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STRONG LAWS OF LARGE NUMBERS FOR r-DIMENSIONAL ARRAYS OF RANDOM VARIABLES

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Let K_r be the set of r-tuples $\mathbf{k} = (k_1, k_2, \cdots, k_r)$ with positive integers for coordinates ($r \geqq 1$). Let $\{X_\mathbf{k} : \mathbf{k} \in K_r\}$ be a set of i.i.d. random variables with mean zero, and let $\leqq$ denote the coordinate-wise partial ordering on K_r. Set $|\mathbf{k}| = k_k k_2 \cdots k_r$ and define, for $\mathbf{k} \in K_r$: $S_\mathbf{k} = \sum_{\mathbf{j} \leqq \mathbf{k}} X_\mathbf{j}$. If $\{E_\mathbf{k} : \mathbf{k} \in K_r\}$ is a set of events indexed by K_r, we say (given ω) "$E_\mathbf{k}$ f.o." if $\exists\, \mathbf{l}(\omega) \in K_r$ such that $\mathbf{k} \nleqq \mathbf{l}$ implies $\omega \in E_\mathbf{k}^c$. We say "$E_\mathbf{k}$ a.l." if given any $\mathbf{l} \in K_r$, $\exists\, \mathbf{k} \geqq \mathbf{l}$ such that $\omega \in E_\mathbf{k}$. We prove:

(i) If $E\{|X_\mathbf{k}| (\log^+ |X_\mathbf{k}|)^{r-1}\} = \infty$, then given any $A > 0$, $P\{|S_\mathbf{k}|/|\mathbf{k}| > A \text{ a.l.}\} = 1$. Using martingale techniques, we also give a new proof of the converse result due to Zygmund:

(ii) If $E\{|X_\mathbf{k}| (\log^+ |X_\mathbf{k}|)^{r-1}\} < \infty$, then given any $\varepsilon > 0$, $P\{|S_\mathbf{k}|/|\mathbf{k}| < \varepsilon \text{ f.o.}\} = 1$.

For non-identically distributed independent random variables with mean zero, the usual conditions sufficient for convergence of S_n/n to zero in the linearly ordered case are also sufficient for matrix arrays.

1. Introduction and notation. We wish to consider the following problem: Given a probability space (Ω, F, P) and an r-dimensional array of independent random variables with zero mean defined on (Ω, F, P), under what conditions will the sample averages converge to zero?

To make this problem more precise it will be convenient to introduce some notation. Let K_r be the set of r-tuples $\mathbf{k} = (k_1, k_2, \cdots, k_r)$ with positive integers for coordinates ($r \geqq 1$). Let $\{X_\mathbf{k} : \mathbf{k} \in K_r\}$ be a set of independent random variables with mean zero and let $\leqq$ denote the coordinate-wise partial ordering on K_r. Set $|\mathbf{k}| = k_1 \cdot k_2 \cdots k_r$ and define, for $\mathbf{k} \in K_r$:

$$S_\mathbf{k} = \sum_{\mathbf{j} \leqq \mathbf{k}} X_\mathbf{j}\,, \qquad Z_\mathbf{k} = \frac{S_\mathbf{k}}{|\mathbf{k}|}\,.$$

If $\{E_\mathbf{k} : \mathbf{k} \in K_r\}$ is a set of events indexed by K_r, we say (given $w \in \Omega$) "$E_\mathbf{k}$ occurs finitely often" (abbreviated "$E_\mathbf{k}$ f.o.") if there exists $\mathbf{l}(w) \in K_r$ such that $\mathbf{k} \nleqq \mathbf{l}$ implies $w \in E_\mathbf{k}^c$. We say "$E_\mathbf{k}$ occurs for arbitrarily large indices" (abbreviated "$E_\mathbf{k}$ a.l.") if, given any $\mathbf{l} \in K_r$, there exists $\mathbf{k} \geqq \mathbf{l}$ such that $w \in E_\mathbf{k}$. (Note that when $r > 1$ this last condition is not equivalent to the condition "$E_\mathbf{k}$ occurs infinitely often", which is weaker.) Our problem can now be stated as follows:

