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ON THE EFFICIENCY OF MULTIVARIATE SPATIAL SIGN AND RANK TESTS

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Asymptotic Pitman efficiencies of multivariate spatial sign and rank methods are considered in the one-sample location case. Limiting distributions of the spatial sign and signed-rank tests under the null hypothesis as well as under contiguous sequences of alternatives are given. Formulae for asymptotic relative efficiencies are found and, under multivariate t distributions, relative efficiencies with respect to Hotelling's T^2 test are calculated.

1. Introduction. The purpose of this paper is to consider the efficiencies of the multivariate spatial sign and rank tests discussed earlier by Brown (1983), Chaudhuri (1992) and Möttönen and Oja (1995). These tests are conditionally and asymptotically distribution-free rotation (but not scale) invariant competitors of the well known Hotelling's T^2 tests. In this paper efficiency results are found in the one-sample location case, but they naturally hold in the multivariate two-sample, multisample and general linear model case.

The lack of means of fully ordering (or ranking) observations in the multivariate setting seems at first sight to be an obstacle to generalizing the concepts of sign and rank and signed-rank. However, these concepts can be defined also by using L_1 criterion functions as follows. Consider k-variate observations $\mathbf{y}_1, \dots, \mathbf{y}_N$. Utilizing the Euclidean distance $|\mathbf{y}| = (y_1^2 + \cdots + y_k^2)^{1/2}$, the three objective functions

$$\sum_i |\mathbf{y}_i| = \sum_i \mathbf{S}^T(\mathbf{y}_i)\mathbf{y}_i,$$

$$\frac{1}{2N} \sum_i \sum_j |\mathbf{y}_i - \mathbf{y}_j| = \sum_i \mathbf{R}_N^T(\mathbf{y}_i)\mathbf{y}_i$$

and

$$\frac{1}{4N} \sum_i \sum_j \{|\mathbf{y}_i - \mathbf{y}_j| + |\mathbf{y}_i + \mathbf{y}_j|\} = \sum_i \mathbf{Q}_N^T(\mathbf{y}_i)\mathbf{y}_i$$

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yield the multivariate (spatial) concepts of sign, centered rank and signed-rank functions $\mathbf{S}(\mathbf{y})$, $\mathbf{R}_N(\mathbf{y})$ and $\mathbf{Q}_N(\mathbf{y}) = \mathbf{R}_N(\mathbf{y}) - \mathbf{R}_N(-\mathbf{y})$. In the univariate case the usual univariate concepts of sign, centered rank and signed-rank are obtained [Hettmansperger and Aubuchon (1988)].

The (vector-valued) *spatial sign function* of $\mathbf{y}$ is then

$$\mathbf{S}(\mathbf{y}) = \mathbf{u}_y,$$

the unit vector in the direction of $\mathbf{y}$. Also analogously with the univariate case the (vector-valued) centered *spatial rank function* of $\mathbf{y}$ with respect to the sample $\mathbf{y}_1, \dots, \mathbf{y}_N$ appears to be $\mathbf{R}_N(\mathbf{y}) = (1/N)\sum_i \mathbf{S}(\mathbf{y} - \mathbf{y}_i)$, that is, the mean of spatial signs of $\mathbf{y} - \mathbf{y}_i$. Note that the ranks $\mathbf{R}_N(\mathbf{y}_i)$ are centered, that is, $\sum_i \mathbf{R}_N(\mathbf{y}_i) = \mathbf{0}$. Finally, the *spatial signed-rank function* of $\mathbf{y}$,

$$\mathbf{Q}_N(\mathbf{y}) = \frac{1}{2N} \sum_i \{\mathbf{S}(\mathbf{y} - \mathbf{y}_i) + \mathbf{S}(\mathbf{y} + \mathbf{y}_i)\},$$

is the spatial rank of $\mathbf{y}$ among both the observations $\mathbf{y}_1, \dots, \mathbf{y}_N$ and their reflections $-\mathbf{y}_1, \dots, -\mathbf{y}_N$. See Möttönen and Oja (1995).

