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BLOCKING IN REGULAR FRACTIONAL FACTORIALS: A PROJECTIVE GEOMETRIC APPROACH[1]

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A projective geometric characterization is given of the existence of any regular main effect s^{n-k} design in s^{γ} blocks. It leads to a constructive method for finding a maximal blocking scheme for any given fractional factorial design. A useful sufficient condition for admissible block designs is given in terms of the minimum aberration property of a certain unblocked design.

1. Introduction and preliminaries. Blocking is an effective method for reducing the variation in the comparison of treatments when the heterogeneity between blocks is much larger than the heterogeneity within blocks. Typical examples of blocking variables include time, location, batch, operator and so on. In the context of factorial and fractional factorial designs, a fundamental theoretical issue is how to choose good blocking schemes and to measure their "goodness." The classic works at the National Bureau of Standards (1957, 1959) contain many useful blocking schemes for two-level and three-level factorial and fractional factorial designs. Because no criterion for choosing these schemes was spelled out, it was not clear whether they are optimal in a reasonable sense. Sun, Wu and Chen (1997) made the first systematic attempt to understand this problem for two-level designs. As pointed out by these authors, the study of blocking in fractional factorial designs is complicated by the presence of two defining contrast subgroups, one for defining the fraction and another for defining the blocking scheme. Somewhat counterintuitively, they found that there are situations where a lower resolution design can be partitioned into more blocks than a higher resolution one. Even for designs with the same resolution, they noted that in some situations a minimum aberration 2^{n-k} design cannot be partitioned into the maximum number (i.e., 2^{n-k-1}) of blocks while a design with worse aberration can.

In the first part of this paper we develop a theory to explain the above phenomena for any s^{n-k} designs with prime power s. Our main tool is to use projective geometry to characterize the existence of an s^{n-k} design arranged

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in s^{γ} blocks so that the main effects are not confounded among each other or with the block effects. The characterization result in Theorem 1 provides a satisfactory explanation of the empirical findings described above. It also suggests a constructive method for studying the maximal blocking for a given unblocked s^{n-k} design. The second part of the paper is concerned with the class of admissible designs. Admissibility is defined in terms of the two defining contrast subgroups for treatments and for blocks. Even though admissibility is a weak criterion, it serves to rule out bad designs. There is no existing result on the characterization of admissible block designs. Theorem 2 of Section 4 gives a very useful sufficient condition for admissibility in terms of the minimum aberration property of a certain unblocked design. Many existing results on minimum aberration designs can be exploited in the search for admissible block designs. Several technical lemmas in Section 3 are useful for proving the two main results as well as being of general value. In the remaining part of this section we give some definitions and preparatory results.

Let $s(\geq 2)$ be a prime or prime power and consider the setup of an s^n factorial design. For $\nu \geq 1$, let Ω_ν denote the set of $\nu \times 1$ vectors defined over $GF(s)$. Then, as usual, a typical level combination x will be a member of Ω_n while a typical pencil b, corresponding to a factorial effect, will be a nonnull member of Ω_n. For nonzero λ in $GF(s)$, b and λb represent the same pencil. A main effect pencil is one that involves exactly one nonzero element.

We shall be concerned with s^{n-k} fractional factorial designs arranged in s^{γ} equal-sized blocks ($k, \gamma \geq 1$; $k + \gamma < n$). A design of this kind will be called a regular (s^{n-k}, s^{γ}) design. It is well known that such a design is specified by a pair of matrices H_1 and H_2, defined over $GF(s)$ and of orders $k \times n$ and $\gamma \times n$, respectively, such that

$$\text{rank}(H_1' \quad H_2') = k + \gamma, \tag{1}$$

where H_i' denotes the transpose of H_i. A typical block of the design consists of level combinations x satisfying

$$H_1 x = 0, \qquad H_2 x = \xi, \tag{2}$$

where ξ is a fixed member of Ω_γ. The s^{γ} blocks correspond to the s^{γ} possible choices of ξ in Ω_γ. A pencil b appears in the defining equation of such a design provided

$$b \in \mathscr{M}(H_1'), \tag{3}$$

where $\mathscr{M}(\cdot)$ denotes the column space of a matrix. A pencil b, not appearing in the defining equation [i..e, not satisfying (3)] is confounded with blocks provided

$$b \in \mathscr{M}(H_1' \quad H_2') \backslash \mathscr{M}(H_1'). \tag{4}$$

Let R be the minimum number of nonzero elements in a pencil satisfying (3) and $v + 1$ be the minimum number of nonzero elements in a pencil satisfying (4). In the unblocked case, the resolution of a design is given by R.

In the same spirit, we define the *resolution* of a block design as R^*, where

$$R^* = \begin{cases} \min(R, 2v + 1), & \text{if } R \text{ is odd,} \\ \min(R, 2v + 2), & \text{if } R \text{ is even.} \end{cases}$$

Then, as in the unblocked case, for any integer u not exceeding $\frac{1}{2}(R^* - 1)$, a block design of resolution R^* keeps all factorial effects involving u or less factors estimable when all effects involving $R^* - u$ or more factors are negligible. For a full factorial block design, the largest integer u^* not exceeding $\frac{1}{2}(R^* - 1)$ is called the *order of estimability* [Sun, Wu and Chen (1997)]. Since in the factorial setting the main effects are of primary interest, we shall hereafter consider only those regular (s^{n-k}, s^{γ}) designs which have resolution at least three. A design of this kind will be called a regular (s^{n-k}, s^{γ}) main effect design.

2. A projective geometric formulation: application to maximal blocking. In the unblocked case, it is well known that a fractional factorial design of resolution three or more can be characterized in terms of a set of distinct points in a finite projective geometry; see, for example, Bose (1947), Chen and Hedayat (1996) and Tang and Wu (1996). The corresponding development in the presence of blocks will be investigated now.

Let P denote the set of distinct points in the finite projective geometry $PG(n - k - 1, s)$. Since the points in $PG(n - k - 1, s)$ are given by the nonnull members of Ω_{n-k}, with mutually proportional members representing the same point, we have $\#P = L_{n-k}$, where # denotes the cardinality of a set and

$$L_{\nu} = (s^{\nu} - 1)/(s - 1), \qquad \nu = 0, 1, 2, \dots . \tag{5}$$

For any nonempty subset C of P, let $V(C)$ be an $(n - k) \times \nu$ matrix with columns given by the points in C, where $\nu = \#C$. A nonempty subset C_0 of P is called a subspace if, up to proportionality, C_0 is closed under the formation of nonnull linear combinations. Clearly, then $\#C_0 = L_u$, where $u = \text{rank}\{V(C_0)\}$. An ordered pair of nonempty subsets (C_0, C) of P, with $\#C_0 = \nu_0$ and $\#C = \nu$, will be referred to as a (ν_0, ν)-pair. Such a pair will be called *eligible* if (1) C_0 and C are disjoint and (2) C_0 is a subspace.

