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LINEAR RANK STATISTICS, FINITE SAMPLING, PERMUTATION TESTS AND WINSORIZING[1]

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Asymptotic normality and a representation of all possible subsequential limiting distributions of a simple linear rank statistic are obtained. This is then applied to finite sampling and permutation tests for slope coefficients. The effects of Winsorizing in these situations are considered carefully. Of particular interest regarding slope coefficients is that either using normal score regression constants or Winsorizing slowly increasing numbers of the population values will guarantee asymptotic normality.

1. Linear rank statistics. Consider numbers $a_{N1}, \dots, a_{NN}$ called *scores* and numbers $c_{N1}, \dots, c_{NN}$ called *regression constants*. Although the problem is mathematically symmetric in a_{Ni} and c_{Ni}, our choices below are guided by the fact that nature may well choose the a_{Ni}'s, while the experimenter chooses the c_{Ni}'s. We let

$$(1.1)\qquad \begin{aligned} c_{N\cdot} &\equiv \frac{1}{N}\sum_{i=1}^{N} c_{Ni},\ \sigma_{c,N}^2 \equiv \frac{1}{N}\sum_{i=1}^{N}(c_{Ni}-c_{N\cdot})^2 \quad \text{and} \\ \overline{c_N^4} &\equiv \frac{1}{N}\sum_{i=1}^{N}\frac{(c_{Ni}-c_{N\cdot})^4}{\sigma_{c,N}^4}. \end{aligned}$$

Let $(R_{N1}, \dots, R_{NN})$ denote a random permutation of $(1, \dots, N)$. We will represent these as the *ranks* of a random sample of independent Uniform(0, 1) r.v.'s $\xi_{N1}, \dots, \xi_{NN}$. (These ξ_{Ni}'s are an *artificial* added ingredient to the *statement* of the problem, but they are the *key* to the *proofs* of our theorems.) We let $(D_{N1}, \dots, D_{NN})$ denote the inverse permutation, or the *antiranks*. Thus $R_{ND_{Ni}} = i$, $\xi_{Ni} = \xi_{N:R_{Ni}}$ and $\xi_{N:i} = \xi_{ND_{Ni}}$. The class of *simple linear rank statistics* is of the form

$$(1.2)\qquad \begin{aligned} T_N &\equiv \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{c_{Ni}-c_{N\cdot}}{\sigma_{c,N}}\,\frac{a_{NR_{Ni}}-a_{N\cdot}}{\sigma_{a,N}} \\ &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{a_{Ni}-a_{N\cdot}}{\sigma_{a,N}}\,\frac{c_{ND_{Ni}}-c_{N\cdot}}{\sigma_{c,N}}. \end{aligned}$$

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Such statistics were studied extensively in Hájek and Šidak (1967). Whereas they were concerned with determining when T_N was asymptotically $N(0,1)$, the main concern here is to identify all possible limiting distributions. Recall that the statistician typically gets to specify the c_{Ni}'s. The choice of normal scores when $c_{Ni} \equiv \Phi^{-1}(i/(N+1))$ is seen in Corollary 1.2 (the corollary to Theorem 1.2) to *guarantee* asymptotic normality for *any* a_{Ni}'s having $a_{N1} < a_{NN}$. For very general c_{Ni}'s, Winsorizing a slowly growing number of a_{Ni}'s is seen in Corollary 1.1 (the corollary to Theorem 1.1) to *guarantee* asymptotic normality for virtually *any* a_{Ni}'s. These last two conclusions are highly practical.

It is elementary that

$$ET_N = 0 \quad \text{and} \quad \operatorname{Var}[T_N] = \frac{N}{(N-1)}. \tag{1.3}$$

To ease notational complication somewhat, we assume

$$c_{N\cdot} = 0 \quad \text{and} \quad \sigma_{c,N}^2 = 1, \quad \text{and then} \quad \overline{c_N^4} = \sum_{i=1}^{N} \frac{c_{Ni}^4}{N}. \tag{1.4}$$

We now consider another representation of T_N. We define the *finite sampling process* $\mathbb{R}_N$ on $[0,1]$ by

$$\begin{aligned} \mathbb{R}_N(t) &\equiv \frac{1}{\sqrt{N}} \sum_{i=1}^{[(N+1)t]} \frac{c_{ND_{Ni}} - c_{N\cdot}}{\sigma_{c,N}} \\ &= \frac{1}{\sqrt{N}} \sum_{i=1}^{[(N+1)t]} c_{ND_{Ni}} \quad \text{for} \quad 0 \le t \le 1 \end{aligned} \tag{1.5}$$

[with $\mathbb{R}_N(t) = 0$ for $0 \le t < 1/(N+1)$ and $N/(N+1) \le t \le 1$]. We now relabel for convenience so that

$$a_{N1} \le \cdots \le a_{NN}, \tag{1.6}$$

and we define an $\nearrow$ (i.e., nondecreasing), left-continuous function h_N on $[0,1]$ by

$$h_N(t) = a_{Ni} \quad \text{for} \quad \frac{i-1}{N} < t \le \frac{i}{N} \text{ and } 1 \le i \le N, \tag{1.7}$$

with $h_N(0) \equiv a_{N1}$. Note that $a_{N\cdot} = Eh_N(\xi)$ and $\sigma_{a,N}^2 = \operatorname{Var}[h_N(\xi)]$ for a generic uniform(0, 1) r.v. ξ. Also

$$T_N = \int_0^1 \frac{h_N \, d\mathbb{R}_N}{\sigma_{a,N}} = -\int_0^1 \frac{\mathbb{R}_N \, dh_N}{\sigma_{a,N}}. \tag{1.8}$$

Since $\mathbb{R}_N$ converges to Brownian bridge $\mathbb{W}$, a likely "limit" for T_N is

$$Z_N \equiv -\int_0^1 \frac{\mathbb{W} \, dh_N}{\sigma_{a,N}} \cong N(0,1). \tag{1.9}$$

We write $T_N =_a Z_N$ whenever $T_N - Z_N \to_p 0$ is true.