In this paper we consider the multivariate one-sample location case, that is, we assume that $\mathbf{y}_1, \dots, \mathbf{y}_N$ is a random sample from a k-variate symmetric distribution with p.d.f. $f(\mathbf{y} - \boldsymbol{\mu})$ (f is symmetric about the origin and $\boldsymbol{\mu}$ is the unknown symmetry center). By a symmetry we mean that the distributions of $\mathbf{y}_i - \boldsymbol{\mu}$ and $\boldsymbol{\mu} - \mathbf{y}_i$ are the same. Without loss of generality we assume that the null hypothesis to be tested is

$$H_0\colon \boldsymbol{\mu} = \mathbf{0}.$$

Write $\mathbf{L} = \mathbf{L}(\mathbf{y})$ for the gradient vector of $\ln(f(\mathbf{y} - \boldsymbol{\mu}))$ w.r.t. $\boldsymbol{\mu}$ at the origin. [$\mathbf{L}(\mathbf{y})$ is the optimal location score function.]

The *spatial sign test statistic* or the angle test statistic for testing the null hypothesis H_0 is

$$\mathbf{T}_{1N} = \sum \mathbf{S}(\mathbf{y}_i),$$

that is, the sum of spatial signs or the spatial rank of $\mathbf{0}$ w.r.t. the sample $\mathbf{y}_1, \dots, \mathbf{y}_N$. The *spatial signed-rank test statistic* for testing the null hypothesis $H_0\colon \boldsymbol{\mu} = \mathbf{0}$ is the V-statistic [Serfling (1980), Chapter 5.1.2]

$$\mathbf{T}_{2N} = \sum \mathbf{Q}_N(\mathbf{y}_i) = \frac{1}{2N} \sum_i \sum_j \mathbf{S}(\mathbf{y}_i + \mathbf{y}_j),$$

that is, the sum of signed-ranks of the observations $\mathbf{y}_i$, $i = 1, \dots, N$, or a constant times the sum of signs of pairwise averages $(\mathbf{y}_i + \mathbf{y}_j)/2$. Under the null hypothesis the test statistics $N^{-1/2}\mathbf{T}_{1N}$ and $N^{-1/2}\mathbf{T}_{2N}$ are asymptotically multinormal $N_k(\mathbf{0}, \mathbf{B}_1)$ and $N_k(\mathbf{0}, \mathbf{B}_2)$, respectively, $\mathbf{B}_1$ and $\mathbf{B}_2$ naturally depending on the unknown distribution f. The squared versions $N^{-1}\mathbf{T}_{iN}^T \mathbf{B}_i^{-1} \mathbf{T}_{iN}$, $i = 1, 2$, have limiting chi-squared distributions with k degrees of freedom. The tests are only conditionally distribution-free. Asymptotically distribution-free versions are obtained when the covariance matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ are replaced by convergent estimates. See Brown (1983), Chaudhuri (1992) and Möttönen and Oja (1995).

In this paper we consider the efficiencies of the multivariate spatial sign and signed-rank tests. Limiting distributions of the test statistics $\mathbf{T}_{1N}$ and $\mathbf{T}_{2N}$ under the contiguous alternative sequences of hypotheses are given in Section 2. Formulae for calculating asymptotic Pitman efficiencies and efficiencies for multivariate t-distribution family are given in the final Section 3. Efficiencies in the multinormal case have been earlier given by Brown [(1983), spatial median and angle test] and by Chaudhuri [(1992), spatial Hodges–Lehmann estimate and spatial signed-rank test].

2. Limiting distributions. Consider first the score test statistics of the general form

$$\mathbf{T}_N = \sum_{i=1}^{N} \mathbf{K}(\mathbf{y}_i)$$

for a fixed vector $(k \times 1)$-valued function $\mathbf{K}(\mathbf{y})$. Assume $\mathbf{K}$ is centered so that the expected value of $\mathbf{T}_N$ under the null hypothesis is the zero vector, that is, $E_0(\mathbf{K}) = E_0(\mathbf{K}(\mathbf{y})) = \mathbf{0}$. It is well known that the asymptotically best choice for the score function $\mathbf{K}(\mathbf{y})$ is the optimal score $\mathbf{L}(\mathbf{y})$. Let

$$\mathbf{U}_N = \sum_{i=1}^{N} \mathbf{L}(\mathbf{y}_i)$$

be this optimal test statistic. Note that $\mathbf{K}(\mathbf{y}) = \mathbf{y}$ gives a test which is asymptotically equivalent with Hotelling's T^2 test and optimal under multinormality.