Given $\varepsilon > 0$, what conditions on the $X_\mathbf{k}$ will guarantee that

$$P\{|Z_\mathbf{k}| \geqq \varepsilon \ \text{ f.o.}\} = 1?$$

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For $r = 1$, the result reduces in the identically distributed case to Kolmogorov's strong law of large numbers. For $r > 1$, the result is of interest in the study of properties of "Brownian sheets" $X_{t_1, t_2, \cdots, t_r}$ with an r-dimensional time parameter (see [5], [7]); for example, if $P\{|Z_{\mathbf{k}}| > \varepsilon \text{ f.o.}\} = 1$, one could deduce that (writing $\mathbf{t}$ for $(t_1, t_2, \cdots, t_r)$ with probability one,

$$\frac{X_{\mathbf{t}}}{|\mathbf{t}|} \to 0 \quad \text{as} \quad |\mathbf{t}| \uparrow \infty .$$

The identically distributed case has been treated in the context of ergodic theory by Dunford ([4]) and by Zygmund ([9]). Dunford proved that the integrability of $|X_{\mathbf{k}}| \log^+ |X_{\mathbf{k}}|$ was sufficient, when $r = 2$, to insure that $P\{|Z_{\mathbf{k}}| > \varepsilon \text{ a.l.}\} = 0$; implicit in Zygmund's treatment is the result that for general r, the integrability of $|X_{\mathbf{k}}| (\log^+ |X_{\mathbf{k}}|)^{r-1}$ is a sufficient condition for convergence.

We show in Section 2 that the sufficient condition given above is also necessary when the random variables are identically distributed. We give also a new proof of the sufficiency via martingale theory, specifically the inequalities established by Cairoli ([1]). In Section 3 we discuss the non-identically distributed case.

2. The identically distributed case.

THEOREM. *Let* $\{X_{\mathbf{k}} : \mathbf{k} \in K_r\}$ *be* i.i.d. *random variables with zero mean.*

(1) *If* $E\{|X_{\mathbf{k}}| (\log^+ |X_{\mathbf{k}}|)^{r-1}\} < \infty$, *then for any* $\varepsilon > 0$, $P\{|Z_{\mathbf{k}}| > \varepsilon \text{ f.o.}\} = 1$.

(2) *If* $E\{|X_{\mathbf{k}}| (\log^+ |X_{\mathbf{k}}|)^{r-1}\} = \infty$, *then for any* $A > 0$, $P\{|Z_{\mathbf{k}}| > A \text{ a.l.}\} = 1$.

PROOF. We prove first the necessity of the condition, i.e., (2). Assume that $E\,|X_{\mathbf{k}}| (\log^+ |X_{\mathbf{k}}|)^{r-1} = \infty$. Then

$$\begin{aligned} &\int_1^\infty \cdots \int_1^\infty P\{|X_{\mathbf{1}}| > |\mathbf{k}|\}\, dk_1 \cdots dk_r \\ &\quad = \int_1^\infty \cdots \int_1^\infty \left(\int_0^\infty 1_{[|\mathbf{k}| \leq \lambda]} F(d\lambda)\right) dk_1\, dk_2 \cdots dk_r \end{aligned}$$

where $F(\cdot)$ is the distribution function of $|X_{\mathbf{1}}|$. By simple integration using Fubini's theorem the above multiple integral is found to be

$$\int_0^\infty \lambda (\log^+ \lambda)^{r-1} F(d\lambda) + (\text{terms of lower order}) = \infty .$$

It therefore follows that $\sum_{k_1} \cdots \sum_{k_r} P\{|X_{\mathbf{1}}| \geqq A|\mathbf{k}|\} = \infty$ for any $A > 0$; hence that $\sum_{k_1} \cdots \sum_{k_r} P\{|X_{\mathbf{k}}| > A|\mathbf{k}|\} = \infty$.
By the (converse) Borel–Cantelli lemma one concludes that $P\{|X_{\mathbf{k}}| > A\,|\mathbf{k}| \text{ i.o.}\} = 1$; by taking r-dimensional differences of the partial sums this implies that $P\{|Z_{\mathbf{k}}| > A/2^r \text{ i.o.}\} = 1$ for any $A > 0$.

