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MINIMAX LINEAR ESTIMATION IN A WHITE NOISE PROBLEM

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Linear estimation of $f(x)$ at a point in a white noise model is considered. The exact linear minimax estimator of $f(0)$ is found for the family of $f(x)$ in which $f'(x)$ is $\mathrm{Lip}(M)$. The resulting estimator is then used to verify a conjecture of Sacks and Ylvisaker concerning the near optimality of the Epanechnikov kernel.

1. Introduction. Consider the following prototypical problem. Observe Y of the form

$$dY(t) = f(t)\,dt + \sigma dB(t),$$

where f belongs to some family $\mathscr{F}$ and B is a standard Weiner process. (The model is called the white noise model.) Let T be a continuous linear functional on $\mathscr{F}$. One wants to estimate $T(f)$ in terms of $Y(t)$. A linear estimator of T is defined as

$$\hat{T}_L = \int g(t)\,dY(t),$$

for some kernal function g. Denote $\mathscr{L}$ to be the set of all linear estimators. Then a linear minimax estimator is the one which attains the following infimum:

$$\inf_{\hat{T}_L \in \mathscr{L}} \sup_{f \in \mathscr{F}} E_f\big(\hat{T}_L - T(f)\big)^2.$$

Much research has been conducted on the above and related problems. Ibragimov and Khasminiskii (1984) described a method which can often be used to find the linear minimax estimator for more general contexts. Donoho and Liu (1991) proved the result that, under certain broad conditions, the ratio of the minimax risk of the linear minimax estimator to that of the minimax estimator is bounded by 1.25. Hence, in addition to its inherent advantage of simplicity, the Donoho and Liu result showed that in terms of efficiency, the linear minimax estimator is nearly optimal in a minimax sense. Moreover, Donoho and Liu (1991) showed that various other common problems such as density estimation and nonparametric regression are as hard as the above white noise model. Further equivalence results of this nature can be found in Low (1992), Brown and Low (1996), and Nussbaum (1996). This makes the model above more useful.

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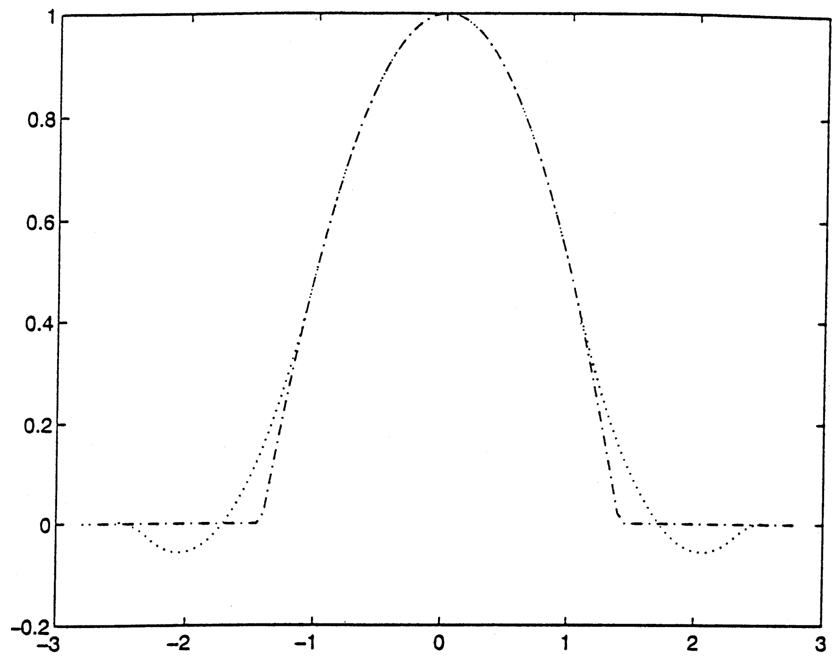

FIG. 1. *Dashed curve*: *Epanechinikov kernel. Dotted curve*: *our kernel whose support is on* $[-2.7, 2.7]$.