Under the null hypothesis the limiting distribution of $N^{-1/2}(\mathbf{T}_N^T\ \mathbf{U}_N^T)^T$ is (according to the central limit theorem)

$$N_{2k}\left(\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}, \begin{pmatrix}\mathbf{B} & \mathbf{A}\\ \mathbf{A}^T & \mathbf{I}_0\end{pmatrix}\right),$$

where

$$B = E_0(\mathbf{K}\mathbf{K}^T) \quad \text{and} \quad \mathbf{A} = E_0(\mathbf{K}\mathbf{L}^T)$$

and $\mathbf{I}_0 = E_0(\mathbf{L}\mathbf{L}^T)$ is the expected Fisher information matrix for a single observation at $\boldsymbol{\mu} = \mathbf{0}$. In the following lemma the limiting distribution of $N^{-1/2}\mathbf{T}_N$ under contiguous alternative sequences is found.

LEMMA 1. *Consider contiguous alternative sequences of the form* $f(\mathbf{y} - N^{-1/2}\boldsymbol{\delta})$ *satisfying under* H_0,

$$L_N = \sum_{i=1}^{N} \left\{\ln f\left(\mathbf{y}_i - N^{-1/2}\boldsymbol{\delta}\right) - \ln f(\mathbf{y}_i)\right\} = N^{-1/2}\mathbf{U}_N^T\boldsymbol{\delta} - \tfrac{1}{2}\boldsymbol{\delta}^T\mathbf{I}_0\boldsymbol{\delta} + o_p(1).$$

Then the limiting distribution of the test statistic $N^{-1/2}\mathbf{T}_N$ *under these alternative sequences is* k*-variate normal with mean vector* $\mathbf{A}\boldsymbol{\delta}$ *and covariance matrix* $\mathbf{B}$.

PROOF. Le Cam's Third Lemma is utilized here. See, for example, Hájek and Šidák [(1967), Chapter VI.1.4]. Under the contiguous sequence of alternatives,

$$P_N\{N^{-1/2}\mathbf{T}_N \in C\} = E_N\{1_C(N^{-1/2}\mathbf{T}_N)\} = E_0\{1_C(N^{-1/2}\mathbf{T}_N)\exp(L_N)\}.$$

$\exp(L_N)$ is under H_0 uniformly integrable, since it is positive and integrable with $E_0(\exp(L_N)) = 1$, for all N; see Theorem 5.4 in Billingsley (1968). Then by Vitali's theorem, the above expected value converges to

$$E\{1_C(\mathbf{T}_0)\exp(\mathbf{U}_0^T\boldsymbol{\delta} - \tfrac{1}{2}\boldsymbol{\delta}^T\mathbf{I}_0\boldsymbol{\delta})\}$$
$$= E\{1_C(\mathbf{T}_0)\exp(\boldsymbol{\delta}^T\mathbf{A}^T\mathbf{B}^{-1}\mathbf{T}_0 - \tfrac{1}{2}\boldsymbol{\delta}^T\mathbf{A}^T\mathbf{B}^{-1}\mathbf{A}\boldsymbol{\delta})\},$$

where the distribution of $(\mathbf{T}_0^T\ \mathbf{U}_0^T)^T$ is the limiting multinormal distribution of $N^{-1/2}(\mathbf{T}_N^T\ \mathbf{U}_N^T)^T$ under the null hypothesis. So the density function of the limiting distribution of $N^{-1/2}\mathbf{T}_N$ is, under contiguous sequences, $\exp\{\boldsymbol{\delta}^T\mathbf{A}^T\mathbf{B}^{-1}\mathbf{y} - \frac{1}{2}\boldsymbol{\delta}^T\mathbf{A}^T\mathbf{B}^{-1}\mathbf{A}\boldsymbol{\delta}\}$ times the density under H_0. This gives the result. See also Shorack and Wellner [(1986), Chapter 4.1] and Lehmann [(1983), Lemma 6.8.2]. □