We will now be more specific about the convergence of $\mathbb{R}_N$ to $\mathbb{W}$. Let I denote the identity function, and let $\|f\|_a^b \equiv \sup\{|f(t)|: a \le t \le b\}$. It is shown in Shorack [(1991a), (2.54)] that the row independent uniform$(0,1)$ r.v.'s $\xi_{N1}, \dots, \xi_{NN}$, and the Brownian bridge $\mathbb{W}$ can be constructed on a common probability space in such a way that, for any $0 \le \nu < \frac{1}{4}$,

$$\Delta_N^{(\nu)} \equiv \left\| \frac{N^\nu(\mathbb{R}_N - \mathbb{W})}{[I(1-I)]^{1/2-\nu}} \right\|_{1/(N+2)}^{1-1/(N+2)} = O_p(1) \tag{1.10}$$

whenever $\limsup \overline{c_N^4} < \infty$, since for all $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that, for all $\lambda > 0$,

$$P\big(\Delta_N^{(\nu)} \ge \lambda\big) \le 2^{-1}\varepsilon + \frac{M_\varepsilon \overline{c_N^4}}{\lambda^2} \quad \text{for all } N. \tag{1.11}$$

[Conclusion (1.10) for values ν near 0 undoubtedly holds more generally than when $\limsup \overline{c_N^4} < \infty$. For this reason, the statements and proofs of all results are made to depend only on (1.10), and not on $\limsup \overline{c_N^4} < \infty$. It would seem that the c_{Ni}'s will at least have to be uan (uniformly asymptotically negligible) for (1.10) to hold and that a $(2+\delta)$-moment might well suffice.]

THEOREM 1.1. *Suppose* (1.10) *holds. Given* $\varepsilon > 0$, *there exists* $\delta_{\varepsilon, M} > 0$ *such that*

$$P\big(|T_N - Z_N| \ge \varepsilon\big) \le \varepsilon \quad \textit{whenever} \max_{1\le i\le N} \frac{|a_{Ni} - a_{N\cdot}|}{\sqrt{N}\,\sigma_{a,N}} < \delta_{\varepsilon, M}. \tag{1.12}$$

The approximation in (1.12) *is uniform over all* a_{Ni}*'s satisfying the requirement.*

We say that the a_{Ni}'s satisfy the *uan condition* if

$$\max_{1\le i\le N} \frac{|a_{Ni} - a_{N\cdot}|}{\sqrt{N}\,\sigma_{a,N}} = \frac{|a_{N1} - a_{N\cdot}| \vee |a_{NN} - a_{N\cdot}|}{\sqrt{N}\,\sigma_{a,N}} \to 0 \quad \text{as } N \to \infty. \tag{1.13}$$

We will be particularly interested in *Winsorizing the population*, as this step by itself can *guarantee* that T_N is asymptotically normal. Let $1 \le k_N \le N + 1 - k_N' \le N$, and consider the (k_N, k_N')-Winsorized population

$$a_{N,k_N}, \dots; a_{N,k_N}, a_{N,k_N+1}, \dots, a_{N,N+1-(k_N'+1)}, a_{N,N+1-k_N'}; \dots, a_{N,N+1-k_N'}. \tag{1.14}$$

Define $\tilde{h}_N$ as in (1.7), but for the population of (1.14); then

$$\tilde{a}_N \equiv \int_0^1 \tilde{h}_N(t)\,dt = E\tilde{h}_N(\xi) \quad \text{and} \quad \tilde{\sigma}_{a,N}^2 \equiv \int_0^1 \big[\tilde{h}_N(t) - \tilde{a}_N\big]^2 dt = \operatorname{Var}\big[\tilde{h}_N(\xi)\big]. \tag{1.15}$$

Let $\tilde{T}_N$ and $\tilde{Z}_N$ denote the r.v.'s of (1.8) and (1.9), but with $\tilde{h}_N$ and $\tilde{\sigma}_{a,N}$ in place of h_N, and $\sigma_{a,N}$.

Let us observe that the uan condition (1.13) applied to the Winsorized population displayed in (1.14) becomes the condition that

$$(1.16)\qquad \frac{|a_{Nk_N} - \tilde{a}_N| \vee |a_{N,N+1-k'_N} - \tilde{a}_N|}{\sqrt{N}\,\tilde{\sigma}_{a,N}} \to 0 \quad \text{as} \quad N \to \infty.$$

Now note that

$$(1.17)\qquad \begin{aligned} &\frac{|a_{Nk_N} - \tilde{a}_N| \vee |a_{N,N+1-k'_N} - \tilde{a}_N|}{\sqrt{N}\,\tilde{\sigma}_{a,N}} \\ &\le \frac{|a_{Nk_N} - \tilde{a}_N| \vee |a_{N,N+1-k'_N} - \tilde{a}_N|}{\left\{k_N\left(a_{Nk_N} - \tilde{a}_N\right)^2 \text{ or } k'_N\left(a_{N,N+1-k'_N} - \tilde{a}_N\right)^2\right\}^{1/2}} \\ &\le \frac{1}{\sqrt{k_N \wedge k'_N}} \quad \text{provided } a_{Nk_N} < a_{N,N+1-k'_N}. \end{aligned}$$

COROLLARY 1.1. *Suppose* (1.10) *holds. Then*

$$(1.18)\qquad \tilde{T}_N =_a \tilde{Z}_N = -\int_0^1 \frac{\mathbb{W}\,d\tilde{h}_N}{\tilde{\sigma}_{a,N}} \cong N(0,1)$$

if condition (1.16) *holds, and condition* (1.16) *holds if*

$$(1.19)\qquad k_N \wedge k'_N \to \infty \quad \text{and} \quad a_{Nk_N} < a_{N,N+1-k'_N} \text{ for all sufficiently large } N.$$

We summarize this result by saying that "normality is guaranteed by Winsorizing a slowly increasing number, provided you do not completely collapse the sample."

We say that T_N is *stochastically compact* if and only if every subsequence N' contains a further subsequence N'' for which $T_{N''}$ converges in distribution to a proper r.v. Then from (1.3) we see that T_N is necessarily stochastically compact. Consider a subsequence N'' on which $T_{N''} \to_d$ (some r.v. T). Since variances are uniformly bounded, the means converge; thus $ET = 0$. Moreover, (1.3) guarantees that $\operatorname{Var}[T] \le 1$ (use Fatou). We now seek to describe all possible subsequential limits of T_N.