To strengthen this to the result that $P\{|Z_{\mathbf{k}}| > A \text{ a.l.}\} = 1$, it suffices to show that for arbitrarily large $\mathbf{k}$, $\sum_{\mathbf{l} \geqq \mathbf{k}} P\{|X_{\mathbf{l}}| > A\,|\mathbf{l}|\} = \infty$. But if $k_i \geqq 2$, $1 \leqq i \leqq r$, the series $\sum_{\mathbf{l} \geqq \mathbf{k}} P\{|X_{\mathbf{l}}| > A\,|\mathbf{l}|\}$ majorizes, term by term, the series $\sum_{\mathbf{l}} P\{|X_{\mathbf{l}}|/|\mathbf{k}| > A\,|\mathbf{l}|\}$ and by the argument above, this latter series diverges. □

We turn now to (1). For any $r \geqq 1$ and $\mathbf{k} \in k_r$, define $\mathscr{G}_{\mathbf{k}} = \bigvee_{\mathbf{j} \geq \mathbf{k}, \mathbf{j} \in K_r} \sigma(S_{\mathbf{j}})$

LEMMA 1. *$(Z_{\mathbf{k}}, \mathscr{G}_{\mathbf{k}})$ is a reversed martingale on K_r.*

PROOF. If $\mathbf{j} \leqq \mathbf{k}$ it is clear that $(E(X_{\mathbf{j}} \mid \mathscr{G}_{\mathbf{k}}), \mathscr{G}_{\mathbf{k}})$ is a reversed martingale. Now $\sum_{\mathbf{j} \leq \mathbf{k}} E(X_{\mathbf{j}} \mid \mathscr{G}_{\mathbf{k}}) = E(S_{\mathbf{k}} \mid \mathscr{G}_{\mathbf{k}}) = S_{\mathbf{k}}$, so it follows by symmetry that $Z_{\mathbf{k}} = E(X_{\mathbf{j}} \mid \mathscr{G}_{\mathbf{k}})$, proving the lemma.

LEMMA 2. *Fix k_r and let $\hat{\mathbf{k}}$ denote the element $(k_1, k_2, \cdots, k_{r-1})$ of K_{r-1}. Let $\hat{\mathbf{k}}, j$ represent the r-tuple $(k_1, k_2, \cdots, k_{r-1}, j)$. Then $Z_{\hat{\mathbf{k}}, k_r}, \mathscr{G}_{\hat{\mathbf{k}},1})$ is a reversed martingale on K_{r-1}.*

PROOF. Let $S_{\hat{\mathbf{k}}|m} = \sum_{\hat{\mathbf{j}} \leq \hat{\mathbf{k}}} X_{\hat{\mathbf{j}},m}$.
We have

$$\mathscr{G}_{\hat{\mathbf{k}},1} = \bigvee_{m=1}^{\infty} \sigma(S_{\hat{\mathbf{k}}|m}) \vee \bigvee_{\hat{\mathbf{n}} \geq \hat{\mathbf{k}}} \sigma(X_{\hat{\mathbf{n}},m}) .$$

Now

$$E(Z_{\hat{\mathbf{l}},k_r} \mid \mathscr{G}_{\hat{\mathbf{k}},1}) = \sum_{m=1}^{k_r} E\left(\frac{S_{\hat{\mathbf{l}}|m}}{|\hat{\mathbf{l}}| \cdot k_r} \,\Big|\, \mathscr{G}_{\hat{\mathbf{k}},1}\right) .$$

Using the independence of the $X_{\mathbf{k}}$, when $\mathbf{l} \leqq \mathbf{k}$ the sum is equal to

$$\sum_{m=1}^{k_r} E\left(\frac{S_{\hat{\mathbf{l}}|m}}{|\hat{\mathbf{l}}| \cdot k_r} \,\Big|\, S_{\hat{\mathbf{k}}|m}\right)$$

which by Lemma 1 is equal to $\sum_{m=1}^{k_r} S_{\hat{\mathbf{k}}|m}/|\hat{\mathbf{k}}|k_r = Z_{\hat{\mathbf{k}},k_r}$

REMARK. The proof of Lemma 2 can easily be extended to show that if any two coordinates k_i, k_j of $(k_1, k_2, \cdots, k_r)$ are fixed, $Z_{\mathbf{k}}$ is a reversed martingale indexed by the r-2-tuples $(k_1, \cdots, k_{i-1}, k_{i+1}, \cdots, k_{j-1}, k_{j+1}, \cdots, k_r)$, with respect to the fields $\mathscr{G}_{k_1, \cdots, k_{i-1}, 1, k_{i+1}, \cdots, k_{j-1}, 1, k_{j+1}, \cdots, k_r}$.