Consequently, under the sequence of contiguous alternatives, the limiting distribution of $N^{-1}\mathbf{T}_N^T\mathbf{B}^{-1}\mathbf{T}_N$ is a noncentral chi-squared distribution with k degrees of freedom and noncentrality parameter $\boldsymbol{\delta}^T\mathbf{A}^T\mathbf{B}^{-1}\mathbf{A}\boldsymbol{\delta}$. See also Hettmansperger, Nyblom and Oja (1994). So only the factor $\mathbf{A}^T\mathbf{B}^{-1}\mathbf{A}$ is needed for efficiency comparisons. For Hotelling's T^2 test [with $\mathbf{K}(\mathbf{y}) = \mathbf{y}$], $\mathbf{A} = \mathbf{I}$ and $\mathbf{B} = \Sigma = E(\mathbf{y}\mathbf{y}^T)$. For the spatial sign test [with $\mathbf{K}(\mathbf{y}) = \mathbf{S}(\mathbf{y}) = \mathbf{u}_y$] we thus have the following theorem:

THEOREM 1. *Under the sequence of contiguous alternatives,* $N^{-1/2}\mathbf{T}_{1N}$ *is asymptotically k-variate normal with mean vector* $\mathbf{A}_1\boldsymbol{\delta}$, *where* $\mathbf{A}_1 = E_0(\mathbf{S}\mathbf{L}^T)$ *and with covariance matrix* $\mathbf{B}_1 = E_0(\mathbf{S}\mathbf{S}^T)$.

Unfortunately, the spatial signed-rank test statistic is not of the above form, since then

$$\mathbf{T}_{2N} = \sum_{i=1}^{N} \mathbf{Q}_N(\mathbf{y}_i),$$

with empirical score function $\mathbf{Q}_N$. As in Brown, Hettmansperger, Nyblom and Oja (1992) and in Hettmansperger, Nyblom and Oja (1994), we first show that, under H_0 as well as under contiguous sequences, $\mathbf{Q}_N$ is a uniformly weakly convergent estimate of the corresponding *theoretical signed-rank function*

$$\mathbf{Q}(\mathbf{y}) = \tfrac{1}{2}E_0(\mathbf{u}_{y-y_i} + \mathbf{u}_{y+y_i}) = E_0\mathbf{u}_{y-y_i}.$$

LEMMA 2. *Under the sequence of contiguous alternatives with uniformly bounded continuous density function f,*

$$\sup_y |\mathbf{Q}_N(\mathbf{y}) - \mathbf{Q}(\mathbf{y})| \to_P 0.$$

PROOF. It is enough to consider the null hypothesis case. Write

$$\mathbf{S}_\varepsilon(\mathbf{y}) = \begin{cases} |\mathbf{y}|^{-1}\mathbf{y}, & \text{if } |\mathbf{y}| > \varepsilon, \\ \varepsilon^{-1}\mathbf{y}, & \text{if } |\mathbf{y}| \le \varepsilon \end{cases}$$

and

$$\mathbf{Q}_{\varepsilon,N}(\mathbf{y}) = \frac{1}{2N}\sum(\mathbf{S}_\varepsilon(\mathbf{y}-\mathbf{y}_i) + \mathbf{S}_\varepsilon(\mathbf{y}+\mathbf{y}_i))$$

for bounded and uniformly continuous approximations ($\varepsilon > 0$) of the sign and signed-rank functions, correspondingly. [$\mathbf{S}_0(\mathbf{y}) = \mathbf{S}(\mathbf{y})$ and $\mathbf{Q}_{0,N}(\mathbf{y}) = \mathbf{Q}_N(y)$.] Moreover, write

$$\mathbf{Q}_\varepsilon(\mathbf{y}) = E(\mathbf{Q}_{\varepsilon,N}(\mathbf{y})) = \tfrac{1}{2}\{E(\mathbf{S}_\varepsilon(\mathbf{y}-\mathbf{y}_i)) + E(\mathbf{S}_\varepsilon(\mathbf{y}+\mathbf{y}_i))\}.$$