Note that the r.v. $\eta_{N1} \equiv c_{ND_{N1}}$, representing the result of the first draw from the urn, has $E\eta_{N1} = 0$ and $\operatorname{Var}[\eta_{N1}] = 1$. Thus η_{N1} is necessarily stochastically compact, and thus any possible subsequential limit η_1 must also satisfy $E\eta_1 = 0$ and $\operatorname{Var}[\eta_1] \le 1$. To ease the notational burden (but causing no loss of generality, other than replacing "limit" by "possible subsequential limit"), *we will state our theorem as though* $\eta_{N1} \to_d$ (*some r.v.* η_1). Some examples should help make things clearer. If, before normalization, the c_{Ni}'s consist of M_N copies of the symbol 1 and $N - M_N$ copies of the symbol 0, where $M_N/N \to p \in (0,1)$, then η_1 is (Bernoulli$(p) - p)/\sqrt{pq}$. If, before normalization, $c_{Ni} = i/N$ for $1 \le i \le N$, then η_1 is uniform$(-\sqrt{3}, \sqrt{3})$. If, before normalization, $c_{Ni} = \Phi^{-1}(i/(N+1))$ for $1 \le i \le N$, then η_1 is $N(0,1)$. If $\sqrt{N}/2$ of the c_{Ni}'s equal each of $\pm N^{1/4}$ while all the other c_{Ni}'s equal 0, then η_1 is point mass at 0; note that these are normed c_{Ni}'s that are uan. Combining the last two examples correctly can give a $N(0, d_c^2)$ limit for any $0 \le d_c^2 < 1$ [note (1.21)].

THEOREM 1.2. *Suppose* (1.10) *holds, at least one* $a_{Ni} - a_{N\cdot} \neq 0$, *and* $c_{ND_{N1}} \to_d \eta_1$.

(i) *Then* $T_N \to_d N(0,1)$ *if and only if either*

$$\max_{1 \le i \le N} \frac{|a_{Ni} - a_{N\cdot}|}{\sqrt{N}\,\sigma_{a,N}} \to 0 \quad \text{or} \quad \eta_1 \cong N(0,1). \tag{1.20}$$

(ii) *Now* $T_N \to_d N(0, d^2)$, *where* $0 \le d < 1$ *is possible. It can happen* [*note part* (iv)] *only if*

$$\text{the } a_{Ni}\text{'s are not uan} \quad \text{and} \quad \eta_1 \cong N\big(0, d_c^2\big) \quad \text{with } 0 \le d_c^2 < 1. \tag{1.21}$$

(iii) *Any subsequential limiting r.v. of the stochastically compact* T_N *must be of the form*

$$T_0 + \tau Z + T_1, \tag{1.22}$$

where T_0, T_1 *and* Z *are independent*, $Z \cong N(0,1)$, $0 \le \tau \le 1$, *and*

$$T_0 = \sum_{i=0}^{i_0} \Phi_0(i)\eta_i \quad \text{and} \quad T_1 \equiv \sum_{j=0}^{j_0} \Phi_1(j)\eta'_j, \tag{1.23}$$

with $\eta_1, \eta_2, \dots$ *and* $\eta'_1, \eta'_2, \dots$ *all iid as* η_1, *and with* i_0 *and* j_0 *each taking a value from* $0, 1, 2, 3, \dots, \infty$. *The numbers* $\Phi_0(i)$ *and* $\Phi_1(j)$ *satisfy the following*:

$$\begin{aligned} &-1 \le \Phi_0(1) \le \cdots \le \Phi_0(i) < 0 \quad \text{for all } i < i_0 + 1 \ (\text{in case } i_0 > 0), \\ &\quad 0 < \Phi_1(j) \le \cdots \le \Phi_1(1) \le 1 \quad \text{for all } j < j_0 + 1 \ (\text{in case } j_0 > 0) \end{aligned} \tag{1.24}$$

[*where* $\Phi_0(0) \equiv 0 \equiv \Phi_1(0)$ *and* $\eta_0 \equiv 0 \equiv \eta'_0$, *for clarity in the summations*]; *and*

$$\sum_{i=0}^{i_0} \Phi_0^2(i) + \tau^2 + \sum_{j=0}^{j_0} \Phi_1^2(j) = 1. \tag{1.25}$$

Now $i < i_0 + 1$ *in* (1.24) *means* $i \le i_0$ *if* i_0 *is finite, and it means* $i < \infty$ *if* i_0 *is infinite.*

(iv) *Suppose* $i_0 \vee j_0 \ge 1$. *Then the r.v. of* (1.22) *is* $N(0,1)$ *if and only if* $\eta_1 \cong N(0,1)$; *and it is* $N(0, d^2)$ *with* $0 \le d < 1$ *if and only if* $\eta_1 \cong N(0, d_c^2)$ *with* $0 \le d_c < 1$, *in which case*

$$d^2 = \tau^2 + d_c^2\left(\sum_{i=0}^{i_0} \Phi_0^2(i) + \sum_{j=0}^{j_0} \Phi_1^2(j)\right) \tag{1.26}$$

while (1.25) *still holds.*

(v) *Omitting the hypothesis* $c_{ND_{N1}} \to_d \eta_1$ *does not change conclusion* (iii), *in that every subsequence N' contains a further subsequence N'' on which the hypothesis does hold.*

(vi) *Let η_1 denote any r.v. compatible with $c_{ND_{N1}} \to_d \eta_1$ and* (1.10). *Then any r.v. of the form* (1.22) *and* (1.23) *subject to* (1.24) *and* (1.25) *is an achievable limiting r.v.*

COROLLARY 1.2. *Suppose $c_{Ni} \equiv \Phi^{-1}(i/(N+1))$ for the $N(0,1)$ d.f. Φ. Of $a_{N1}, \dots, a_{NN}$ we require only $a_{N1} < a_{N\cdot} < a_{NN}$. Then the T_N of* (1.2) *satisfies $T_N \to_d N(0,1)$.*

Note the following caveat: Currently, hypothesis (1.10) requires $\limsup \overline{c_N^4} < \infty$ in order to be known to be true. However, $\limsup \overline{c_N^4} < \infty$ implies $d_c^2 = 1$; that is, bounded fourth moments plus convergence in distribution imply convergence of second moments. However, assuming (1.10) is shown to hold in situations that allow $c_{ND_{N1}} \to_d \eta_1$ with $0 \le d_c^2 \equiv \mathrm{Var}[\eta_1] < 1$, then these η_1 can also appear as achievable limits of type (1.26) (with *no change* in the proof given below). [In the proof given below, (1.10) is used *only* in line (3.32).] Bear in mind that the statistician is typically free to specify the C_{Ni}'s, while nature supplies the a_{Ni}'s. Thus the implications of what is established here are considerable.