THEOREM 1. *The existence of an eligible (L_{γ}, n)-pair of subsets (C_0, C) of P, with $V(C)$ having full row rank, is equivalent to that of a regular (s^{n-k}, s^{γ}) main effect design such that*:

(i) *A pencil b appears in the defining equation of the design if and only if $V(C)b = 0$.*

(ii) *A pencil b does not appear in the defining equation of the design but is confounded with blocks if and only if $V(C)b$ is identical with some point in C_0.*

PROOF. First suppose an eligible (L_γ, n)-pair of subsets (C_0, C) of P, with $V(C)$ having full row rank, is available. Then $V(C)$ is an $(n-k)\times n$ matrix with rank $n-k$, which implies the existence of a $k\times n$ matrix H_1, defined over $GF(s)$, such that

$$\text{rank}(H_1) = k, \qquad V(C)H_1' = 0. \tag{6}$$

Since C_0 is a subspace with cardinality L_γ, there exists an $(n-k)\times\gamma$ matrix V_0, having full column rank, such that the columns of V_0 span C_0. As $V(C)$ has full row rank, we can find a $\gamma\times n$ matrix H_2 such that

$$V(C)H_2' = V_0. \tag{7}$$

From (6) and (7), one can check that the matrix $(H_1' \;\; H_2')$, with H_1 and H_2 defined as above, has full column rank; compare (1). Hence, as in the last section, starting from H_1 and H_2, a regular (s^{n-k}, s^γ) design can be constructed. By (2), (6) and (7), a typical block of this design will be of the form

$$\{x\colon x = V(C)'l, \text{ where } l\in\Omega_{n-k} \text{ and } V_0'l = \xi\}, \tag{8}$$

ξ being any fixed member of Ω_γ. It remains to show that this will be a main effect design for which (i) and (ii) hold.

The truth of (i) is obvious from (3) and (6). Next consider (ii) and observe that by (4), a pencil b is confounded with blocks without appearing in the defining equation if and only if $b - H_2'\xi \in \mathscr{M}(H_1')$ for some $\xi\neq 0$. By (6) and (7), this is equivalent to $V(C)b = V_0\xi$ for some $\xi\neq 0$, which, by the definition of V_0, happens if and only if $V(C)b$ is identical with some point in C_0. This proves (ii). It remains to prove that it is a main effect design. Since the points in C are distinct, by (i), each pencil appearing in the defining equation has at least three nonzero elements. Furthermore, for any main effect pencil b, $V(C)b\in C$ and, as C_0 and C are disjoint, by (ii) no main effect pencil is confounded with blocks. Thus the pair (C_0, C) leads to a main effect design for which (i) and (ii) hold.

The converse can be proved by reversing the above steps. □

In view of Theorem 1, studying regular (s^{n-k}, s^γ) main effect designs is equivalent to considering eligible (L_γ, n)-pairs of subsets (C_0, C) of P with $V(C)$ having full row rank. The main effect design arising from any such eligible pair (C_0, C) [cf. (8)] will be denoted by $d(C_0, C)$. Considering the cardinalities of C_0, C and P in Theorem 1 [see (5)], the following corollary is evident. Hereafter, n, k, γ and s will be assumed to be such that condition (9), as stated below, holds.

COROLLARY 1. *Let $s(\geq 2)$ be a prime or prime power and n, k, γ be positive integers such that $k+\gamma<n$. Then for the existence of a regular (s^{n-k}, s^γ) main effect design, it is necessary and sufficient that*

$$L_\gamma + n \leq L_{n-k}, \text{ that is, } n\leq (s^{n-k}-s^\gamma)/(s-1). \tag{9}$$

Identifying the blocks as the levels of another "factor", the result given above is anticipated also from Theorem 2 of Wu, Zhang and Wang (1992).

Given an unblocked s^{n-k} fractional factorial design d_{un} of resolution at least three, Theorem 1 can be employed to find the *maximum* possible blocking of d_{un} retaining the estimability of the main effect pencils. The search can be done in two steps.

1. Represent d_{un} by a subset C of P such that $V(C)$ has full row rank and $\#C = n$.
2. Given C, find the maximal γ such that $P - C$ contains a subspace of cardinality L_γ. The maximal blocking scheme has its generators in the subspace.

We present some illustrative examples, the first two of which settle the cases $k = 1$ and $k = 2$, respectively. In the sequel, for $n - k \ge 3$, the subspaces C_0^*, $C_i^*, C_{i_1 i_2}^*$, with cardinalities L_{n-k-1}, L_{n-k-2}, L_{n-k-2}, respectively, are defined as

$$
\begin{aligned}
C_0^* &= \{\alpha = (\alpha_1, \dots, \alpha_{n-k})' \colon \alpha \in P,\ \alpha_1 + \cdots + \alpha_{n-k} = 0\},\\
C_i^* &= \{\alpha = (\alpha_1, \dots, \alpha_{n-k})' \colon \alpha \in C_0^*,\ \alpha_i = 0\}, \qquad 1 \le i \le n-k,\\
C_{i_1 i_2}^* &= \{\alpha = (\alpha_1, \dots, \alpha_{n-k})' \colon \alpha \in P,\ \alpha_1 + \cdots + \alpha_{n-k} = \alpha_{i_1} = \alpha_{i_2}\},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \le i_1 < i_2 \le n-k.
\end{aligned} \tag{10}
$$

EXAMPLE 1. Let $k = 1$ and, to avoid trivialities, let $n \ge 4$. Without loss of generality, let d_{un} be represented by $C = \{e_1, \dots, e_{n-1}, y\}$, where $e_1, \dots, e_{n-1}$ are the unit vectors, of order $n - 1$, defined over $GF(s)$ and $y = (y_1, \dots, y_{n-1})'$. By Corollary 1, if $P - C$ contains a subspace of cardinality L_γ, then

$$\gamma \le n - 2. \tag{11}$$

(a) First consider the case $s \ge 3$ and, without loss of generality, let $y_1 \ne 0$. As $s \ge 3$, there exists $\lambda(\ne 0) \in GF(s)$ such that $\lambda \ne y_1^{-1}(y_2 + \cdots + y_{n-1})$. The $n - 2$ points $e_1 + \lambda e_i$, $2 \le i \le n - 1$, are linearly independent and it can be seen that the subspace spanned by them does not contain any member of C. Hence for $s \ge 3$, the bound (11) is always attainable.

(b) If $s = 2$ and $y \notin C_0^*$ then $C_0^* \subset P - C$ and the bound (11) is attainable.