Then clearly

$$P\left\{\lim_N \left(\sup_y |\mathbf{Q}_{\varepsilon,N}(\mathbf{y}) - \mathbf{Q}_\varepsilon(\mathbf{y})|\right) = 0\right\} = 1$$

for every fixed $\varepsilon > 0$ [Ranga Rao (1962)]. Further note that

$$|\mathbf{Q}_\varepsilon(\mathbf{y}) - \mathbf{Q}(\mathbf{y})| \le \tfrac{1}{2}\int_{|y-x|<\varepsilon} f(\mathbf{x})\,d\mathbf{x} + \tfrac{1}{2}\int_{|y+x|<\varepsilon} f(\mathbf{x})\,d\mathbf{x}$$

and

$$|\mathbf{Q}_{\varepsilon,N}(\mathbf{y}) - \mathbf{Q}_N(\mathbf{y})| \le \frac{1}{2N}\left(\#\{i|0 < |\mathbf{y}-\mathbf{y}_i| < \varepsilon\} + \#\{i|0 < |\mathbf{y}+\mathbf{y}_i| < \varepsilon\}\right).$$

The right-hand side of the first inequality is "uniformly small" if $f(\mathbf{x})$ is uniformly bounded and ε is small. The right-hand side of the second inequality converges, with probability 1, uniformly to the right-hand side of the first inequality. As

$$|\mathbf{Q}_N(\mathbf{y}) - \mathbf{Q}(\mathbf{y})| \le |\mathbf{Q}_N(\mathbf{y}) - \mathbf{Q}_{\varepsilon,N}(\mathbf{y})| + |\mathbf{Q}_{\varepsilon,N}(\mathbf{y}) - \mathbf{Q}_\varepsilon(\mathbf{y})| + |\mathbf{Q}_\varepsilon(\mathbf{y}) - \mathbf{Q}(\mathbf{y})|,$$

the result follows. □

Next note that the V-statistic $N^{-1/2}\mathbf{T}_{2N}$ and the projection of the corresponding U-statistic (with a finite kernel) $N^{-1/2}\Sigma\mathbf{Q}(\mathbf{y}_i)$ are then asymptotically equivalent with the same asymptotic properties, and one can just apply the above formulae for $\mathbf{A}$ and $\mathbf{B}$ utilizing the "limit score function" $\mathbf{Q}$. See Theorem 5.3.2 in Serfling (1980). Therefore, the following theorem is true.

THEOREM 2. *Under the sequence of contiguous alternatives given in Lemma 1, $N^{-1/2}\mathbf{T}_{2N}$ is asymptotically k-variate normal with mean vector $\mathbf{A}_2\boldsymbol{\delta}$, where $\mathbf{A}_2 = E(\mathbf{Q}\mathbf{L}^T)$, and with covariance matrix $\mathbf{B}_2 = E(\mathbf{Q}\mathbf{Q}^T)$.*

3. Efficiency. According to the previous section, the efficiency factors for Hotelling's test, for the spatial sign test and for the spatial rank test are the inverses of Σ, $\mathbf{A}_1^{-1}\mathbf{B}_1(\mathbf{A}_1^T)^{-1}$ and $\mathbf{A}_2^{-1}\mathbf{B}_2(\mathbf{A}_2^T)^{-1}$, the asymptotic covariance matrices of the corresponding one-sample location estimates, namely, the mean vector, the spatial median and the spatial HL-estimate. See Chaudhuri (1992) for Bahadur-type representations of the estimates which nicely show the connection between the tests and corresponding estimates. See also Möttönen and Oja (1995).

THEOREM 3. *The Pitman asymptotic relative efficiencies of the spatial sign test* ($i = 1$) *and the spatial signed-rank test* ($i = 2$) *with respect to Hotelling's* T^2 *test are*

$$\mathrm{ARE}_i = \frac{\boldsymbol{\delta}^T \mathbf{A}_i^T \mathbf{B}_i^{-1} \mathbf{A}_i \boldsymbol{\delta}}{\boldsymbol{\delta}^T \Sigma^{-1} \boldsymbol{\delta}}, \qquad i = 1, 2.$$