Following Cairoli we will now establish the basic inequalities.

LEMMA 3.

$$\lambda P\{\sup_{\mathbf{k}} |Z_{\mathbf{k}}| \geqq \lambda\} \leqq A_r \sup_{\mathbf{k}} E\{|Z_{\mathbf{k}}| (\log^+ |Z_{\mathbf{k}}|)^{r-1}\} + B_r \tag{3}$$

$$E\{\sup_{\mathbf{k}} |Z_{\mathbf{k}}|^p\} \leqq A_{p,r} \sup_{\mathbf{k}} E\{|Z_{\mathbf{k}}|^p\} , \tag{4}$$

where A_r, B_r and $A_{p,r}$ are universal constants and $p > 1$.

PROOF. We consider first the case $r = 2$ and proceed by induction.

From Lemma 2 it follows that $(\sup_{k_2} |Z_{(k_1,k_2)}|, \mathscr{G}_{k_1,1})$ is a positive reversed submartingale, indexed by k_1 (assuming the expectations are finite). By Doob's inequality, we have, for every $\lambda \geqq 0$:

$$\lambda P\{\sup_{k_1,k_2} |Z_{(k_1,k_2)}| \geqq \lambda\} \leqq \sup_{k_1} E\{\sup_{k_2} |Z_{(k_1,k_2)}|\} . \tag{5}$$

But it now follows by a second inequality of Doob ([3] page 317), since $(Z_{(k_1,k_2)}, \mathscr{G}_{1,k_2})$ is a reversed martingale indexed by k_2 for fixed k_1, that

$$E\{\sup_{k_2} |Z_{(k_1,k_2)}|\} \leqq A \sup_{k_2} E\{|Z_{(k_1,k_2)}| \log^+ |Z_{(k_1,k_2)}|\} + A . \tag{6}$$

In a similar fashion it is shown (see again [3] page 317) that

(7) $$E\{\sup_{k_1,k_2} |Z_{(k_1,k_2)}|^p\} \leqq B_p \sup_{k_1,k_2} E\{|Z_{(k_1,k_2)}|^p\}$$ for

all $p > 1$, where B_p is a universal constant. This establishes the lemma for $r = 2$.

For general r let

$$H_r = \text{all positive submartingales } (Y_{\mathbf{k}}, \mathscr{G}_{\mathbf{k}}) \text{ such that for } k_i \text{ fixed}$$
$$(Y_{\mathbf{k}}, \mathscr{G}_{k_1,\cdots,k_{i-1},1,k_{i+1},\cdots,k_r}) \text{ is a positive submartingale.}$$

By (6) and (7) (and their proofs), the inequalities (3) and (4) are valid for $r = 2$ for all submartingales in H_2.

Assume now the validity of the inequalities for all submartingales in H_{r-1}. If $\hat{\mathbf{k}} = (k_1, \cdots, k_{r-1})$, then

$$\lambda P\{\sup_{\hat{\mathbf{k}},k_r} |Z_{\hat{\mathbf{k}},k_r}| \geqq \lambda\} = \lambda P\{\sup_{\hat{\mathbf{k}}} (\sup_{k_r} |Z_{\hat{\mathbf{k}},k_r}| \geqq \lambda)\} .$$

By the remark following Lemma 2, $\sup_{k_r} |Z_{\hat{\mathbf{k}},k_r}| \in H_{r-1}$.

Hence by the induction assumption:

(8) $$\lambda P\{\sup_{\hat{\mathbf{k}},k_r} |Z_{\hat{\mathbf{k}},k_r}| \geqq \lambda\} \leqq A_{r-1} \sup_{\hat{k}} E\{\sup_{k_r} |Z_{\hat{k},k_r}| (\log^+ |Z_{\hat{k},k_r}|)^{r-2}\} + B_{r-1}$$

But it is clear that for fixed $\hat{\mathbf{k}}$, $|Z_{\hat{\mathbf{k}},k_r}|$ is a positive supermartingale indexed by k_r; since $t(\log^+ t)^{r-2}$ is a convex increasing function for $r > 2$, $|Z_{\hat{\mathbf{k}},k_r}| (\log^+ |Z_{\hat{\mathbf{k}},k_r}|)^{r-2}$ is also a positive submartingale.