It is appropriate to consult Hájek and Šidák [(1967), Exercises 2 and 8, page 193] in relation to the result in Theorem 1.2. Theorem 1.2 is a more reasonable formulation since the statistician specifies the c_{Ni}'s. The work of Csörgő, Haeussler and Mason (1988) and Mason and Shorack (1992) inspired its proof. Work related to the present paper is found in Deheuvels, Mason and Shorack (1993). The referee points out Zolaterev (1967) and Pardzhanadze and Khmaladze (1986), especially in regard to non-uan situations.

2. Other examples. The scope of these results is actually much broader, because, besides linear rank statistics, the theorems of Section 1 apply to the following situations.

EXAMPLE 2.1 (Finite sampling). Let $a_{N1} \le \cdots \le a_{NN}$ denote a finite population. Let $n \equiv n_N$, and suppose $X_{N1}, \dots, X_{Nn}$ are a random sample from the finite population. Let

$$\bar{X}_n \equiv \frac{1}{n} \sum_{i=1}^{n} X_{Ni} \quad \text{and} \quad S_n^2 \equiv \frac{1}{n} \sum_{i=1}^{n} \left(X_{Ni} - \bar{X}_n \right)^2 \tag{2.1}$$

denote the sample mean and variance. It is elementary that

$$\left(\frac{S_n}{\sigma_{a,N}} - 1 \right) = o_p(1), \tag{2.2}$$

provided $\liminf n/N > 0$ and the a_{Ni}'s are uan.

Suppose now that the c_{Ni}'s satisfy $0 < \liminf n/N \le \limsup n/N < 1$. When this is so, Theorem 1.2 implies (since η_1 is not normal in this example) that

$$\frac{\sqrt{n}\left(\overline{X}_n - a_{N\cdot}\right)}{\sigma_{a,N}\sqrt{1-(n-1)/(N-1)}} \to_d N(0,1) \tag{2.3}$$

if and only if the a_{Ni}'s are uan. Moreover, we have from (2.2) and (2.3) that

$$\frac{\sqrt{n}\left(\overline{X}_n - a_{N\cdot}\right)}{S_n\sqrt{1-(n-1)/(N-1)}} \to_d N(0,1) \quad \text{for uan } a_{Ni}\text{'s}. \tag{2.4}$$

[To apply the earlier results, set the first n of the c_{Ni}'s equal to 1 and the rest to 0 (so that the η_1 of Theorem 1.2 is a normed Bernoulli r.v.). Then $\limsup \overline{c_N^4} < \infty$, since $0 < \liminf n/N < \limsup n/N < 1$.] More interesting is the following:

(2.5) conclusions (2.2), (2.3) and (2.4) apply to the (k_N, k'_N)-Winsorized population (even if k_N and k'_N are random, though dependent only on the order statistics),

provided only that condition (1.19) holds with probability approaching 1 as $N \to \infty$.

EXAMPLE 2.2 (Regression tests). Suppose $X_{N1}, \dots, X_{NN}$ are iid with nondegenerate d.f. F. Let $X_{N:1} \le \cdots \le X_{N:N}$ denote the order statistics, and let

$$X_{N\cdot} \equiv \frac{1}{N}\sum_{i=1}^{N} X_{Ni} \quad \text{and} \quad S_N^2 \equiv \frac{1}{N}\sum_{i=1}^{N}\left(X_{Ni} - X_{N\cdot}\right)^2 \tag{2.6}$$

denote the sample mean and variance. It is shown in Csörgő and Mason (1989) that [for D(Normal) the domain of attraction of the Normal]

$$\max_{1\le i\le N} \frac{|X_{Ni} - X_{N\cdot}|}{\sqrt{N}S_N} \begin{cases} 018030 \to_p 0, & \text{if and only if } F \in D\ (\text{Normal}), \\ \to 0, \text{ a.s.}, & \text{if and only if } F \text{ has finite variance.} \end{cases} \tag{2.7}$$

Let $c_{N1}, \dots, c_{NN}$ denote regression constants as in (1.1) that satisfy (1.10). Then form the statistic

$$\begin{aligned} T_N &\equiv \frac{1}{\sqrt{N}}\sum_{i=1}^{N} \frac{c_{Ni} - c_{N\cdot}}{\sigma_{c,N}} \frac{X_{Ni} - X_{N\cdot}}{S_N} \\ &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N} \frac{X_{N:i} - X_{N\cdot}}{S_N} \frac{c_{ND_{Ni}} - c_{N\cdot}}{\sigma_{c,N}}. \end{aligned} \tag{2.8}$$

Even if $F \in D$ (Normal), each subsequence N' contains a further subsequence N'' on which condition (2.7) holds a.s. Thus Theorem 1.1 implies that $T_N \to_d N(0,1)$ on N''. Thus

$$T_N \to_d N(0,1) \quad \text{as } N \to \infty, \text{ for all } F \in D\ (\text{Normal}), \tag{2.9}$$

for c_{Ni}'s satisfying (1.10).