Compare now the above three tests in the case of the k-variate spherical t distribution with ν degrees of freedom [Johnson and Kotz (1972)]. For spherical distributions, the efficiency factors for Hotelling's T^2 test and the spatial sign test are

$$kE^{-1}(r^2)\mathbf{I}_k \quad \text{and} \quad \frac{(k-1)^2 E^2(r^{-1})}{k}\mathbf{I}_k,$$

correspondingly. The relative efficiency of the spatial sign test with respect to Hotelling's T^2 test in the general spherical case then is

$$\mathrm{ARE}_1 = \left(\frac{k-1}{k}\right)^2 E(r^2) E^2(r^{-1}),$$

which coincides with the efficiency of the affine invariant sign test based on the Oja median. See Hettmansperger, Nyblom and Oja (1994). In the multivariate t distribution case, r^2/k has a $F(k, \nu)$ distribution and in the multinormal case, r^2 has a χ^2_k distribution.

For spherical distributions the theoretical signed-rank function is

$$\mathbf{Q}(r\mathbf{u}) = q(r)\mathbf{u},$$

where $\mathbf{u}$ is the direction vector ($\mathbf{u}^T\mathbf{u} = 1$), r is the radius and

$$q(r) = E_s\left\{(r - s_k)/\left(s_1^2 + \cdots + s_{k-1}^2 + (r - s_k)^2\right)^{1/2}\right\},$$

$\mathbf{s} = (s_1 \cdots s_k)^T$ coming from the distribution under consideration. Under the multivariate t distribution with ν degrees of freedom, the Pitman asymptotic relative efficiency of the signed-rank test with respect to the Hotelling's T^2 test then is

$$\mathrm{ARE}_2 = \frac{\nu(\nu+k)^2}{k(\nu-2)}\left[E\left\{q(r)\frac{r}{\nu+r^2}\right\}\right]^2 \left[E(q^2(r))\right]^{-1}$$

[r^2/k has a $F(k, \nu)$ distribution]. If the observations come from $N_k(\mathbf{0}, \mathbf{I})$, the asymptotic relative efficiency is

$$\mathrm{ARE}_2 = \frac{1}{k}\left[E\{q(r)r\}\right]^2\left[E(q^2(r))\right]^{-1}$$

(squared radius r^2 has a χ_k^2 distribution). The efficiencies ARE_2 are quite tedious to compute. The properties of the hypergeometric series can be utilized here [Erdélyi, Magnus, Oberhettinger and Tricomi (1953)]. For details, see the Appendix and Möttönen and Oja (1994).

Table 1 lists some efficiencies of the spatial sign test and the spatial signed-rank test with respect to the Hotelling's T^2 test for t distributions with selected values of degrees of freedom ν and with selected dimensions k. In the multinormal case ($\nu = \infty$), the efficiencies of the spatial signed-rank test dominate the efficiencies of the spatial sign test, but for small values of degrees of freedom (heavy-tailed distributions) with high dimensions the sign test is better. Both tests seem to have good efficiencies over the broad class of t distributions, and the higher the dimension, the higher the efficiency.

Efficiencies for the spatial sign test in Table 1 agree with efficiencies of the affine invariant multivariate sign test based on the Oja median [Oja (1983); Brown and Hettmansperger (1989); Hettmansperger, Nyblom and Oja (1994)]. For sign methods in the bivariate case, see Oja and Nyblom [(1989), Table 6]; for the univariate case, see Lehmann [(1983), Table 5.3.1 and 5.6.1]; for the sign methods in the multivariate normal case ($\nu = \infty$), see Oja and Niinimaa [(1985), Table 1] and Brown (1983); and for rank methods in the multinormal case ($\nu = \infty$), see Chaudhuri (1992).

Randles (1989) and Jan and Randles (1994) introduced similar affine invariant sign and signed-rank tests (with signs based on so-called interdirections) which are asymptotically equivalent with their spatial counterparts in

TABLE 1
Asymptotic efficiencies of multivariate spatial signed-rank (sign) test relative to Hotelling's test under multivariate t distribution