Employing Doob's inequality noted earlier and the inequality

$$t(\log^+ t)^{r-2} \log^+ ([t \log^+ t]^{r-2} \leqq (r-1)t[\log^+ t]^{r-1} ,$$

which holds for integers $r \geqq 2$ and all $t \geqq 0$, we have

(9) $$\begin{aligned} &E\{\sup_{k_r} |Z_{\hat{\mathbf{k}},k_r}| (\log^+ |Z_{\hat{\mathbf{k}},k_r}|)^{r-2}\} \\ &\quad \leqq A \sup_{k_r} E\{|Z_{\hat{\mathbf{k}},k_r}| (\log^+ |Z_{\hat{\mathbf{k}},k_r}|)^{r-2} \log^+ [|Z_{\hat{\mathbf{k}},k_r}| (\log^+ |Z_{\hat{\mathbf{k}},k_r}|)^{r-2}]\} + A \\ &\quad \leqq A(r-1) \sup_{k_r} E\{|Z_{\hat{\mathbf{k}},k_r}| (\log^+ |Z_{\hat{\mathbf{k}},k_r}|)^{r-1}\} + A . \end{aligned}$$

From (8) and (9) we immediately deduce (3) and (4).

LEMMA 4. *If the right-hand side of* (3) (*resp.* (4)) *is finite, then* $\lim_{\mathbf{k}\to\infty} Z_{\mathbf{k}}$ *exists in the* L^1 (*resp.* L^p) *sense* (*with the usual definition of convergence along a directed set*).

PROOF. It is well known that uniform integrability of a (forward or reversed) martingale implies L^1 convergence along a directed set ([4] page 86).

We can now deduce a.s. convergence by a modification of Cairoli's argument. Let α be a positive constant and set

$$\begin{aligned} X^{(\alpha)} &= -\alpha & X &< -\alpha \\ &= X & |X| &\leqq \alpha \\ &= \alpha & X &> \alpha . \end{aligned}$$

Then $\{X_{\mathbf{k}} - X_{\mathbf{k}}^{(\alpha)} : \mathbf{k} \in K_r\}$ is an array of i.i.d. random variables.

Let $S_{\mathbf{k}}^{(\alpha)} = \sum_{\mathbf{j} \leqq \mathbf{k}} X_{\mathbf{j}}^{(\alpha)}$ and define $Z_{\mathbf{k}}^{(\alpha)}$ accordingly. Let $\mathbf{1}$ denote the r-tuple $(1, 1, \cdots, 1)$. By retracing the arguments leading to (3) for the array $\{X_{\mathbf{k}} - X_{\mathbf{k}}^{(\alpha)}\}$ we get

$$(10) \qquad \begin{aligned} &P\{\sup_{\mathbf{k}} |Z_{\mathbf{k}} - Z_{\mathbf{k}}^{(\alpha)}| \geqq \lambda\} \\ &\leqq (A/\lambda) \sup_{\mathbf{k}} E\{|Z_{\mathbf{k}} - Z_{\mathbf{k}}^{(\alpha)}| (\log^+ |Z_{\mathbf{k}} - Z_{\mathbf{k}}^{(\alpha)}|)^{r-1}\} + A \\ &\leqq (A/\lambda) E\{|X_{\mathbf{1}} - X_{\mathbf{1}}^{(\alpha)}| (\log^+ |X_{\mathbf{1}} - X_{\mathbf{1}}^{(\alpha)}|)^{r-1}\} + A . \end{aligned}$$

The last inequality is a consequence of Jensen's (conditional) inequality, since

$$E\{X_{\mathbf{1}} - X_{\mathbf{1}}^{(\alpha)} \mid \sigma(S_{\mathbf{k}} - S_{\mathbf{k}}^{(\alpha)})\} = \frac{S_{\mathbf{k}}}{|\mathbf{k}|} - \frac{S_{\mathbf{k}}^{(\alpha)}}{|\mathbf{k}|} .$$

