 v_n. \end{aligned}$$

Similarly $E[S^2_{[k_2(1\pm\varepsilon)]}] = [k_2(1\pm\varepsilon)]v_n$. Hence

$$\frac{E\left[S_{k_2}^2\right] - E\left[S^2_{[k_2(1\pm\varepsilon)]}\right]}{k_2} = \frac{v_n\left(k_2 - [k_2(1\pm\varepsilon)]\right)}{k_2} \sim \pm v_n\varepsilon,$$

when n is large. Therefore it suffices to show that v_n is asymptotically bounded:

$$\begin{aligned} v_n &= E\left[E\left[\left(\log\frac{Z_{n-k_2+1}}{Z_{n-k_2}} - E\left[\log\frac{Z_{n-k_2+1}}{Z_{n-k_2}}\middle|Z_{n-k_2}\right]\right)^2\middle|Z_{n-k_2}\right]\right] \\ &= \theta^{-2} + 2D_1(\theta+\beta)^{-1}\left((\theta+\beta)^{-1} - \theta^{-1}\right)\int_{x_0}^{\infty} z_{n-k_2}^{-\beta}\,dH_\theta \\ &\qquad + \cdots + \int_{x_0}^{\infty} o\left(z_{n-k_2}^{-l\beta}\right)dH_\theta \\ &\le \theta^{-2} + 2D_1(\theta+\beta)^{-1}\left((\theta+\beta)^{-1} + \theta^{-1}\right)x_0^{-\beta} + \cdots + o(1), \end{aligned}$$

where $x_0 > 0$ is such that $F_\theta(x_0) = 0$ and H_θ is the distribution function of Z_{n-k_2}.

We now prove that $\sqrt{k_2}\,o(Z_{n-\hat{k}_2}^{-l\beta}) \to_P 0$ as $n \to \infty$ or $P(\sqrt{k_2}\,Z_{n-\hat{k}_2}^{-l\beta} > M) \to 0$ as $n \to \infty$ and $M \to \infty$. The above probability is

$$\begin{aligned} P\left(\sqrt{k_2}\,Z_{n-\hat{k}_2}^{-l\beta} > M\right) &= P\left(\sqrt{k_2}\,Z_{n-\hat{k}_2}^{-l\beta} > M, \left|\frac{\hat{k}_2}{k_2} - 1\right| \le \varepsilon\right) \\ &\qquad + P\left(\sqrt{k_2}\,Z_{n-\hat{k}_2}^{-l\beta} > M, \left|\frac{\hat{k}_2}{k_2} - 1\right| > \varepsilon\right) \\ &\le P\left(\sqrt{k_2}\,Z_{n-\hat{k}_2}^{-l\beta} > M, k_2(1-\varepsilon) \le \hat{k}_2 \le k_2(1+\varepsilon)\right) \\ &\qquad + P\left(\left|\frac{\hat{k}_2}{k_2} - 1\right| > \varepsilon\right) \end{aligned}$$

$$\le P\Big(\sqrt{k_2} \sup_{k_2(1-\varepsilon)\le k'\le k_2(1+\varepsilon)} Z_{n-k'}^{-l\beta} > M\Big) + P\left(\left|\frac{\hat{k}_2}{k_2} - 1\right| > \varepsilon\right)$$

$$= P\Big(\sqrt{k_2}\, Z_{n-k_2(1+\varepsilon)}^{-l\beta} > M\Big) + P\left(\left|\frac{\hat{k}_2}{k_2} - 1\right| > \varepsilon\right).$$

Since $Z_{n-k}/F_\theta^{-1}(1-k/n) \to_{\mathrm{P}} 1$, it suffices to show that

$$\sqrt{k_2}\left(F_\theta^{-1}\left(1 - \frac{k_2(1+\varepsilon)}{n}\right)\right)^{-l\beta} \to C,$$

where $C < \infty$ is a constant. By the definition of F_θ we have

$$\sqrt{k_2}\left(F_\theta^{-1}\left(1 - \frac{k_2(1+\varepsilon)}{n}\right)\right)^{-l\beta} \sim \sqrt{k_2}\left(\frac{k_2(1+\varepsilon)}{n}\right)^{l\beta/\theta},$$

as $n \to \infty$ [see Haeusler and Teugels (1985)]. Since $\sqrt{k_2} = n^{2\beta/(4\beta+\theta)}$, we have

$$\sqrt{k_2}\left(\frac{k_2(1+\varepsilon)}{n}\right)^{l\beta/\theta} = n^{2\beta/(4\beta+\theta)}(1+\varepsilon)^{l\beta/\theta} n^{-l\beta/(4\beta+\theta)} \to C,$$

as $n \to \infty$, where $C = (1+\varepsilon)^{l\beta/\theta}$, if $l = 2$, and $C = 0$, if $l > 2$. Since the denominator of II goes to θ^{-2} in probability, the term II converges to $N(0, \theta^2)$ in distribution.