Sacks and Ylvisaker (1981) found the linear minimax estimator of $f(0)$, where f is a function in a class of functions described in terms of their Taylor series expansion about 0. In the case of restriction on the remainder in the first order Taylor expansion, they discovered that the familiar Epanechinikov kernel [Epanechinikov (1969)] yields the linear minimax estimator. As they noted, their family of functions, while mathematically convenient, is not statistically natural. The most natural family is undoubtedly that under which second derivatives are bounded, that is, $\{f\colon |f''(t)| \le B\}$. For technical reasons we instead consider the closure of this family, which is that f' satisfies a Lipschitz condition. Sacks and Ylvisaker conjectured that the linear minimax estimator for this problem should have a minimax risk not much smaller than that given by the Epanechinikov estimator. In Theorem 3 we derive the kernel for the minimax linear estimator over this Lipschitz class (Figure 1 displays the kernel we found and the Epanechinikov kernel) and in Section 4 we use this to find that the Epanechinikov kernel is 99% efficient.

2. Main results. The aim of this paper is to find a linear minimax estimator for

$$T(f) = f(0),$$

where f is continuously differentiable and

$$f \in \mathscr{F}_M = \{ f \mid f \in L_2(-\infty, +\infty),\ |f'(x) - f'(y)| \le M|x - y| \}.$$

According to Ibragimov and Khas'minskii (1984) and Donoho and Liu (1991), if f_1 attains

$$\sup_{f \in \mathscr{F}} \frac{T^2(f)}{\sigma^2 + \|f\|^2}, \tag{1}$$

then $\hat{T} = \int \psi(t)\, dY(t)$ is the linear minimax estimator, where

$$\psi(t) = \frac{T(f_1)}{\sigma^2 + \|f_1\|^2} f_1(t).$$

Furthermore the linear minimax risk is

$$\mathscr{M} = \frac{\sigma^2 T^2(f_1)}{\sigma^2 + \|f_1\|^2}. \tag{2}$$

Now let

$$b(\varepsilon) = \sup\{ |T(f)|, f \in \mathscr{F}, \|f\|^2 \le \varepsilon^2 \}. \tag{3}$$

It will be seen in (8) that $b(\varepsilon) = A\varepsilon^r$ with $r = 4/5$, as expected from Donoho and Liu (1991). Then as shown there,

$$\sup_{f \in \mathscr{F}} \frac{T^2(f)}{\sigma^2 + \|f\|^2} = \sup_{\varepsilon > 0} \frac{b^2(\varepsilon)}{\sigma^2 + \varepsilon^2} \tag{4}$$

and the second supremum is attained at $\varepsilon_0 = \sqrt{r\sigma^2/(1 - r)}$.

This implies that once the solution to (4) is known, the problem (1) can be solved. Notice that the problems of finding

$$\sup_{\|f\|^2 \le \varepsilon^2} \{ |f(0)|, f \in \mathscr{F}_{\mathscr{M}} \}$$

and of finding

$$\inf_{|f(0)| = b} \{ \|f\|^2, f \in \mathscr{F}_{\mathscr{M}} \}, \tag{5}$$

are equivalent; hence it suffices to solve (5) with $b > 0$.

This same minimization problem is the key to some nonparametric estimation problems other than the quadratic risk estimation problem described above. See Donoho (1994a, b) for more details.

We first prove an existence theorem for problem (5).

Theorem 1. *There exists a unique function f which solves*

$$\inf_{f(0) = b,\, f \in \mathscr{F}_{\mathscr{M}}} \int f^2\, dx,$$

and the minimum function is an even function.

PROOF. (i) *Existence*. $S_b = \{f \in \mathscr{F}_M, f(0) = b\}$ is compact in L_2 and $\int f^2\,dx$ is continuous on L_2. Hence there exists an f which attains the infimum.

(ii) *Uniqueness*. S_b is convex and $\int f^2\,dx$ is a convex function. Hence the infimum is attained at a unique point in S_b. If $f \in S_b$ is not an even function then it cannot attain this minimum since $\bar{f} = (f(x) + f(-x))/2 \in S_b$ and $\bar{f} \neq f$ so that

$$\int \left(\frac{f(x) + f(-x)}{2}\right)^2 dx < \frac{1}{2}\left[\int f^2(x)\,dx + \int f^2(-x)\,dx\right]$$
$$= \int f^2(x)\,dx. \qquad \square$$

It is possible to describe how the solution to (5) depends on b and M. Let $f_{b,M}$ and $I_{b,M}$ denote the minimizer and minimum value, respectively, for (5). We have the following lemma.