We now seek to extend (at the statistician's discretion) the conclusion (2.9) beyond the case of uan a_{Ni}'s. From Corollary 1.2 we obtain

$$(2.10)\quad T_N^0 \equiv \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \Phi^{-1}\left(\frac{i}{N+1}\right) \frac{[X_{Ni} - X_{N\cdot}]}{S_N} \to_d N(0,1) \quad \text{for all } F;$$

that is, the choice $c_{Ni} = \Phi^{-1}(i/(N+1))$ [or $c_{Ni} = \Phi^{-1}((3i-1)/(3N+1))$ etc.] *always* works. [For emphasis, the *only* requirement on the $a_{Ni} \equiv X_{N:i}$ in (2.10) is that $P(X_{N:1} < X_{N:N}) \to 1$ as $N \to \infty$, and this does indeed hold for all nondegenerate d.f.'s F.]

Instead of choosing special c_{Ni}'s, the same effect follows if the statistician Winsorizes the sample, where k_N and k'_N are integer-valued r.v.'s that are dependent on the observations only through the order statistics and satisfy (1.16). Condition (1.16) necessarily holds if

$$(2.11)\qquad k_N \wedge k'_N \to_p \infty \quad \text{and} \quad P\big(X_{N:k_N} < X_{N:N+1-k'_N}\big) \to 1,$$

or if

$$(2.12)\qquad (k_n \vee k'_N)/N \to_p 0 \quad \text{and} \quad F \in D\,(\text{Normal}).$$

Then (go to subsequences to) apply Corollary 1.1, giving

$$(2.13)\qquad \tilde{T}_N \equiv \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{c_{Ni} - c_{N\cdot}}{\sigma_{c,N}} \frac{\tilde{X}_{Ni} - \tilde{X}_{N\cdot}}{\tilde{S}_N} \to_d N(0,1) \quad \text{for all } F,$$

provided the c_{Ni}'s satisfy (1.10); here $\tilde{X}_{N:1} \le \cdots \le \tilde{X}_{N:N}$ is the (k_N, k'_N)-Winsorized sample, with mean $\tilde{X}_{N\cdot}$ and variance $\tilde{S}_N^2$. Of course, if we require "$\to \infty$ a.s." and $P(\ ) = 1$, in (2.11) and (2.12), then (2.13) holds for a.e. realization of the r.v.'s. We note that if $c_{N\cdot} = 0$ (i.e., in the regression situation), then "*Winsorizing does what Winsorizing was supposed to do*."

Another application of these ideas appears in Deheuvels, Mason and Shorack (1993).

3. Proofs. Let $\sigma_N \equiv \sigma_{a,N}$. Fix $0 < \nu < \frac{1}{4}$, and define

$$(3.1)\qquad M_N[a,b] \equiv \int_{[a,b]} N^{-\nu}[t(1-t)]^{1/2-\nu}\, dh_N(t).$$

$$(3.2)\qquad \sigma_N^2[a,b] = \int_{[a,b]}\int_{[a,b]} (s \wedge t - st)\, dh_N(s)\, dh_N(t).$$

Then, akin to Shorack [(1991b), (2.32) and (2.34)] from estimates used earlier in Csörgő, Haeussler and Mason (1988) and Mason and Shorack (1992), for arbitrary quantile functions:

$$\frac{M_N[r/(N+1), 1-r'/(N+1)]}{\sigma_N[r/(N+1), 1-r'/(N+1)]} \le \frac{3}{\sqrt{\nu}(r \wedge r')^{\nu}}; \tag{3.3}$$

$$\begin{aligned} M_N\left[\frac{1}{N+1}, \frac{r}{N+1}\right] &\le \sqrt{r}\,\frac{|a_{N1}-a_{N\cdot}|}{\sqrt{N}} \quad \text{and} \\ M_N\left[1-\frac{r'}{N+1}, 1-\frac{1}{N+1}\right] &\le \sqrt{r'}\,\frac{|a_{NN}-a_{N\cdot}|}{\sqrt{N}} \end{aligned} \tag{3.4}$$

for any $r, r' \ge 1$ and any a_{Ni}'s having $a_{N1} - a_{N\cdot} < 0 < a_{NN} - a_{N\cdot}$.

Let $\varepsilon > 0$ be given. Then (1.10) allows us to choose an M_ε so large that

$$A_{N\varepsilon} \equiv \left[\Delta_N^{(\nu)} < M_\varepsilon\right] \quad \text{has} \quad P(A_{N\varepsilon}^c) < \varepsilon \quad \text{for all } N; \tag{3.5}$$

let $1_{N\varepsilon}$ denote the indicator function of $A_{N\varepsilon}$.

Proof of Theorem 1.1. Now (1.8), (1.9), (1.10), (3.1), (3.3) and (3.4) show that

$$\begin{aligned} |T_N - Z_N| &= \frac{1}{\sigma_N}\left|\int_{1/(N+1)}^{N/(N+1)}\right. \\ &\le \Delta_N^{(\nu)} \frac{M_N[1/(N+1), N/(N+1)]}{\sigma_N} \qquad (3.6) \\ &\le \Delta_N^{(\nu)}\left\{\frac{\sqrt{r}\,|a_{N1}-a_{N\cdot}|}{\sqrt{N}\,\sigma_N} + 3\nu^{-1/2}(r\wedge r')^{-\nu} + \frac{\sqrt{r'}\,|a_{NN}-a_{N\cdot}|}{\sqrt{N}\,\sigma_N}\right\} \\ &\le \Delta_N^{(\nu)}\left\{3\nu^{-1/2}(r\wedge r')^{-\nu} + \left(\sqrt{r}+\sqrt{r'}\right)\max_{1\le i\le N}\frac{|a_{Ni}-a_{N\cdot}|}{\sqrt{N}\,\sigma_N}\right\}. \qquad (3.7) \end{aligned}$$

Thus, if $r = r' = r_\varepsilon$ is so large that $M_\varepsilon 3\nu^{-1/2} r_\varepsilon^{-\nu} < \varepsilon/3$, then

$$1_{N\varepsilon}|T_N - Z_N| \le \frac{\varepsilon}{3} + M_\varepsilon 2\sqrt{r_\varepsilon}\max_{1\le i\le N}\frac{|a_{Ni}-a_{N\cdot}|}{\sqrt{N}\,\sigma_N} \le \varepsilon \quad \text{for } N \ge N_\varepsilon, \tag{3.8}$$

provided N_ε is chosen so large that

$$\frac{\max_{1\le i\le N}|a_{Ni}-a_{N\cdot}|}{\sqrt{N}\,\sigma_N} \le \delta_{\varepsilon,M} \equiv \frac{\varepsilon}{3M_\varepsilon\sqrt{r_\varepsilon}} \quad \text{for all } N \ge N_\varepsilon. \tag{3.9}$$