	Degrees of Freedom							
Dimension	**3**	**4**	**6**	**8**	**10**	**15**	**20**	**∞**
1	1.900	1.401	1.164	1.089	1.054	1.014	0.997	0.955
	(1.621)	(1.125)	(0.879)	(0.798)	(0.757)	(0.710)	(0.690)	(0.637)
2	1.953	1.435	1.187	1.108	1.071	1.029	1.011	0.967
	(2.000)	(1.388)	(1.084)	(0.984)	(0.934)	(0.877)	(0.851)	(0.785)
3	1.994	1.453	1.200	1.119	1.081	1.038	1.019	0.973
	(2.162)	(1.500)	(1.172)	(1.063)	(1.009)	(0.947)	(0.920)	(0.849)
4	2.018	1.467	1.208	1.127	1.087	1.044	1.025	0.978
	(2.250)	(1.561)	(1.220)	(1.107)	(1.051)	(0.986)	(0.958)	(0.884)
6	2.050	1.484	1.219	1.136	1.095	1.051	1.031	0.984
	(2.344)	(1.626)	(1.271)	(1.153)	(1.094)	(1.027)	(0.997)	(0.920)
10	2.093	1.503	1.229	1.144	1.103	1.058	1.038	0.989
	(2.422)	(1.681)	(1.313)	(1.192)	(1.131)	(1.062)	(1.031)	(0.951)

the case of circular distributions. For efficiencies in the elliptically symmetric power family case, see tables in Randles (1989, 1992), Randles and Peters (1990) and Jan and Randles (1994). Also signs and data-driven coordinates proposed by Chaudhuri and Sengupta (1993) can be used to construct different affine invariant asymptotically equivalent versions of the sign and signed-rank test statistics.

APPENDIX

Efficiency of the spatial signed-rank test in the multivariate *t* distribution case. We say that the distribution of $\mathbf{y}$ is k-variate t distribution with ν degrees of freedom ($\mathbf{y} \sim t_{\nu,k}$) if the p.d.f. of $\mathbf{y}$ is

$$f(\mathbf{y}) = \frac{\Gamma((k+\nu)/2)}{\Gamma(\nu/2)(\pi\nu)^{k/2}}\left[1 + \frac{1}{\nu}\mathbf{y}^T\mathbf{y}\right]^{-(k+\nu)/2}.$$

The optimal score function then is

$$\mathbf{L}(\mathbf{y}) = \frac{\nu + k}{\nu + \mathbf{y}^T\mathbf{y}}\mathbf{y}.$$

First note that the theoretical signed-rank function $\mathbf{Q}(\mathbf{y})$ is the gradient of

$$\varrho(\mathbf{y}) = E_s\{|\mathbf{y} - \mathbf{s}|\} = E_s(D),$$

where $\mathbf{s}$ comes from F, that means

$$\mathbf{s} = \begin{pmatrix} s_1 \\ \vdots \\ s_k \end{pmatrix} = \begin{pmatrix} z_1(x/\nu)^{-1/2} \\ \vdots \\ z_k(x/\nu)^{-1/2} \end{pmatrix} \sim t_{\nu,k}$$

with independent $\mathbf{z} \sim N_k(\mathbf{0}, \mathbf{I})$ and $x \sim \chi^2_\nu(0)$. Our plan is to find first $\varrho(\mathbf{y})$ and then its gradient to obtain the theoretical signed-rank function $\mathbf{Q}(\mathbf{y})$.

Since now f is spherical, $\varrho(\mathbf{y}) = \varrho_0(r)$ depends on $\mathbf{y} = r\mathbf{u}$ only through its length r, and it is enough to study only the case $\mathbf{y} = (0 \quad \cdots \quad 0 \quad r)^T$.

The calculations now proceed as follows. See Möttönen and Oja (1994) for detailed calculations in the t distribution as well as in the multinormal case. As above, write $D = |\mathbf{y} - \mathbf{s}|$.

LEMMA A1. *The conditional distribution of $(x/\nu)D^2$ given x is a noncentral $\chi^2_k(r^2x/\nu)$.*

LEMMA A2.

$$\varrho_0(r) = E(D) = E(E(D|x)) = \sum_{i=0}^{\infty} c_i \nu^{1/2} \frac{[r^2/\nu]^i}{[1 + r^2/\nu]^{(\nu+2i-1)/2}},$$

where

$$c_i = \frac{\Gamma((k+2i+1)/2)\Gamma((\nu+2i-1)/2)}{i!\Gamma((k+2i)/2)\Gamma(\nu/2)}.$$