Combining the above results, we have (11) for $i = 1$. The proof for i under the assumption that the theorem is true for $i-1$ is similar and omitted. □

In fact we may let $\hat{k}_1$ be the same as what Hall and Welsh (1985) proposed and let $\hat{k}_j = n^{2j\beta/(2j\beta+\hat{\theta}_{k_{j-1}})}$, $j = 2, 3, \dots, l$. It is easy to see that $\hat{k}_j/k_j \to_P 1$ since

$$\log\left(\frac{\hat{k}_j}{k_j}\right) = \frac{2j\beta\left(\hat{\theta}_{\hat{k}_{j-1}} - \theta\right)}{(2j\beta+\theta)\left(2j\beta+\hat{\theta}_{\hat{k}_{j-1}}\right)}\log n \to_P 0,$$

as $n \to \infty$. This implies that $\hat{k}_j/k_j \to_P 1$.

Now let us assume that $F_\theta(x)$ is regularly varying at ∞ with the following form:

$$(12)\quad 1 - F_\theta(x) = Cx^{-\theta}\left(1 + D_1x^{-\theta} + D_2x^{-2\theta} + \cdots + D_{l-1}x^{-l\theta} + o(x^{-l\theta})\right)$$

as $x \to \infty$, where $\theta > 0$, $C > 0$ and the D_j's may be known and differentiable functions of θ. We have the following theorem, which is similar to Theorem 3.

THEOREM 5. *Suppose* $\{Z_{n-k+j},\ j = 0,1,2,\dots,k\}$ *are the order statistics corresponding to i.i.d. random variables from* (12) *which are larger than* Z_{n-k}. *Let* $k_i = \lambda_i n^{2i/(2i+1)}$, $i = 1,2,\dots,l$, *and let* $\hat{\theta}_{k_1}$ *be Hill's estimator of* θ *using the* k_1 *largest order statistics. Let*

$$\hat{\theta}_{k_{i+1}} = \hat{\theta}_{k_i} - \frac{\Delta^*_{k_{i+1}}\left(\hat{\theta}_{k_i}\right)}{\partial\Delta^*_{k_{i+1}}\left(\hat{\theta}_{k_i}\right)/\partial\theta}, \qquad i = 1,2,\dots,l-1,$$

where

$$\Delta^*_k(\theta) = \frac{1}{k}\sum_{j=1}^{k}\left(\log\frac{Z_{n-k+j}}{Z_{n-k}} - \sum_{q=0}^{l}\sum_{m=0}^{q} p_{q,m}(D_1,\dots,D_l)(m+1)^{-1}\theta^{-1}Z_{n-k}^{-q\theta}\right),$$

where $p_{q,m}(D_1,\dots,D_l)$*'s are polynomials of* $D_1, D_2,\dots,D_l$. *Then*

$$\sqrt{k_{i+1}}\left(\left(\hat{\theta}_{k_{i+1}} - \theta\right) - \Delta_{k_{i+1}}(\theta)\right) \to_P 0$$

as $n \to \infty$ *for* $i = 1,2,\dots,(l-1)$, *where* $\Delta_{k_{i+1}}(\theta)$ *is the score function as defined before.*

PROOF. Similar to the proof of Theorem 3 with only a few obvious modifications. □

We conclude the paper by comparing two estimators of θ when $L_\theta(x)$ has the following form: $L_\theta(x) = C(1 + Dx^{-\beta} + o(x^{-\beta}))$, where $C > 0$, $\beta > 0$ and D is a known constant. Hall (1982) showed that when $k = n^{2\beta/(2\beta+\theta)}$ Hill's estimator is asymptotically biased and asymptotically normal, that is, $\sqrt{k}\,(\theta^*_k - \theta) \to_D N(DC^{-\beta/\theta}\theta\beta(\theta+\beta)^{-1}, \theta^2)$. The one-step iterative estimator obtained as in Section 3 for $k_2 = k_1 = n^{2\beta/(2\beta+\theta)}$ is, however, asymptotically unbiased, that is, $\sqrt{k}\,(\hat{\theta}_k - \theta) \to_D N(0, \theta^2)$. Applying the results about conditional MLE, it is easy to see that this iterative estimator has the same asymptotic behavior as a conditional MLE when $k = n^{2\beta/(2\beta+\theta)}$. The same is true for Hill's estimator only when $k = o(n^{2\beta/(2\beta+\theta)})$. It seems that the price we paid for lack of knowledge about the functional form of $L_\theta(x)$ is a slower convergence rate.

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MERCK RESEARCH LABORATORIES
P.O. BOX 2000
RAHWAY, NEW JERSEY 07065