LEMMA 1.
$$I_{b,M} = \frac{b^{5/2}}{\sqrt{M}} I_{1,1}, \qquad f_{b,M}(x) = b f_{1,1}\left(\sqrt{\frac{M}{b}}\, x\right).$$

PROOF. Suppose $g(0) = b, |g'(x) - g'(y)| \le M|x - y|$. Let

$$g_1(x) = g\left(\sqrt{\frac{b}{M}}\, x\right)/b,$$

then $g_1(0) = 1$ and $|g_1'(x) - g_1'(y)| \le |x - y|$. On the other hand, if $g_1(0) = 1$ and $|g_1'(x) - g_1'(y)| \le |x - y|$, then $g(x) = b g_1(\sqrt{(M/b)}\, x)$ has $g(0) = b, |g'(x) - g'(y)| \le M|x - y|$. So

$$\min_{g(0)=b,\,|g'(x)-g'(y)|\le M|x-y|} \int g^2(x)\,dx$$
$$= \min_{g(0)=1,|g'(x)-g'(y)|\le|x-y|} \frac{b^{5/2}}{\sqrt{M}} \int g^2(x)\,dx$$

and

$$f_{b,M} = b f_{1,1}\left(\sqrt{\frac{M}{b}}\, x\right). \qquad \square$$

Lemma 1 can be viewed as a special case of more general results derived in Donoho and Low (1992).

The lemma shows that it is adequate to solve

$$\inf_{f(0)=1} \left\{\int f^2(x)\,dx, |f'(x) - f'(y)| \le |x - y|\right\}. \tag{6}$$

In order to solve (6), Theorem 2 provides the first step of that solution.

THEOREM 2. *There is* $0 < k_0 < \sqrt{2}$, *such that the minimizer of* (6), *denoted by* $f_0(x)$, *satisfies*

$$f_0(x) = \begin{cases} 1 - \dfrac{x^2}{2}, & 0 \le x \le k_0, \\ 1 - k_0^2 + \dfrac{(x - 2k_0)^2}{2}, & k_0 \le x \le 2k_0 \end{cases}$$

and

$$f_0'(2k_0) = 0.$$

PROOF. We will restrict our attention to $x \ge 0$.

(i) Let $P_1(x) = 1 - x^2/2$. We claim that $f_0(x) \ge P_1(x)$.

Since $f_0(x)$ is even, then $f_0'(0) = 0$. Moreover, $f_0(0) = 1$, $P_1(0) = 1$. We know that $f_0''(x)$ exists almost everywhere, and $f_0'' \ge P_1''(x) = -1$. Hence

$$\begin{aligned} P_1(x) &= \int_0^x \left(\int_0^t (-1)\, d\tau \right) dt \\ &\le \int_0^x \left(\int_0^t f_0''(\tau)\, d\tau \right) dt \\ &= f_0(x) \end{aligned}$$

(ii) Suppose that $f_0(x) \ge 0\ \forall\, x$. Then $f_0(x) = f_1(x)$ where

$$f_1(x) = \begin{cases} 1 - \dfrac{x^2}{2}, & 0 \le x \le 1, \\ \dfrac{(x-2)^2}{2}, & 1 < x \le 2, \\ 0, & 2 < x. \end{cases}$$

Suppose that $f_0(x)$ does not cross $f_1(x)$. By (i) we know that $f_1(x) \le f_0(x)$, when $0 \le x \le 1$. Hence $f_1(x) \le f_0(x)$ and $f_1 = f_0$. So $k_0 = 1$ in this case.

If $f_0(x)$ crosses $f_1(x)$ at some points, let x_1 be the first nonzero cross point; then $x_1 \ge 1$ by (i) and $f_0(x) \ge f_1(x)$ when $0 \le x \le x_1$. Hence $f_0'(x_1) \le f_1'(x_1)$. Assume $x_1 \le 2$.

Since $f_0(x_1) = f_1(x_1)$, $f_0''(x) \le f_1''(x)$ a.e. between 1 and 2; integrating twice, we get that

$$f_0(x) \le f_1(x) \quad \text{when } x_1 \le x \le 2.$$

So $f_0(x)$ has to be 0 at some point less than or equal to 2, say x_2. Since $f_0(x_2) = 0$ is the minimum value of $f_0(x)$, so $f_0'(x_2) = 0$. Similarly to (i), we also have that

$$f_0(x) \le \frac{(x - x_2)^2}{2}$$

and $f_0(x) \geq 1 - x^2/2$. But $(x - x_2)^2/2$ and $1 - x^2/2$ will intersect twice if $x_2 < 2$. This is impossible since no curve can then lie above $1 - x^2/2$ and below $(x - x_2)^2/2$. So the only possible case is that $x_1 = x_2 = 2$, and $f_0(x) = f_1(x), 0 \leq x \leq 2$.