□

Proof of Theorem 1.2. In the left tail of T_N we have

$$T_N[1,r] \equiv \sum_{i=1}^{r}\left(\frac{a_{Ni}-a_{N\cdot}}{\sqrt{N}\,\sigma_N}\right)c_{ND_{Ni}} \equiv \sum_{i=1}^{r}\Phi_{0N}(i)c_{ND_{Ni}}. \tag{3.10}$$

Note that

$$(3.11)\qquad \sum_{i=1}^{r} \Phi_{0N}^2(i) = \frac{\sum_{i=1}^{r}(a_{Ni} - a_{N\cdot})^2}{\sum_{i=1}^{N}(a_{Ni} - a_{N\cdot})^2} \le 1 \quad \text{for all } N \text{ and all } r.$$

Thus on every subsequence N' there exists (by diagonalization) a further N'' on which

$$(3.12)\qquad \Phi_{0N}(i) \to (\text{some } \Phi_0(i)) \quad \text{for } i = 1, 2, \dots \text{ on } N''.$$

Recall that $\Phi_{0N}(1) \le \cdots \le \Phi_{0N}(N)$ with an average value of 0 and at least one strictly positive and one strictly negative value. Thus $\Phi_0(1) \le 0$ and $\Phi_0(1) \le \Phi_0(2) \le \cdots$. Let i_0 denote the supremum of the integers i such that $\Phi_0(i) < 0$, where any of the values $0, 1, 2, \dots, \infty$ is possible. To keep the notation from getting out of hand, we will act as though (3.12) holds on the whole sequence N. Note from (3.11) that

$$(3.13)\qquad \sum_{i=1}^{r} \Phi_0^2(i) \le 1 \quad \text{for all } r;$$

and we just learned that

$$(3.14)\qquad -1 \le \Phi_0(1) \le \Phi_0(2) \le \cdots \le \Phi_0(i) < 0 \quad \text{for all } 0 \le i < i_0 + 1,$$

where i_0 may equal $0, 1, 2, \dots, \infty$. It is elementary for the reader to show that if $i_0 = \infty$, then

$$(3.15)\qquad i\Phi_0^2(i) \to 0 \quad \text{as } i \to \infty.$$

We will also require below (at the very end) the fact that $d_i \equiv \Phi_0^2(i)$ satisfies

$$(3.16)\qquad \sum_{i=1}^{\sqrt{N}} \frac{\sqrt{d_i}}{\sqrt{N}} \to 0 \quad \text{as } N \to \infty.$$

Let i_ε be so large that $\sqrt{d_i} \le \varepsilon/\sqrt{i}$ for all $i \ge i_\varepsilon$. For $N \ge (\text{some } N_\varepsilon)$, (3.16) follows from

$$\sum_{1}^{\sqrt{N}} \frac{\sqrt{d_i}}{\sqrt{N}} \le \left(\sum_{1}^{i_\varepsilon} \frac{\sqrt{d_i}}{\sqrt{N}} \right) + \frac{\varepsilon}{\sqrt{N}} \sum_{i_\varepsilon}^{\sqrt{N}} i^{-1/2} \le \varepsilon \left(1 + N^{-1} \sum_{i_\varepsilon}^{\sqrt{N}} \frac{1}{\sqrt{i/N}} \right)$$

$$\le \varepsilon \left(1 + 2\int_0^1 x^{-1/2}\, dx \right) \le 5\varepsilon.$$

That η_{N1} has mean 0 and variance 1 and $\eta_{N1} \to_d \eta_1$, gives $E\eta_1 = 0$ and $\operatorname{Var}[\eta_1] \le 1$. Now the effect of 1 (or even r) draws on an urn of size N is negligible as $N \to \infty$. Thus

$$(3.17)\qquad T_N[1, r] \to_d \sum_{i=1}^{r} \Phi_0(i)\eta_i \quad \text{on } N'' \text{ as } N'' \to \infty,$$

for each fixed $r < i_0 + 1$. Applying (3.13) and Breiman [(1968), page 46] shows that T_0 is a well-defined r.v., where

$$(3.18)\qquad T_0 \equiv \sum_{i=1}^{i_0} \Phi_0(i)\eta_i.$$

Let $(1/2) \geq \varepsilon_m \searrow 0$ be a given sequence. If $i_0 < \infty$, define $\ell_m = i_0$. Suppose now that $i_0 = \infty$, and let $\ell_m \nearrow \infty$ and then let $r_m \nearrow \infty$ be so large that

$$(3.19) \qquad \sum_{i=l_m+1}^{\infty} \Phi_0^2(i) \leq \varepsilon_m^2, \quad \text{and then} \quad r_m - 1 \geq m(\ell_m + 1).$$

Then specify nondecreasing N_m so large that we have all five of

$$(3.20) \quad \frac{\ell_m^2}{N_m} < \frac{1}{m}, \qquad \frac{r_m}{N_m} < \frac{1}{m} \text{and} \quad \Phi_{0N}(r_m) < 0 \qquad \text{for all } N \geq N_m,$$

as well as the "*uniform*" bound

$$(3.21) \quad |\Phi_{0N}(i) - \Phi_0(i)| < \varepsilon_m \frac{|\Phi_0(i)|}{r_m} \qquad \text{for } 1 \leq i \leq r_m \text{ for all } N \geq N_m;$$

and, finally [applying a crude variance bound, using (3.21), and then (3.19)], for all $N \geq N_m$,

$$(3.22) \quad \begin{aligned} \operatorname{Var}\big[T_N[\ell_m + 1, r_m - 1]\big] &\leq 2\sigma_{c,N}^2 \sum_{i=\ell_m+1}^{r_m-1} \Phi_{0N}^2(i) \\ &\leq 3 \sum_{i=\ell_m+1}^{\infty} \Phi_0^2(i) \leq 3\varepsilon_m^2. \end{aligned}$$