(c) Now suppose $s = 2$ and $y \in C_0^*$ and, if possible, let the bound (11) be attainable. Then $P - C$ contains a subspace, say $\tilde{C}$, of cardinality L_{n-2}. As P itself has cardinality L_{n-1}; P can be spanned by y, which is outside $\tilde{C}$, and points in $\tilde{C}$, which implies that $e_i = y + q_i$, $1 \le i \le n - 1$, for some $q_1, \dots, q_{n-1} \in \tilde{C}$. Since $y \in C_0^*$, the points $q_i(= e_i + y)$, $1 \le i \le n - 1$, are linearly independent. But this is impossible as they belong to a subspace of cardinality L_{n-2}. Hence for $s = 2$ and $y \in C_0^*$, equality cannot hold in (11), that is, $\gamma \le n - 3$. Since $P - C \supset C_i^*$ for any i such that $y_i \ne 0$, the attainability of the bound $\gamma \le n - 3$ is evident.

Continuing with $s = 2$, let $d_{\text{un}}^{(1)}$ and $d_{\text{un}}^{(2)}$ denote the unblocked 2^{n-1} designs with the highest and second highest resolution, respectively. Then, without loss of generality, $d_{\text{un}}^{(1)}$ and $d_{\text{un}}^{(2)}$ correspond to $y = y^{(1)}$ and $y = y^{(2)}$, respectively, where $y^{(1)} = (1, 1, \dots, 1)'$ and $y^{(2)} = (0, 1, \dots, 1)'$ are both of order $n - 1$. For *even* n, $y^{(1)} \notin C_0^*$ so that by (b) above $d_{\text{un}}^{(1)}$ *allows* a partitioning into 2^{n-2} blocks. On the other hand for *odd* n, $y^{(1)} \in C_0^*$ and $y^{(2)} \notin C_0^*$. Hence, by (b) and (c) above, the maximum resolution design $d_{\text{un}}^{(1)}$ *does not allow* a partitioning into 2^{n-2} blocks while the next best design $d_{\text{un}}^{(2)}$ does so. This provides a theoretical explanation for a conflict, for *odd* n, between maximum resolution and maximal blocking that was earlier noted by Sun, Wu and Chen (1997) in the special cases $n = 5, 7$.

EXAMPLE 2. Let $k = 2$ and, to avoid trivialities, suppose $n \geq 5$. Without loss of generality, represent d_{un} by $C = \{e_1, \dots, e_{n-2}, y, z\}$, where $e_1, \dots, e_{n-2}$ are the unit vectors, of order $n - 2$, over $GF(s)$ and $y = (y_1, \dots, y_{n-2})'$, $z = (z_1, \dots, z_{n-2})'$. By Corollary 1, if $P - C$ contains a subspace of cardinality L_γ, then

$$\gamma \leq n - 3. \tag{12}$$

We note that the bound (12) is attainable if there exist nonzero elements $\lambda_1, \dots, \lambda_{n-2}$ of $GF(s)$ such that

$$\sum_{i=1}^{n-2} \lambda_i y_i \neq 0, \qquad \sum_{i=1}^{n-2} \lambda_i z_i \neq 0. \tag{13}$$

This is because then the $n - 3$ points $\lambda_i e_1 - \lambda_1 e_i$, $2 \leq i \leq n - 2$, are linearly independent and the subspace spanned by them does not contain any member of C. We consider various cases below.

(a) First suppose $s \geq 4$ and, without loss of generality, let $y_1 \neq 0$. If $z_1 \neq 0$, then as $s \geq 4$, there exist nonzero elements $\lambda_1, \dots, \lambda_{n-2}$ of $GF(s)$ such that $\lambda_2, \dots, \lambda_{n-2}$ are arbitrary and λ_1 is different from both $-y_1^{-1}\sum_{i=2}^{n-2}\lambda_i y_i$ and $-z_1^{-1}\sum_{i=2}^{n-2}\lambda_i z_i$. On the other hand, if $z_1 = 0$, then without loss of generality, $z_2 \neq 0$ and there exist nonzero elements $\lambda_1, \dots, \lambda_{n-2}$ of $GF(s)$ such that $\lambda_3, \dots, \lambda_{n-2}$ are arbitrary and $\lambda_2 \neq -z_2^{-1}\sum_{i=3}^{n-2}\lambda_i z_i$, $\lambda_1 \neq -y_1^{-1}\sum_{i=2}^{n-2}\lambda_i y_i$. In either case, we get nonzero $\lambda_1, \dots, \lambda_{n-2}$ satisfying (13) and hence, for $s \geq 4$, the bound (12) is attainable.

(b) Consider next the case $s = 3$ and, without loss of generality, again let $y_1 \neq 0$. If $z_1 = 0$ then exactly as in (a) above the bound (12) is attainable. Now let $z_1 \neq 0$ and suppose both y_i and z_i are nonzero for two other choices of i, say $i = 2$ and 3. Then there exist nonzero elements $\lambda_1, \dots, \lambda_{n-2}$ of $GF(3)$ such that $\lambda_4, \dots, \lambda_{n-2}$ are arbitrary and

$$\lambda_3 \neq -z_3^{-1}\sum_{i=4}^{n-2}\lambda_i z_i, \qquad \lambda_2 = -z_2^{-1}\sum_{i=3}^{n-2}\lambda_i z_i, \qquad \lambda_1 \neq -y_1^{-1}\sum_{i=2}^{n-2}\lambda_i y_i.$$

Then (13) is satisfied and the bound (12) is attainable. Thus, for $s = 3$, the bound (12) can be attained unless $y_i = z_i = 0$ for exactly $n - 4$ choices of i. In the latter case, writing, without loss of generality, $y = (1, 1, 0, \dots, 0)'$, $z = (1, 2, 0, \dots, 0)'$, it can be shown that equality cannot hold in (12), though $\gamma = n - 4$ is attainable as $P - C \supset C_{12}^*$ [see (10)]. While any unblocked design corresponding to this latter case has resolution three, it is easily seen that for $n \geq 6$, unblocked 3^{n-2} designs of resolution greater than three exist. Hence for $n \geq 6$, all 3^{n-2} unblocked designs with highest resolution attain the bound (12). For $n = 5$, the 3^{5-2} unblocked minimum aberration design is given by $y = (1, 1, 0)$ and $z = (1, 2, 1)$. It is easily seen that (12) holds for this design.

(c) Now suppose $s = 2$. If $y \notin C_0^*$, $z \notin C_0^*$, then $C_0^* \subset P - C$ and the bound (12) is attainable. On the other hand, if either of y or z belongs to C_0^*, then as in Example 1, equality cannot hold in (12). In this case, $\gamma = n - 4$ is attainable. If $y_i \neq 0$, $z_i \neq 0$ for some i, say $i = 1$, then this follows by noting that $P - C \supset C_1^*$. Otherwise, there exist i_1, i_2 ($i_1 \neq i_2$) such that $y_{i_1} \neq 0$, $z_{i_1} = 0$, $y_{i_2} = 0$, $z_{i_2} \neq 0$ and $P - C \supset C_{i_1 i_2}^*$ [see (10)].