(iii) If $f_0(x)$ has negative values, let x_0 be the first local minimum point with negative local minimum value. Let $y_0 < 0$ denote this local minimum. Then

$$f_0'(x_0) = 0.$$

By (i), we know that

$$f_0(x) \geq 1 - \frac{x^2}{2}.$$

An argument similar to (i) yields

$$f_0(x) \leq y_0 + \frac{(x - x_0)^2}{2}.$$

We claim that

$$f_0(x) = \begin{cases} 1 - \dfrac{x^2}{2}, & 0 \leq x \leq x_0/2, \\ y_0 + \dfrac{(x - x_0)^2}{2}, & x_0/2 < x \leq x_0. \end{cases}$$

Let

$$f_2(x) = \begin{cases} 1 - \dfrac{x^2}{2}, & 0 \leq x \leq \sqrt{1 - y_0}, \\ y_0 + \dfrac{\left(x - 2\sqrt{1 - y_0}\right)^2}{2}, & \sqrt{1 - y_0} \leq x \leq 2\sqrt{1 - y_0}. \end{cases}$$

Since $f_0(x)$ has to be between $1 - x^2/2$ and $y_0 + (x - x_0)^2/2$, so

$$2\sqrt{1 - y_0} \leq x_0.$$

Let

$$x_1 = \sup_z \{z : f_0(x) = P_1(x), 0 \leq x \leq z\}.$$

We claim that $x_1 \leq \sqrt{1 - y_0}$. Suppose $x_1 > \sqrt{1 - y_0}$, then $f_0(x_1) < f_2(x_1)$ and $f_0'(x_1) \leq f_2'(x_1)$. By the fact that $f_0''(x) \leq f_2''(x)$ when $\sqrt{1 - y_0} \leq x \leq 2\sqrt{1 - y_0}$, we get $f_0(x) < f_2(x)$ when $\sqrt{1 - y_0} \leq x \leq 2\sqrt{1 - y_0}$. This contradicts the assumption that (x_0, y_0) is the first negative local minimum.

Now, we show that $f_0(x) = f_2(x)$, when $0 \leq x \leq 2\sqrt{1 - y_0}$. First we claim that $f_0(x)$ has to intersect with $f_2(x)$ between $\sqrt{1 - y_0}$ and $2\sqrt{1 - y_0}$. Let

$x_2 = 2\sqrt{1-y_0} + \sqrt{-2y_0}$ be the x-intercept of $f_2(x)$. Suppose the above claim fails. Then $f_2(x) < f_0(x)$ when $\sqrt{1-y_0} \le x \le x_2$. Take

$$f_3(x) = \begin{cases} f_2(x), & 0 \le x \le 2\sqrt{1-y_0}\,, \\ f_0\left(x - 2\sqrt{1-y_0} + x_0\right), & 2\sqrt{1-y_0} < x. \end{cases}$$

Then $\int f_3^2(x)dx < \int f_0^2(x)dx$. Contradiction.

Suppose x_3 is the first intersection of $f_0(x)$ and $f_2(x)$ in $[\sqrt{1-y_0}, 2\sqrt{1-y_0}]$. Then by a similar argument to (ii) we know that

$$f_0(x) \le f_2(x) \quad \text{if } x_2 \le x \le 2\sqrt{1-y_0}\,.$$

This leads to the conclusion that $x_0 = 2\sqrt{1-y_0}$. Hence,

$$f_0(x) = \begin{cases} 1 - \dfrac{x^2}{2}, & 0 \le x \le x_0/2, \\ y_0 + \dfrac{(x-x_0)^2}{2}, & x_0/2 \le x \le x_0. \end{cases} \qquad \square$$

Now it is possible to construct the function f_0.

THEOREM 3. *We have*

$$f_0(x) = 1 + \int_0^{|x|}\left(\int_0^t \sum_{k=1}^{\infty} (-1)^k I_{[l_{k-1}, l_k)}(\tau)\, d\tau\right) dt,$$

where $[l_{k-1}, l_k)$ *are disjoint. Furthermore,* $k_0 = 1.028, \dots, l_0 = 0, l_1 = k_0$,

$$l_k = l_{k-1} + k_0\left(|1-k_0^2|^{(k-2)/2} + |1-k_0^2|^{(k-1)/2}\right), \qquad k \ge 2$$

and $\mathrm{I}_{1,1} = 2 \times 0.76402\dots$.