We now specify (whether $i_0 < \infty$ or $i_0 = \infty$) that, for each $m \geq 1$ (with $N_0 \equiv 1$),

$$(3.23) \qquad \varepsilon_N = \varepsilon_m, \qquad \ell_N = \ell_m \quad \text{and} \quad r_N = r_m \text{ for all} \quad N_m \leq N < N_{m+1}.$$

Then, in the case $i_0 = \infty$ (only $\ell_N \to \infty$ fails if $i_0 < \infty$, since then $\ell_N = i_0 < \infty$),

$$(3.24) \quad \ell_N \to \infty, \qquad \frac{l_N^2}{N} \to 0, \qquad \frac{\ell_N}{r_N} \to 0, \ \frac{r_N}{N} \to 0 \quad \text{and} \quad (3.22) \text{ still holds.}$$

At the time we choose r_m above, we can also insist that r_m is large enough that

$$(3.25) \qquad \frac{M_{\varepsilon_m} 3\nu^{-1/2}}{r_{\varepsilon_m}^{\nu}} < \frac{\varepsilon_m}{3},$$

as leading to (3.8).

Let us now introduce an *associated situation* involving sampling *with replacement*. To describe it, we will slightly enlarge the probability space by introducing a triangular array of independent r.v.'s $\kappa_{N1}, \dots, \kappa_{N,[N\theta]}$ that are also independent of all other r.v.'s in this paper. We suppose that κ_{N_j} equals j with probability $(1 - (j-1)/N)$ and that it equals each of $1, \dots, j-1$ with probability $1/N$. Then for each N we define $\eta_{N_j} = c_{ND_{N\kappa_{Nj}}}$ for $1 \leq j \leq [N\theta]$. Thus $\eta_{N1}, \dots, \eta_{N[N\theta]}$ represent $[N\theta]$ independent samplings *with replacement* from an urn containing $c_{N1}, \dots, c_{NN}$. (The value $[N\theta]$ is not crucial, it is just a safe upper limit.) Note that, for ℓ_N satisfying $\ell_N^2/N \to 0$ as in (3.24),

$$P\big(\eta_{Ni} \neq c_{ND_{Ni}} \text{ for some } 1 \le i \le \ell_N\big) \le \frac{2(1+\cdots+\ell_N)}{N} \le \frac{2\ell_N^2}{N} \to 0. \tag{3.26}$$

It is trivial [in light of $\eta_{N1} \cong c_{ND_{N1}} \to_d \eta_1$ from hypotheses, and of (3.19) and the uniform approximation of (3.21)] that

$$\sum_{i=1}^{\ell_N} \Phi_{0N}(i)\eta_{Ni} \to_d T_0 = \sum_{i=1}^{i_0} \Phi_0(i)\eta_i. \tag{3.27}$$

Then (3.26) and (3.22) give immediately that

$$T_N[1,\ell_N] = \sum_{i=1}^{\ell_N} \Phi_{0N}(i) c_{ND_{Ni}} \to_d T_0 = \sum_{i=1}^{i_0} \Phi_0(i)\eta_i \quad \text{and} \quad T_N[\ell_N+1, r_N-1] \to_p 0. \tag{3.28}$$

A symmetric argument holds in the right tail for analogous j_0, ℓ_N' and r_N'. [Statement (3.28) actually attains on N'' as $N'' \to \infty$.]

We now turn to consideration of the middle. Define the following:

$$\tilde{Z}_N \equiv -\frac{1}{\tilde{\sigma}_N}\int_{[r_N/(N+1),\,1-r_N'/(N+1)]} \mathbb{W}\, dh_N \cong N(0,1). \tag{3.29}$$

$$\tilde{\sigma}_N^2 \equiv \sigma_N\left[\frac{r_N}{N+1}, 1-\frac{r_N'}{N+1}\right]. \tag{3.30}$$

Integration by parts gives

$$\begin{aligned} T_N[r_N, N+1-r_N'] &\equiv \sum_{i=r_N}^{N+1-r_N'} \Phi_{0N}(i) c_{ND_{Ni}} \\ &= \frac{1}{\sigma_N}\int_{[r_N/(N+1),\,1-r_N'/(N+1)]} [h_N - a_{N\cdot}]\, d\mathbb{R}_N \\ &= -\frac{[h_N(r_N/(N+1)) - a_{N\cdot}]\mathbb{R}_N(r_N/(N+1)-0)}{\sigma_N} \\ &\quad + \left(\frac{\tilde{\sigma}_N}{\sigma_N}\right)\left(-\frac{1}{\tilde{\sigma}_N}\int_{[r_N/(N+1),\,1-r_N'/(N+1)]} \mathbb{R}_N\, dh_N\right) \\ &\quad + \frac{[h_N(1-r_N'/(N+1)) - a_{N\cdot}]\mathbb{R}_N(1-r_N'/(N+1))}{\sigma_N} \\ &\equiv \gamma_N + \tau_N \tilde{T}_N + \gamma_N'. \end{aligned} \tag{3.31}$$

Fix a small $\nu > 0$. Then (3.3) and (3.5) show that

$$
\begin{aligned}
1_{N\varepsilon_N}|\tilde{T}_N - \tilde{Z}_N| &= 1_{N\varepsilon_N}\left| -\int_{[r_N/(N+1),\,1-r'_N/(N+1)]} \frac{(\mathbb{R}_N - \mathbb{W})\,dh_N}{\tilde{\sigma}_N}\right| \\
&\le 1_{N\varepsilon_N}\left\| \frac{N^{\nu}(\mathbb{R}_N - \mathbb{W})}{[t(1-t)]^{1/2-\nu}}\right\|_{1/(N+2)}^{1-1/(N+2)} \\
&\quad \times \int_{[r_N/(N+1),\,1-r'_N/(N+1)]} N^{-\nu}[t(1-t)]^{1/2-\nu}\frac{dh_N(t)}{\tilde{\sigma}_N} \\
&= 1_{N\varepsilon_N}\Delta_N^{(\nu)}\frac{M_N[r_N/(N+1),1-r'_N/(N+1)]}{\tilde{\sigma}_N} \\
&\le 1_{N\varepsilon_N}\Delta_N^{(\nu)}\frac{3\nu^{-1/2}}{(r_N \wedge r'_N)^{\nu}} \le M_{\varepsilon_N}\frac{3\nu^{-1/2}}{(r_N \wedge r'_N)^{\nu}} \qquad (3.32) \\
&\le \frac{\varepsilon_N}{3} \quad \text{for all } N, \qquad (3.33)
\end{aligned}
$$