LEMMA A3. $\mathbf{Q}(r\mathbf{u}) = q(r)\mathbf{u}$, *where*

$$\begin{aligned} q(r) &= \varrho_0'(r) \\ &= -c_0(\nu-1)\left(\frac{k}{\nu}\right)^{1/2}\frac{F^{1/2}}{[1+(k/\nu)F]^{(\nu+1)/2}} \\ &\quad+\sum_{i=1}^{\infty}\left[2ic_i\left(\frac{k}{\nu}\right)^{i-1/2}\frac{F^{i-1/2}}{[1+(k/\nu)F]^{(\nu+2i+1)/2}}\right. \\ &\qquad\left.-c_i(\nu-1)\left(\frac{k}{\nu}\right)^{i+1/2}\frac{F^{i+1/2}}{[1+(k/\nu)F]^{(\nu+2i+1)/2}}\right] \end{aligned}$$

with $F = r^2/k$.

LEMMA A4. *Under a* $t_{\nu,k}$ *distribution*, $F = r^2/k \sim F(k,\nu)$ *and*

$$\begin{aligned} &E\{q^2(r)\} \\ &\quad= \frac{\Gamma((k+\nu)/2)\Gamma(3\nu/2)}{\Gamma(k/2)\Gamma^3(\nu/2)} \\ &\qquad\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\big[\Gamma((k+1)/2+i)\Gamma((k+1)/2+j)\Gamma((\nu-1)/2+i) \\ &\qquad\qquad\times\Gamma((\nu-1)/2+j)\Gamma(k/2+i+j)\big] \\ &\qquad\times\big[i!j!\Gamma(k/2+i)\Gamma(k/2+j)\Gamma((k+3\nu+2)/2+i+j)\big]^{-1} \\ &\qquad\times\Big[(5\nu^2+8\nu+2)ij-(\nu-1)(2\nu+1)(i^2+j^2) \\ &\qquad\quad-(\nu-1)\{2^{-1}(k-4)\nu+k-1\}(i+j)+4^{-1}k(k-2)(v-1)^2\Big] \\ &\qquad\times\big[i+j+2^{-1}(k-2)\big]^{-1} \end{aligned}$$

and

$$E\left\{q(r)\frac{r}{\nu+r^2}\right\} = \frac{2\Gamma((2\nu+1)/2)\Gamma((k+1)/2)\Gamma^2((\nu+1)/2)}{\nu^{1/2}\Gamma(k/2)\Gamma^2(\nu/2)\nu!(k+\nu)}.$$

LEMMA A5. *Under a* $t_{\nu,k}$ *distribution*,

$$\mathbf{A}_2 = E(\mathbf{Q}(\mathbf{y})\mathbf{L}^T(\mathbf{y})) = -\frac{\nu+k}{k}E\left\{q(r)\frac{r}{\nu+r^2}\right\}\mathbf{I},$$

$$\mathbf{B}_2 = E(\mathbf{Q}(\mathbf{y})\mathbf{Q}^T(\mathbf{y})) = \frac{1}{k}E\{q^2(r)\}\mathbf{I}$$

and

$$\mathbf{A}_2^T\mathbf{B}_2^{-1}\mathbf{A}_2 = \frac{(\nu+k)^2}{k}\left[E\left\{q(r)\frac{r}{\nu+r^2}\right\}\right]^2\big[E\{q^2(r)\}\big]^{-1}\mathbf{I}.$$

LEMMA A6. *The Pitman asymptotic relative efficiency of the multivariate spatial rank test relative to the Hotelling's test under a $t_{\nu,k}$ distribution is*

$$\mathrm{ARE}_2 = \frac{\boldsymbol{\delta}^T\big((\nu+k)^2/k\big)\big[E\{q(r)r/(\nu+r^2)\}\big]^2\big[E\{q^2(r)\}\big]^{-1}\mathbf{I}\boldsymbol{\delta}}{\boldsymbol{\delta}^T((\nu-2)/\nu)\mathbf{I}\boldsymbol{\delta}}$$

$$= \frac{\nu(\nu+k)^2}{k(\nu-2)}\left[E\left\{q(r)\frac{r}{\nu+r^2}\right\}\right]^2\big[E\{q^2(r)\}\big]^{-1}.$$

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