PROOF. (i) Since

$$f_0(x) = \begin{cases} 1 - \dfrac{x^2}{2}, & 0 \le x \le k, \\ 1 - k^2 + \dfrac{(x-2k)^2}{2}, & k \le x \le 2k, \end{cases}$$

for some k, and $f_0'(2k) = 0$, then by Lemma 1, when $x \ge 2k_0$, $f_0(x) = f_{b,M}(x - 2k_0)$, with $b = |1-k^2|$.

So, by Lemma 1,

$$\begin{aligned} I_{1,1}^k &= \int_0^k (1 - x^2/2)^2\, dx + \int_k^{2k}\left(1 - k^2 + \frac{(x-2k)^2}{2}\right)^2 dx + I_{|1-k^2|,1} \\ &= \left(\frac{23}{30}k^5 - 2k^3 + 2k\right) + |1-k^2|^{5/2} I_{1,1}^k. \end{aligned}$$

Hence

$$I_{1,1}^{k} = \frac{((23/30)k^5 - 2k^3 + 2k)}{1 - |1 - k^2|^{5/2}}.$$

It is obvious that k_0 has to minimize $I_{1,1}^{k}$.

(ii) We know that $0 < k < \sqrt{2}$, since

$$\lim_{k \to 0+} I_{1,1}^{k} = \lim_{k \to 0+} \frac{((23/30)k^5 - 2k^3 + 2k)}{1 - |1 - k^2|^{5/2}} = +\infty$$

and

$$\lim_{k \to \sqrt{2}^-} \frac{((23/30)k^5 - 2k^3 + 2k)}{1 - |1 - k^2|^{5/2}} = +\infty.$$

So $I_{1,1}^{k}$ attains its minimum value at some point between 0 and $\sqrt{2}$. We solved this simple minimization problem numerically. It turns out that

$$k_0 = 1.028\ldots.$$

Plugging in $k = 1.028$, we have

$$\inf_k I_{1,1}^{k} = \left.\frac{((23/30)k^5 - 2k^3 + 2k)}{1 - |1 - k^2|^{5/2}}\right|_{k=1.028} = 0.764\ldots.$$

The formula for f_0 can now be obtained by induction, as follows.

(iii) From 0 to k_0, $f_0''(x) = -1$. Then it changes to $+1$ between k_0 and $2k_0$ and $f_0'(2k_0) = 0$.

Since $f_{b,1}(x) = bf_0(x/\sqrt{b})$, $f_{b,1}''(x) = f_0''(x/\sqrt{b})$. Hence $f_{b,1}''(x) = -1$ from 0 to $\sqrt{bk_0}$, then switches to 1 from $\sqrt{bk_0}$ to $2k_0\sqrt{b}$, and $f_{b,1}(2k_0\sqrt{b}) = bf_0(2k_0) = b(1 - k_0^2)$.

Let k_1 be the next turning point at which $f_{b,1}''(x)$ switches from 1 to -1. By the previous argument, we know that the difference between $2k_0\sqrt{b}$ and k_1 is

$$k_0\sqrt{|b(1 - k_0^2)|}.$$

So the distance between $k_0\sqrt{b}$ and k_1 is

$$k_0\sqrt{b} + k_0\sqrt{|b(1 - k_0^2)|} = k_0\left(\sqrt{b} + \sqrt{|b(1 - k_0^2)|}\right).$$

From the above general formula, we have

$$\begin{aligned}
&l_0 = 0,\\
&l_1 = k_0,\\
&l_2 = l_1 + k_0\left(1 + \sqrt{|1 - k_0^2|}\right),\\
&l_3 = l_2 + k_0\left(\sqrt{|1 - k_0^2|} + \sqrt{|1 - k_0^2|^2}\right),\\
&\quad\vdots
\end{aligned}$$

$$l_k = l_{k-1} + k_0\left(|1 - k_0^2|^{(k-2)/2} + |1 - k_0^2|^{(k-1)/2}\right), \qquad k \geq 2, \tag{7}$$

and

$$f_0''(x) = \sum_{k=1}^{\infty} (-1)^k I_{[l_{k-1}, l_k)}(x).$$

(iv) Since also $f_0'(0) = 0$, $f_0(0) = 1$, so

$$f_0(x) = 1 + \int_0^{|x|} \left(\int_0^t \sum_{k=1}^{\infty} (-1)^k I_{[l_{k-1}, l_k)}(\tau)\, d\tau \right) dt. \qquad \square$$

Gabushin (1968) solved some inequalities between norms of derivatives of functions. One of these is more or less equivalent to problem (5). The methods used by Gabushin are different from ours and it appears they could be used to provide an alternate derivation for the results in our paper. We provide solutions explicitly and our proofs are more intuitive and easier to understand.