Continuing with $s = 2$, we now consider unblocked 2^{n-2} designs of maximum resolution.

(i) Let $n = 3t + 1$, $t \geq 2$. Then, up to renaming of factors, there are three distinct unblocked 2^{n-2} designs with maximum resolution $2t$. These designs, namely $d_{\text{un}}^{(i)}$, can be represented by $C^{(i)} = \{e_1, \dots, e_{n-2}, y, z^{(i)}\}$, $1 \leq i \leq 3$, where $y = e_1 + \cdots + e_{2t-1}$, $z^{(1)} = e_t + \cdots + e_{3t-2}$, $z^{(2)} = e_{t+1} + \cdots + e_{3t-1}$, $z^{(3)} = e_t + \cdots + e_{3t-1}$. It can be verified that $d_{\text{un}}^{(3)}$ is the unique minimum aberration unblocked design and that $d_{\text{un}}^{(2)}$ has less aberration than $d_{\text{un}}^{(1)}$. Since $z^{(3)} \in C_0^*$, by (c) above, *the minimum aberration design* $d_{\text{un}}^{(3)}$ *can be partitioned into* 2^{n-4} *blocks but not into* 2^{n-3} *blocks*. On the other hand, $d_{\text{un}}^{(1)}$ and $d_{\text{un}}^{(2)}$, which do not have minimum aberration, can be partitioned into 2^{n-3} blocks because $z^{(1)} \notin C_0^*$ and $z^{(2)} \notin C_0^*$.

(ii) Let $n = 3t + 2$, $t \geq 1$. Then, up to renaming of factors, the unique unblocked 2^{n-2} design with maximum resolution $2t + 1$ is d_{un} which corresponds to $y = e_1 + \cdots + e_{2t}$, $z = e_{t+1} + \cdots + e_{3t}$. By (c) above, d_{un} *can be partitioned into at most* 2^{n-4} *blocks* while, as in Example 1, for $t \geq 2$, one can find designs with resolution $2t$ for which partitioning into 2^{n-3} blocks is possible.

(iii) Let $n = 3t$, $t \geq 2$. Then, up to renaming of factors, the unique unblocked 2^{n-2} design with maximum resolution $2t$ is d_{un} which corresponds to $y = e_1 + \cdots + e_{2t-1}$, $z = e_t + \cdots + e_{3t-2}$. By (c) above, (12) is attained and d_{un} *can be partitioned into* 2^{n-3} *blocks*.

The theoretical results in (i) and (ii) above confirm the empirical findings in Sun, Wu and Chen (1997) that for $n = 5$, 7 and 8 minimum aberration 2^{n-2} designs cannot be partitioned into the maximum number (i.e., 2^{n-3}) of blocks.

EXAMPLE 3. For nearly saturated unblocked designs, the cardinality of $P - C$, namely, $f^* = L_{n-k} - n$, is small and hence Theorem 1 can considerably simplify the study of maximal blocking. We briefly discuss this with reference to the minimum aberration unblocked designs reported in Tang and Wu (1996) for $s = 2$, $f^* \leq 11$ and Suen, Chen and Wu (1997) for $s = 3$,

$f^* \leq 13$. The main innovation in these two papers is to characterize the minimum aberration property of an unblocked s^{n-k} design, which is represented by C in the present paper, in terms of its complementary design, which is represented by $P - C$. In this context, Corollary 1 can be interpreted as saying that "an unblocked design can be partitioned into the maximum number of blocks if and only if its complementary design $P - C$ contains a subspace of maximum dimension." Verifying the above result on the complementary designs in these two papers, it can be seen that all the designs given there, except the one corresponding to $s = 2$, $f^* = 10$ allow maximum possible blocking.

For example, with $s = 3$, $f^* = 4$, the minimum aberration unblocked design is given by C such that $P - C = \{e_1, e_2, e_1 + e_2, e_1 + 2e_2\}$, which is itself a subspace, and thus the design can be partitioned into 3^2 blocks which is obviously the maximum possible; see (9). In this particular case, any other design which does not have minimum aberration can be partitioned into at most three blocks.

Turning to the exceptional situation $s = 2$, $f^* = 10$, Tang and Wu (1996) noted that the minimum aberration unblocked design is given by C such that $P - C = \{e_1, e_2, e_3, e_4, e_1 + e_2, e_1 + e_3, e_1 + e_4, e_2 + e_3, e_2 + e_4, e_3 + e_4\}$. Since $P - C$ does not include any point like $e_1 + e_2 + e_3$, it does not contain a subspace of cardinality L_3. Therefore, *any minimum aberration* 2^{n-k} *design with* $f^*(= 2^{n-k} - 1 - n) = 10$ *can be partitioned into* 2^2 *blocks but not into* 2^3 *blocks*. The next best unblocked design, according to the aberration criterion, corresponds to $P - C = \{e_1, e_2, e_3, e_1 + e_2, e_1 + e_3, e_2 + e_3, e_1 + e_2 + e_3, e_4, e_1 + e_4, e_2 + e_4\}$. Because $P - C$ contains a subspace of cardinality L_3, it can be partitioned into 2^3 blocks, which is the maximum possible.

3. Design criteria. The following notation will be helpful. For integers i and $\nu (0 \leq i \leq \nu,\ \nu \geq 1)$, let $\Omega_{i\nu}$ be the set of $\nu \times 1$ vectors over $GF(s)$ which involve exactly i nonzero elements. For any subset C, with cardinality $\nu (\geq 1)$, of P, let

$$A_i(C) = (s-1)^{-1} \#\{\beta: \beta \in \Omega_{i\nu}, V(C)\beta = 0\}, \qquad 0 \leq i \leq \nu. \tag{14}$$

Similarly, for any eligible pair of subsets (C_0, C) of P, where $\#C = \nu$, define

$$B_i(C_0, C) = (s-1)^{-1} \#\{\beta: \beta \in \Omega_{i\nu}, V(C)\beta \text{ is identical with some point in } C_0\}, \qquad 0 \leq i \leq \nu. \tag{15}$$

It is easily seen that for $i \geq 1$, both $A_i(C)$ and $B_i(C_0, C)$ are integers. Also,

$$\begin{aligned} A_0(C) = (s-1)^{-1}, \qquad A_1(C) = A_2(C) = 0, \\ B_0(C_0, C) = B_1(C_0, C) = 0. \end{aligned} \tag{16}$$

In particular, if (C_0, C) represents an eligible (L_γ, n)-pair of subsets with $V(C)$ having full row rank, then by Theorem 1, with reference to the design