Now take a sequence $\{\varepsilon_k\} \downarrow 0$ and a sequence $\{\alpha_k\} \uparrow \infty$ such that

$$(11) \qquad P\{\sup_{\mathbf{k}} |Z_{\mathbf{k}} - Z_{\mathbf{k}}^{(\alpha_j)}| > \varepsilon_j\} \leqq 2^{-j} .$$

This is possible by (10), since

$$\begin{aligned} &P\{\sup_{\mathbf{k}} |Z_{\mathbf{k}} - Z_{\mathbf{k}}^{(\alpha_j)}| > \varepsilon_j\} = P\{\sup_{\mathbf{k}} c\,|Z_{\mathbf{k}} - Z_{\mathbf{k}}^{(\alpha_j)}| > c\varepsilon_j\} \\ &\leqq (A/c\varepsilon_j)E\{c\,|X_{\mathbf{1}} - X_{\mathbf{1}}^{(\alpha_j)}| (\log^+ c\,|X_{\mathbf{1}} - X_{\mathbf{1}}^{(\alpha_j)}|)^{r-1} + (A/c\varepsilon_j) \end{aligned}$$

which for fixed ε_j can be made $\leqq 2^{-j}$ by choosing c and α_j sufficiently large.

By (11) and the Borel-Cantelli lemma, $P\{\sup_{\mathbf{k}} |Z_{\mathbf{k}} - Z_{\mathbf{k}}^{(\alpha_j)}| > \varepsilon_j \text{ i.o.}\} = 0$. Thus if we can show that $Z_{\mathbf{k}}^{(\alpha)}$ converges a.s. it will follow that $Z_{\mathbf{k}}$ converges a.s.

By Lemma 4 we know that $Z_{\mathbf{k}}^{(\alpha)}$ converges in L^p for $p > 1$. Denote the limit by $X^{(\alpha)}$; then $X^{(\alpha)} \in L^p$. Choose a sequence of indices $\mathbf{k}^{(j)}$ such that as $j \to \infty$, $k_1^j \to \infty, \cdots, k_r^j \to \infty$. Fix j; if $\mathscr{G}_{\mathbf{k}}^{(\alpha)}$ denotes the field $\bigvee_{\mathbf{l} \geqq \mathbf{k}} S_{\mathbf{l}}^{(\alpha)}$, then for $\mathbf{k} \geqq \mathbf{k}^{(j)}$, $\{Z_{\mathbf{k}}^{(\alpha)} - X^{(\alpha)}, \mathscr{G}_{\mathbf{k}}^{(\alpha)}\}$ is a martingale. By (4):

$$\begin{aligned} &E\{\sup_{\mathbf{k} \geqq \mathbf{k}^{(j)}} |Z_{\mathbf{k}}^{(\alpha)} - X^{(\alpha)}|^p\} \leqq B^p\{\sup_{\mathbf{k} \geqq \mathbf{k}^{(j)}} E\,|Z_{\mathbf{k}}^{(\alpha)} - X^{(\alpha)}|^p\} \\ &\leqq B^p E\{|Z_{\mathbf{k}^{(j)}}^{(\alpha)} - X^{(\alpha)}|\, p\} , \end{aligned}$$

(since $E\{Z_{\mathbf{k}^{(\alpha)}}^{(\alpha)} - X^{(\alpha)} \mid \mathscr{G}_{\mathbf{k}}^{(\alpha)}\} = Z_{\mathbf{k}}^{(\alpha)} - X^{(\alpha)}$). But $E\{|Z_{\mathbf{k}^{(j)}}^{(\alpha)} - X^{(\alpha)}|^p\} \to_{j \uparrow \infty} 0$ and $\sup_{\mathbf{k} \geqq \mathbf{k}^{(j)}} |Z_{\mathbf{k}}^{(\alpha)} - X^{(\alpha)}|^p$ is decreasing in j; hence $\sup_{\mathbf{k} \geqq \mathbf{k}^{(j)}} |Z_{\mathbf{k}}^{(\alpha)} - X^{(\alpha)}| \to 0$ a.s., as $j \uparrow \infty$ so $Z_{\mathbf{k}} \to X^{(\alpha)}$ a.s. Since $Z_{\mathbf{k}}$ clearly $\to 0$ in probability, it follows that $X^{(\alpha)} = 0$ a.s.