using (3.25) (whether $i_0 < \infty$ or $i_0 = \infty$). Since $P(A^c_{N\varepsilon_N}) \to 0$ from (3.5), we have

$$\tilde{T}_N =_a \tilde{Z}_N. \qquad (3.34)$$

Also $0 \le \tau_N = \tilde{\sigma}_N/\sigma_N \le 1$, so that (by going to a further subsequence if need be) we may suppose

$$\tau_N \to (\text{some } \tau) \in [0,1] \text{ on } N'' \quad \text{as } N'' \to \infty. \qquad (3.35)$$

We now turn to consideration of γ_N. Now (3.31) and finite sampling [see, e.g., Shorack and Wellner (1986), (13) on page 135] and (3.21) show that, for $N_m \le N < N_{m+1}$ [as in (3.23)],

$$
\begin{aligned}
\operatorname{Var}[\gamma_N] &= \Phi_{0N}^2(r_N)\operatorname{Var}\left[\sqrt{N}\,\mathbb{R}_N\left(\frac{r_N}{N+1}\right)\right] \le r_N\Phi_{0N}^2(r_N) \\
&= r_m\Phi_{0N}^2(r_m) \le r_m\Phi_0^2(r_m) + o(1) = r_N\Phi_0^2(r_N) + o(1) \qquad (3.36)\\
&\to 0;
\end{aligned}
$$

by (3.15) if $i_0 = \infty$ and $j_0 = \infty$, by proper choice of N_m in conjunction with

$$\max_{i_0+1 \le i \le N+1-(j_0+1)} \frac{|a_{Ni} - a_{N\cdot}|}{\sqrt{N}\,\sigma_{a,N}} \to 0, \qquad (3.37)$$

if $i_0 < \infty$ and $j_0 < \infty$, and by combining these reasons if one of i_0 and j_0 is finite and the other infinite. Thus conclusion (3.36) (and a symmetric argument in the right tail) give

$$\gamma_N \to_p 0 \quad (\text{and } \gamma'_N \to_p 0). \qquad (3.38)$$

Combining (3.28), (3.29), (3.31), (3.34), (3.35) and (3.38), we have

$$T_N = T_N[1, \ell_N] + \tau_N \tilde{Z}_N + T'_N[1, \ell'_N] + o_p(1), \tag{3.39}$$

where [recall (3.28) for T_0 and a symmetric argument for T_1]

$$\begin{aligned} &T_N[1, \ell_N] \to_d T_0, \qquad T'_N[1, \ell'_N] \to_d T_1, \\ &\qquad \tau_N \to \tau \quad \text{and} \quad \tilde{Z}_N \cong N(0,1) \text{ on } N''. \end{aligned} \tag{3.40}$$

Thus, (1.22) holds, provided

$$\begin{aligned} &T_N[1, \ell_N], T_N[r_N, N+1-r'_N] \text{ and} \\ &\qquad T'_N[1, \ell'_N] \text{ are asymptotically independent.} \end{aligned} \tag{3.41}$$

However, this follows as the r.v.'s η_{Ni} of (3.26) and (3.27) are asymptotically equivalent to the $c_{ND_{Ni}}$'s of (3.26), which are asymptotically independent of the middle term of (3.41) as in Rossberg (1967).

Recall from Cramér's theorem that the sum (1.22) of independent r.v.'s is normal if and only if all the components are normal. Thus (i)–(v) are now clear.

We now turn to (vi). Let η_1 denote any r.v. with $E\eta_1 = 0$ and $0 \le \operatorname{Var}[\eta_1] \le 1$ for which (1.10) and $c_{ND_{N1}} \to_d \eta_1$ both hold. Consider any $\Phi_0(i)$'s, $\Phi_1(i)$'s and $0 \le \tau \le 1$ that satisfy (1.25) and (1.26). We must show that the r.v. of (1.22) is an achievable limit. Just define

$$a_{Ni} \equiv \begin{cases} \sqrt{N}\,\Phi_0(i) - \tau, & \text{for } 1 \le i \le \left(i_0 \wedge \sqrt{N}\right), \\ -\tau, & \text{for } \left(i_0 \wedge \sqrt{N}\right) < i < (N+1)/2, \\ 0, & \text{for } i = (N+1)/2, \text{ if } N \text{ is odd}, \\ \tau, & \text{for } (N+1)/2 < i \\ & \quad < (N+1) - \left(j_0 \wedge \sqrt{N}\right), \\ \sqrt{N}\,\Phi_1(i) + \tau, & \text{for } (N+1) - \left(j_0 \wedge \sqrt{N}\right) \le i \le N. \end{cases} \tag{3.42}$$

Then (3.16) gives

$$a_{N\cdot} = \frac{\sum_{i=0}^{i_0 \wedge \sqrt{N}} \Phi_0(i) + \sum_{j=0}^{j_0 \wedge \sqrt{N}} \Phi_1(j)}{\sqrt{N}} \to 0, \tag{3.43}$$

and

$$\overline{a_N^2} = \sum_{i=0}^{i_0 \wedge \sqrt{N}} \Phi_0^2(i) + \sum_{j=0}^{i_0 \wedge \sqrt{N}} \Phi_1^2(j) + \tau^2\left(1 - \frac{1}{N} 1_{[N = \text{odd}]}\right) + o(1) \to 1. \tag{3.44}$$

Thus

$$\begin{aligned} &\sigma_{a,N}^2 \to 1, \qquad \Phi_{0N}(i) \to \Phi_0(i) \quad \text{for } i < i_0 + 1 \quad \text{and} \\ &\qquad \Phi_{1N}(i) \to \Phi_1(i) \quad \text{for } j < j_0 + 1, \end{aligned} \tag{3.45}$$

and the r.v. of (1.22) is achieved. □

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