$d(C_0, C)$, one can interpret $A_i(C)(i \geq 3)$ and $B_i(C_0, C)(i \geq 2)$, respectively, as the numbers of distinct i-factor interaction pencils which appear in the defining equation and which are confounded with blocks without appearing in the defining equation. Consider regular (s^{n-k}, s^{γ}) main effect designs $d_1 = d(C_{01}, C_1)$ and $d_2 = d(C_{02}, C_2)$, where (C_{01}, C_1) and (C_{02}, C_2) are eligible pairs of subsets. The design d_1 is said to have less aberration than the design d_2 with respect to the defining equation (written $d_1 \succ_{\mathrm{eq}} d_2$) if $A_i(C_1) = A_i(C_2)$ whenever $i < u$, and $A_u(C_1) < A_u(C_2)$, for some $u(3 \leq u \leq n)$. Similarly, d_1 is said to have less aberration than d_2 with respect to blocking (written $d_1 \succ_{\mathrm{bl}} d_2$) if $B_i(C_{01}, C_1) = B_i(C_{02}, C_2)$ whenever $i < u$, and $B_u(C_{01}, C_1) < B_u(C_{02}, C_2)$, for some $u(2 \leq u \leq n)$. We shall also write $d_1 \succeq_{\mathrm{eq}} d_2$ if either $d_1 \succ_{\mathrm{eq}} d_2$ or $A_i(C_1) = A_i(C_2)(3 \leq i \leq n)$. Similarly, we shall write $d_1 \succeq_{\mathrm{bl}} d_2$ if either $d_1 \succ_{\mathrm{bl}} d_2$ or $B_i(C_{01}, C_1) = B_i(C_{02}, C_2)(2 \leq i \leq n)$.

A design has *minimum aberration* of either type if there is no other design having less aberration of that type. Ideally, one should look for a design which has minimum aberration of both types simultaneously. However, as noted by Sun, Wu and Chen (1997), there often does not exist any such design. Hence, following them, one may consider the notion of admissibility. A design d is called *admissible* if there exists no other design d' such that both $d' \succeq_{\mathrm{eq}} d$ and $d' \succeq_{\mathrm{bl}} d$ hold with at least one of $d' \succ_{\mathrm{eq}} d$ and $d' \succ_{\mathrm{bl}} d$ being true. We now present some lemmas which are needed in the sequel and proved in the Appendix.

LEMMA 1. *Let C_0 be a subspace of P with cardinality L_γ.*

(a) *Then for $0 \leq i \leq L_\gamma$, $A_i(C_0) = M_i$ where the M_i's are constants which may depend on γ but not on the specific choice of the subspace C_0.*

(b) *Furthermore, for any $\alpha \in C_0$, $\lambda(\neq 0) \in GF(s)$ and $0 \leq i \leq L_\gamma$, the cardinality of the set $\{\beta\colon \beta \in \Omega_{iL_\gamma}, V(C_0)\beta = \lambda\alpha\}$ equals ψ_i, where*

$$\psi_i = L_\gamma^{-1}\left\{(s-1)^{i-1}\binom{L_\gamma}{i} - M_i\right\}. \tag{17}$$

In particular, by (16), (17) and Lemma 1(a),

$$M_0 = (s-1)^{-1},\ M_1 = 0,\ \psi_0 = 0,\ \psi_1 = 1,\ \psi_2 = \tfrac{1}{2}(s^\gamma - s). \tag{18a}$$

Also, define

$$M_i = \psi_i = 0 \quad \text{for } i > L_\gamma. \tag{18b}$$

LEMMA 2. *Let (C_0, C) be an eligible (L_γ, n)-pair of subsets of P where $n \geq 3$. Then*

$$A_i(C_0 \cup C) = A_i(C) + B_{i-1}(C_0, C) + \sum_{u=2}^{i}\{(s-1)M_u A_{i-u}(C) + \psi_u B_{i-u}(C_0, C)\}, \quad 3 \leq i \leq n+1,$$

where $A_{n+1}(C) = 0$.

For any nonempty proper subset F of P, let $\sigma(F)$ denote the subspace (of P) spanned by the points in F and write $\rho(F) = \text{rank}\{V(F)\}$. Also, let

$$\omega(F) = \begin{cases} 0, & \text{if } \sigma(F) = F, \\ \nu, & \text{otherwise, where } \mathrm{L}_\nu \text{ is the cardinality of} \\ & \text{the largest subspace contained in } \sigma(F) - F. \end{cases} \tag{19}$$

LEMMA 3. (a) *Let F be a nonempty proper subset of P. Then $P - F$ contains a subspace of cardinality L_γ if and only if*

$$\gamma + \rho(F) - \omega(F) \le n - k. \tag{20}$$

(b) *Suppose* (20) *and* (21) *hold, where*

$$\rho(F) \le n - k - 1 \quad \text{and} \quad (s, \gamma, \rho(F)) \ne (2, n-k-1, n-k-1). \tag{21}$$

Then for any choice of a subspace $C_0 (\subset P - F)$ of cardinality L_γ, the matrix $V(C)$ has full row rank where $C = P - (C_0 \cup F)$.

Examples can be given to demonstrate that the conclusion of part (b) of Lemma 3 may not hold without the condition (21).

4. Admissible block designs via minimum aberration unblocked designs. From Theorem 1, recall that a regular (s^{n-k}, s^γ) main effect design $d(C_0, C)$ is represented by an eligible (L_γ, n)-pair of subsets (C_0, C) of P such that $V(C)$ has full row rank. With $N = L_\gamma + n$ and $K = L_\gamma + k$, so that $N - K = n - k$, the set $C_0 \cup C$ consists of N distinct points of $PG(N - K - 1, s)$ and $V(C_0 \cup C)$, like $V(C)$, has full row rank. Hence, following Bose (1947) or Tang and Wu (1996), $C_0 \cup C$ represents an s^{N-K} unblocked design of resolution at least three (hereafter, called a resolution III$^+$ design) and this design will be denoted by $d_{\text{un}}(C_0 \cup C)$. We now present the following useful result which yields admissible block designs via minimum aberration unblocked designs.