COROLLARY. *The extremal function $f_0(x)$ in Theorem 3 has finite support on* $[-2.7, 2.7]$.

PROOF. From Theorem 3 we have

$$\begin{aligned}\sum_{k=1} (l_k - l_{k-1}) &= k_0 \left(1 + \sum_{k=2} \left(1 - k_0^2\right)^{(k-2)/2} + \sum_{k=2} \left(1 - k_0^2\right)^{(k-1)/2} \right) \\ &= 2.699. \qquad \square\end{aligned}$$

3. The linear minimax risk. By Lemma 1,

$$\inf_{f(0)=b} \left\{ \int f^2\, dx,\ f \in \mathscr{F}_{\mathscr{M}} \right\} = \frac{I_{1,1}}{\sqrt{M}} b^{5/2}.$$

Hence we set $(I_{1,1}/\sqrt{M}) b^{5/2} = \varepsilon^2$, and solve for b. We have

$$\begin{aligned}b(\varepsilon) &= \sup_{\|f\|^2 \le \varepsilon^2} \{|f(0)|,\ f \in \mathscr{F}\} \\ &= \left(\frac{\sqrt{M}}{I_{1,1}} \right)^{2/5} \varepsilon^{4/5}.\end{aligned} \tag{8}$$

Once we know $b(\varepsilon)$, we can apply (2) and (4) to get the linear minimax risk

$$\frac{\sigma^2 b^2(\varepsilon_0)}{\sigma^2 + \varepsilon_0^2},$$

where $\varepsilon_0 = \sqrt{r\sigma^2/(1-r)} = \sqrt{(4/5\sigma^2)/(1-4/5)}$.

After some tedious calculations, we have the linear minimax risk

$$\mathscr{M} = \frac{1}{5} \left\{ \left(\frac{4M}{I_{1,1}} \right)^{2/5} \sigma^{4/5} \right\}^2,$$

and $f_1(x)$ in (1)

$$f_1(x) = b(\varepsilon_0) f_0\left(\sqrt{\frac{M}{b(\varepsilon_0)}}\, x\right).$$

Particularly, for $M = 1$ the linear minimax risk is $0.431896\sigma^{8/5}$.

4. The relative efficiency of the Epanechnikov estimator. Sacks and Ylvisaker (1981) showed that the Epanechnikov kernel is the asymptotically optimum kernel for density estimation at a point when the family of f is

$$\mathscr{F}'_{\mathrm{SY}} = \Big\{ f | f \geq 0, \int f = 1, f(x) = f(0) + f'(0)x + r(x),$$
$$f(0) \leq \alpha_1, |r(x)| \leq 1/2x^2 \text{ for} |x| \leq s \Big\},$$

where s is some suitable small number. As remarks, they conjectured that there is not much loss of efficiency when one uses Epanechnikov kernels as suboptimal solutions for density estimation problems on

$$\mathscr{F}_1' = \mathscr{F}_1 \cap \{ f \textit{ is a density function and } \sup |f| \leq \alpha_1 \}.$$

The conjecture is verified as follows: Donoho and Liu (1991) showed the linear minimax risk of the white noise model at noise level $\sigma = \sqrt{\alpha_1/n}$ corresponding to $\mathscr{F}_1$ is asymptotically the same as the one in the density estimation problem over $\mathscr{F}_1'$. So this minimax risk will be $0.432\alpha_1^{4/5} n^{-4.5}$ as we calculated at the end of Section 3. The minimax risk over $\mathscr{F}'_{\mathrm{SY}}$ was calculated to be $0.436\alpha_1^{4/5} n^{-4/5}$ in Sacks and Ylvisaker (1981). Furthermore the maximum risk is attained at a density which is also in $\mathscr{F}_1'$. Hence $0.436\alpha_1^{4/5} n^{-4/5}$ is also the maximum risk over $\mathscr{F}_1'$ of the best Epanechnikov kernel. Consequently in a minimax sense the Epanechnikov kernel is $0.432/0.436 = 99\%$ efficient.

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