We have now shown a.s. convergence in the following sense:

$$(12) \qquad \text{For any } \varepsilon > 0,\ P\{|Z_{\mathbf{k}}| > \varepsilon \text{ a.l.}\} = 0 .$$

To strengthen this to the statement that $P\{|Z_{\mathbf{k}}| > \varepsilon \text{ f.o.}\} = 1$ we proceed as follows. Let $\varepsilon > 0$ be given, and take $r = 2$. On each of the lines $x = l$ and $y = j$, the one-dimensional strong law holds a.s.; denoting by Ω_0 the subset of Ω formed by deleting the null sets of non-convergence, we have $P(\Omega_0) = 1$. For $\omega \in \Omega_0$, find $\mathbf{k} = (k_1, k_2)$ satisfying (12); by taking $\mathbf{k}' > \mathbf{k}$ if necessary we can (since there are only a finite number of lines $x = l$ for $l < k_1$ and $y = j$ for $j < k_2$) conclude that

$$|Z_l(w)| < \varepsilon \qquad \text{for all } \mathbf{l} \nleq \mathbf{k}' .$$

Now assume inductively that $P\{|Z_\mathbf{k}| > \varepsilon \text{ f.o.}\} = 1$ for $r - 1$ dimensions. Then for each of the $r - 1$ dimensional hyperplanes defined by $k_i = j \, (j = 1, 2, \cdots)$ there is a null set on whose complement $|Z_\mathbf{k}| > \varepsilon$ f.o.; again let Ω_0 be the intersection of the sets of convergence over these hyperplanes. For $\omega \in \Omega_0$, find $\mathbf{k} = (k_1, k_2, \cdots, k_r)$ such that (12) holds; by taking $\mathbf{k}' > \mathbf{k}$ if necessary we again get

$$|Z_l(\omega)| < \varepsilon \qquad \text{for } \mathbf{l} \nleq \mathbf{k}' ,$$

hence that $P\{|Z_\mathbf{k}| > \varepsilon \text{ f.o.}\} = 1$, proving (1).

3. The non-identically distributed case. In this section we consider independent random variables with zero mean, but we no longer assume identical distributions. We will simply outline some of the results here, without proof.

One of the keys to an analysis of this case is the following analog of Kolmogorov's inequality for the one-dimensional case ([6]):

THEOREM (Wichura) *Let*

$$M_\mathbf{n} = \max_{\mathbf{j} \leq \mathbf{n}} |S_\mathbf{j}| , \qquad \sigma^2 = E(S_\mathbf{n}^2) .$$

Then

$$E(M_\mathbf{n}^2) \leqq 4^r \sigma^2 .$$

Using this result one can be prove half of the Three Series Theorem for matrix arrays, i.e. that the convergence of the three series is sufficient to insure a.s. convergence of the random series $\sum_\mathbf{k} X_\mathbf{k}$. (The other half is valid also, but we do not have an elementary proof—what is needed is an analog of Kolmogorov's second inequality.) Proceeding as in the one-dimensional case one can then prove, for example, the following analogue of a well-known theorem (e.g. [2] page 117):

THEOREM. *Let φ be a positive, even continuous function on R^1 such that as $|x|$ increases, $\varphi(x)/x$ increases, and $\varphi(x)/x^2$ decreases. Then if*

$$\sum_\mathbf{k} \frac{E(\varphi(X_\mathbf{k}))}{\varphi\, |\mathbf{k}|} < \infty ,$$

it follows that $P\{|Z_\mathbf{k}| > \varepsilon \text{ f.o.}\} = 1$ for any $\varepsilon > 0$.

The usual sufficient conditions for convergence in the one-dimensional case are thus sufficient for n-dimensional arrays as well, e.g., if $\sup_\mathbf{k} E\,|X_\mathbf{k}|^p < \infty$ for for some $p > 1$, we have convergence.

Finally we remark that the approach via Wichura's inequality and the Three Series Theorem can be used to give a different and more classical proof of the strong law in Section 2. Similar techniques can be employed to yield strong laws for more general partially ordered arrays.

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