THEOREM 2. *Let (C_0, C) be an eligible (L_γ, n)-pair of subsets of P such that $V(C)$ has full row rank and suppose $d_{\text{un}}(C_0 \cup C)$ is an s^{N-K} unblocked resolution* III$^+$ *design with minimum aberration. Then the regular (s^{n-k}, s^γ) main effect design $d(C_0, C)$ is admissible.*

PROOF. Suppose $d(C_0, C)$ is not admissible and is dominated by another design $d(C_0^*, C^*)$, where (C_0^*, C^*) is an eligible (L_γ, n)-pair of subsets with $V(C^*)$ having full row rank. Then, with reference to the statements (i) $A_i(C^*) = A_i(C)$ whenever $i < u_1$ and $A_{u_1}(C^*) < A_{u_1}(C)$ for some $u_1 (3 \le u_1 \le n)$, and (ii) $B_i(C_0^*, C^*) = B_i(C_0, C)$ whenever $i < u_2$ and $B_{u_2}(C_0^*, C^*) < B_{u_2}(C_0, C)$ for some $u_2 (2 \le u_2 \le n)$, one of the following mutually exclusive possibilities must arise:

(a) Statements (i) and (ii) both hold.
(b) Statement (i) holds and $B_i(C_0^*, C^*) = B_i(C_0, C)$, $2 \le i \le n$.
(c) Statement (ii) holds and $A_i(C^*) = A_i(C)$, $3 \le i \le n$.

Under (a), with $u = \min(u_1, u_2 + 1)$, by (16) and Lemma 2,

$$A_i(C_0^* \cup C^*) = A_i(C_0 \cup C) \text{ whenever } i < u \text{ and } A_u(C_0^* \cup C^*) < A_u(C_0 \cup C).$$

But then $d_{\text{un}}(C_0 \cup C)$ does not remain an unblocked minimum aberration design. The same contradiction arises under (b) and (c). This proves the result. □

REMARK 1. (a) Let $n - k = 2$. Then $\gamma = 1$ (as $n > k + \gamma$), $N - K = 2$ and, as noted in Cheng and Mukerjee (1998), all s^{N-K} unblocked resolution III$^+$ designs are equivalent under the criterion of aberration. Hence by Theorem 2, all regular (s^{n-k}, s^{γ}) main effect designs are then admissible.

(b) Let $f = L_{n-k} - (L_\gamma + n)(= L_{N-K} - N)$. If $f = 0$, 1 or 2, then for same reason as above [see Tang and Wu (1996), Suen, Chen and Wu (1997)] all regular (s^{n-k}, s^{γ}) main effect designs are admissible. As the following example illustrates, this phenomenon, however, does not hold for $f \geq 3$.

EXAMPLE 4. Let $s = 2$, $n = 11$, $k = 7$, $\gamma = 1$. Then $f = 3$. Let C_0, C_0^* and C be subsets of P such that $C_0 = \{(0,0,1,0)'\}$, $C_0^* = \{(1,0,0,0)'\}$, and

$$V(C) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then $V(C)$ has full row rank and $B_2(C_0, C) = 4$, $B_2(C_0^*, C) = 5$. Hence $d(C_0^*, C)$, being dominated by $d(C_0, C)$, is inadmissible. Incidentally, by Theorem 2, $d(C_0, C)$ itself is admissible since $P - (C_0 \cup C)$ is a subspace so that $d_{\text{un}}(C_0 \cup C)$ has minimum aberration; compare Example 1 of Tang and Wu (1996).

Theorem 2 serves as a powerful tool for obtaining admissible block designs. As Example 4 illustrates, it enables us to consider the use of every available result or table on unblocked minimum aberration designs in this context. Recently, there has been considerable interest in the characterization of unblocked minimum aberration designs in terms of complementary sets; see Chen and Hedayat (1996), Tang and Wu (1996) and Suen, Chen and Wu (1997). The problem of characterizing an unblocked minimum aberration design as being part of an eligible (L_γ, n)-pair remains unresolved.

Theorem 2, when employed jointly with Lemma 3, makes it particularly easy to obtain admissible block designs from such results. The steps in this regard are as follows:

(i) For given $n - k$, let F be a proper subset of P such that $d_{\text{un}}(P - F)$ is an unblocked resolution III$^+$ design with minimum aberration. Let $f = \#F$.

(ii) Find $\sigma(F)$, $\rho(F)$ and $\omega(F)$ as in the context of Lemma 3.

(iii) Suppose there exists a positive integer γ satisfying (20). Then $P - F$ contains a subspace C_0 with cardinality L_γ and C_0 can be actually found following the proof of the "if" part of Lemma 3 which is constructive.

(iv) Obtain $C = P - F - C_0$. Suppose (21) holds. Then $V(C)$ has full row rank and, by Theorem 2, the block design $d(C_0, C)$, representing a regular (s^{n-k}, s^{γ}) main effect plan, is admissible. Here $n = L_{n-k} - f - L_{\gamma}$ and $k = L_{n-k} - f - L_{\gamma} - (n-k)$.

Examples 5–10 below illustrate the above steps. In view of Remark 1, we are primarily concerned with the situation $n - k \geq 3$ and $f \geq 3$ in these examples.

EXAMPLE 5. Let $n - k \geq 3$ and F be a subspace of P with cardinality L_{ρ}, where $2 \leq \rho \leq n - k - 1$. Then $\sigma(F) = F$, $\omega(F) = 0$, $\rho(F) = \rho$. Since F is a subspace, using Rule 1 of Suen, Chen and Wu (1997), $d_{\text{un}}(P - F)$ has minimum aberration. Since $n - k \geq 3$, both (20) and (21) hold if and only if $\gamma \leq n - k - \rho$. For any such γ, steps (i)–(iv) yield an admissible regular (s^{n-k}, s^{γ}) main effect design where $n = L_{n-k} - L_{\rho} - L_{\gamma}$ and $k = L_{n-k} - L_{\rho} - L_{\gamma} - (n-k)$.

For numerical illustration, let $s = 3$, $n - k = 4$, $\rho = 2$, $\gamma = 2$. Then we get an admissible $(3^{32-28}, 3^2)$ design.

EXAMPLE 6. Let $n - k \geq 3$ and $F(\subset P)$ be such that $\rho(F) = 2$. Suppose F is not a subspace since this situation is already covered by Example 5. Then $\omega(F) = 1$ and as noted in Section 5 of Cheng and Mukerjee (1998), $d_{\text{un}}(P - F)$ has minimum aberration. Both (20) and (21) hold if and only if (a) $\gamma = 1$ when $(s, n-k) = (2, 3)$, (b) $\gamma \leq n - k - 1$ when $(s, n-k) \neq (2, 3)$. For any such γ, steps (i)–(iv) yield an admissible regular (s^{n-k}, s^{γ}) main effect design where $n = L_{n-k} - f - L_{\gamma}$ and $k = L_{n-k} - f - L_{\gamma} - (n-k)$, with $f = \#F$.

For numerical illustration, let $s = 4$, $n - k = 3$, $f = 4$, $\gamma = 2$. Then we get an admissible $(4^{12-9}, 4^2)$ design.

EXAMPLE 7. Let $s = 2$, $n - k \geq 4$ and $F(\subset P)$ be obtained by deleting any u distinct point(s) from a subspace with cardinality L_{ρ}, where $3 \leq \rho \leq n - k - 1$, $u \in \{1, 2, 3\}$ and if $u = 3$ the three deleted points are noncollinear. Then $\rho(F) = \rho$, $\omega(F) = 1$ and, following Section 4 of Cheng and Mukerjee (1998), $d_{\text{un}}(P - F)$ has minimum aberration. Both (20) and (21) hold if and only if $\gamma \leq n - k - \rho + 1$. For any such γ, steps (i)–(iv) yield an admissible regular $(2^{n-k}, 2^{\gamma})$ main effect design where, by (5), $n = 2^{n-k} - 2^{\rho} - 2^{\gamma} + u + 1$ and $k = 2^{n-k} - 2^{\rho} - 2^{\gamma} + u + 1 - (n-k)$.

For numerical illustration, let $s = 2$, $n - k = 5$, $\rho = 3$, $u = 3$, $\gamma = 3$. Then we get an admissible $(2^{20-15}, 2^3)$ design.

EXAMPLE 8. Continuing with $s = 2$, we now consider some more admissible block designs arising from unblocked minimum aberration designs with small $f(= \#F)$. Specifically, we are interested in the situations $n - k \geq 4$, $3 \leq f \leq 9$ and $n - k \geq 5$, $10 \leq f \leq 15$ (note that $n - k = 4$ leads to trivialities for $10 \leq f \leq 15$). Examples 5 and 7 cover the cases $n - k \geq 4$, $3 \leq f \leq 7$ and $n - k \geq 5$, $12 \leq f \leq 15$.

Now, for $n-k \geq 4$, $f=8,9$ or $n-k \geq 5$, $f=10,11$, from Tang and Wu (1996), one can find $F(\subset P)$, with cardinality f, such that $d_{\mathrm{un}}(P-F)$ has minimum aberration. For each such f, it can be seen from their results that $\rho(F)=4$ and $\omega(F)=1$. Hence (20) holds provided $\gamma \leq n-k-3$ and for any such γ, (21) also holds unless $(n-k, f, \gamma)=(4,8,1)$ or $(4,9,1)$. However, from the details on F recorded in Tang and Wu (1996), one can check that even with these two exceptional triplets, $V(C)$ has full row rank for every choice of C_0. Thus for $\gamma \leq n-k-3$, steps (i)–(iv) yield an admissible regular $(2^{n-k}, 2^{\gamma})$ main effect design where $n=2^{n-k}-f-2^{\gamma}$ and $k=2^{n-k}-f-2^{\gamma}-(n-k)$.

For numerical illustration, let $s=2$, $n-k=5$, $f=10$, $\gamma=2$. Then we get an admissible $(2^{18-13}, 2^2)$ design.

EXAMPLE 9. In the spirit of the last example, we take $s=3$ and explore admissible block designs arising from minimum aberration unblocked designs for $n-k \geq 3$, $3 \leq f \leq 8$ and $n-k \geq 4$, $9 \leq f \leq 13$ (note that $n-k=3$ leads to trivialities for $9 \leq f \leq 13$), where $f=\#F$. Examples 5 and 6 cover the cases $n-k \geq 3$, $f=3,4$ and $n-k \geq 4$, $f=13$.

For $n-k \geq 3$, $5 \leq f \leq 8$ or $n-k \geq 4$, $9 \leq f \leq 12$, from Suen, Chen and Wu (1997), one can find $F(\subset P)$, with cardinality f, such that $d_{\mathrm{un}}(P-F)$ has minimum aberration. For any such f, it can be seen from their results that $\rho(F)=3$ and $\omega(F)=1$. Hence (20) holds, provided $\gamma \leq n-k-2$ and for any such γ, (21) also holds unless $(n-k, f, \gamma)=(3,5,1)$, $(3,6,1)$, $(3,7,1)$, or $(3,8,1)$. However, from the details on F given by Suen, Chen and Wu (1997), one can check that even with these four exceptional triplets $V(C)$ has full row rank for every choice of C_0. Hence for $\gamma \leq n-k-2$, steps (i)–(iv) yield an admissible regular $(3^{n-k}, 3^{\gamma})$ main effect design where $n=\frac{1}{2}(3^{n-k}-3^{\gamma})-f$ and $k=\frac{1}{2}(3^{n-k}-3^{\gamma})-f-(n-k)$.

For numerical illustration, let $s=3$, $n-k=3$, $f=5$, $\gamma=1$. Then we get an admissible $(3^{7-4}, 3^1)$ design.

EXAMPLE 10. We now illustrate how, given s, n and k, Remark 1 and Examples 5–9 can help in obtaining admissible designs for various values of γ. Let $s=2$, $n=47$ and $k=41$; then by (10), $1 \leq \gamma \leq 4$. If $\gamma=1$ then for any regular $(2^{47-41}, 2^1)$ main effect design $d(C_0, C)$, the cardinality of $P-(C_0 \cup C)$ equals 15 and an admissible $(2^{47-41}, 2^1)$ design can be obtained from Example 5 with $\rho=4$. Similarly, for $\gamma=2,3,4$, admissible designs are given, respectively, by Example 7 (with $\rho=4$, $u=2$), Example 8 (with $f=9$) and Remark 1(b) (with $f=1$).

REMARK 2. Examples can be given to show that the condition in Theorem 2 is sufficient but not necessary for a block design to be admissible. Notwithstanding this, Theorem 2 is very useful for several reasons. First, in most situations complete characterization of admissible block designs is extremely difficult but Theorem 2 can potentially yield at least one admissible design. Second, in the unblocked case, quite often only minimum aberration designs give the smallest possible value of A_3 and, as such, in many situations, admissible block designs for which $d_{\text{un}}(C_0 \cup C)$ has minimum aberration have smaller values of $A_3(C_0 \cup C)$ than those for which $d_{\text{un}}(C_0 \cup C)$ does not have minimum aberration; this happens in all the cases covered by Examples 5–9 except for Example 8 with $f = 10, 11$. Since $A_3(C_0 \cup C) - M_3 = A_3(C) + B_2(C_0, C)$ by Lemma 2, in such situations admissible designs arising from Theorem 2 have an appeal in the sense of exercising a control on "overall" aberration. Furthermore, in view of the recent findings in Cheng and Mukerjee (1998) in the unblocked case, we believe that Theorem 2 should be capable of producing admissible block designs that perform well with regard to the estimation capacity criterion given in their paper.

REMARK 3. Admissible designs generated by Theorem 2 can be further discriminated by using a minimum aberration criterion such as the one proposed by Sitter, Chen and Feder (1997) for blocked 2^{n-k} designs.

APPENDIX

Proofs of Lemmas.

PROOF OF LEMMA 1. The proof of (a) is not hard and we present only the proof of (b). As C_0 is a subspace, given any $\beta \in \Omega_{iL_\gamma}$, the vector $V(C_0)\beta$ is either null or equals $\lambda\alpha$ for some $\alpha \in C_0$ and some $\lambda(\neq 0) \in GF(s)$. Also, by symmetry argument, the cardinality of the set under consideration does not depend on α and λ as long as $\alpha \in C_0$ and $\lambda(\neq 0) \in GF(s)$. Hence, considering all possible choices of β in Ω_{iL_γ}, by (14), we have

$$\binom{L_\gamma}{i}(s-1)^i = (s-1)A_i(C_0) + \psi_i(s-1)L_\gamma,$$

whence using part (a) above, the result follows. Note that the ψ_i's, like the M_i's, may depend on γ but not on the specific choice of C_0. □

PROOF OF LEMMA 2. Let $N = L_\gamma + n$ and Σ^* denote double summation with respect to α and λ such that $\alpha \in C_0$ and $\lambda(\neq 0) \in GF(s)$. Also, for fixed $i(3 \leq i \leq n+1)$, let Σ'_u denote summation with respect to u over the range $\max(i-n, 0) \leq u \leq \min(L_\gamma, i)$. Then for $3 \leq i \leq n+1$, noting that C_0 is a subspace and using (14), (15) and Lemma 1,

$$\begin{aligned}&(s-1)A_i(C_0 \cup C)\\ &\quad = \Sigma'_u \#\left\{(\beta_1', \beta_2')' \colon \beta_1 \in \Omega_{uL_\gamma},\ \beta_2 \in \Omega_{(i-u)n},\ V(C_0)\beta_1 + V(C)\beta_2 = 0\right\}\end{aligned}$$

$$
\begin{aligned}
&= \Sigma'_u\Big[\#\big\{\beta_1\colon \beta_1 \in \Omega_{uL_\gamma}, V(C_0)\beta_1 = 0\big\}\#\big\{\beta_2\colon \beta_2 \in \Omega_{(i-u)n}, V(C)\beta_2 = 0\big\} \\
&\qquad + \Sigma^*\#\big\{\beta_1\colon \beta_1 \in \Omega_{uL_\gamma}, V(C_0)\beta_1 = \lambda\alpha\big\} \\
&\qquad\quad \times\#\big\{\beta_2\colon \beta_2 \in \Omega_{(i-u)n}, V(C)\beta_2 = -\lambda\alpha\big\}\Big] \\
&= \Sigma'_u\Big[(s-1)^2 A_u(C_0)A_{i-u}(C) \\
&\qquad + \psi_u\Sigma^*\#\big\{\beta_2\colon \beta_2 \in \Omega_{(i-u)n}, V(C)\beta_2 = -\lambda\alpha\big\}\Big] \\
&= \Sigma'_u\big\{(s-1)^2 M_u A_{i-u}(C) + \psi_u(s-1)B_{i-u}(C_0, C)\big\},
\end{aligned}
$$

whence, using (18a, b) and recalling the definition of Σ'_u, the result follows. □

PROOF OF LEMMA 3. (a) We write $\rho = \rho(F)$ and $\omega = \omega(F)$ for notational simplicity and proceed as with ordinary finite dimensional vector spaces in proving this part.

If. Consider the case $\omega > 0$ (the treatment is similar for $\omega = 0$). Let F^* be a subspace, of cardinality L_ω, contained in $\sigma(F) - F$ [see (19)]. Let T_1, T_2 and T_3 be subsets of P, with respective cardinalities ω, ρ and $n-k$, such that $T_1 \subset T_2 \subset T_3$ and T_1, T_2 and T_3 span F^*, $\sigma(F)$ and P, respectively. Clearly, then the points in T_3 are linearly independent so that $(T_3 - T_2) \cup T_1$ contains $(n - k - \rho + \omega)$ linearly independent points. If (20) holds, that is, if $\gamma \le n - k - \rho + \omega$, then any γ-subset of $(T_3 - T_2) \cup T_1$ will span a subspace which is contained in $P - F$ and has cardinality L_γ.

Only If. Let $P - F$ contain a subspace C_0 of cardinality L_γ. Note that $C_0 \cap \sigma(F)$ is a subspace if it is nonempty. Hence $\#\{C_0 \cap \sigma(F)\} = L_u$, for some nonnegative integer u. Since $C_0 \cap \sigma(F) \subset \sigma(F) - F$, by (19), $u \le \omega$. Hence

$$\gamma + \rho - \omega \le \gamma + \rho - u = \operatorname{rank}\big\{V(C_0 \cup \sigma(F))\big\} \le n - k,$$

which proves (20).

(b) Let (20) and (21) hold. Since F is nonempty, we have $\rho > \omega$, so that by (20),

$$\gamma \le n - k - 1. \tag{A.1}$$

Now, $C \supset C_1$ where $C_1 = P - (C_0 \cup \sigma(F))$ and it is enough to show that $V(C_1)$ has full row rank. Assume the contrary. Then there exists $l(\ne 0) \in \Omega_{n-k}$ such that $V(C_1)'l = 0$. Then $C_1 \subset C_2$ where C_2 is a subspace, of cardinality L_{n-k-1}, defined as $C_2 = \{\alpha\colon \alpha \in P,\ l'\alpha = 0\}$. Hence, recalling the definition of C_1,

$$
\begin{aligned}
L_{n-k} - L_{n-k-1} &= \#(P - C_2) = \#(P - C_1) - \#(C_2 - C_1) \\
&= \#(C_0 \cup \sigma(F)) - \#\{C_2 \cap (C_0 \cup \sigma(F))\} \qquad \text{(A.2)} \\
&\le \#C_0 - \#(C_2 \cap C_0) + \#\sigma(F) - \#(C_2 \cap \sigma(F)).
\end{aligned}
$$

We can write $C_0 = \{\alpha\colon \alpha \in P,\ Q\alpha = 0\}$, where Q is an $(n-k-\gamma) \times (n-k)$ matrix having full row rank. If C_0 is not a subspace of C_2, then l' does not belong to the row space of Q and, therefore, $\#(C_2 \cap C_0) = L_{\gamma-1}$. On the other hand, if $C_0 \subset C_2$ then $\#(C_2 \cap C_0) = \#C_0 = L_\gamma$. Hence $\#(C_2 \cap C_0) \geq L_{\gamma-1}$. Similarly, $\#(C_2 \cap \sigma(F)) \geq L_{\rho-1}$. Thus by (A.2),

$$L_{n-k} - L_{n-k-1} \leq L_\gamma - L_{\gamma-1} + L_\rho - L_{\rho-1}. \tag{A.3}$$

Using (5), (A.3) implies $s^{n-k-1} \leq s^{\gamma-1} + s^{\rho-1}$, which is, however, impossible by (21) and (A.1). This contradiction proves part (b). □

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