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SAMPLE FUNCTIONS OF THE N-PARAMETER WIENER PROCESS[1]

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Let $W^{(N)}$ denote the N-parameter Wiener process, that is a real-valued Gaussian process with zero means and covariance $\prod_{i=1}^{N} (s_i \wedge t_i)$ where $s = \langle s_i \rangle$, $t = \langle t_i \rangle$, $s_i \geqq 0$, $t_i \geqq 0$, $i = 1, 2, \cdots N$. Then $W^{(N,d)}$ is to be the process with values in R^d determined by making each component an N-parameter Wiener process, the components being independent. Our concern is with continuity and recurrence properties of the sample functions. In particular we give integral tests for upper functions which reduce in the case $N = d = 1$ to the integral tests of Kolmogorov, and of Chung-Erdös-Sirao. We formulate and prove precise statements of the fact that $W^{(N,d)}$ is interval recurrent (point recurrent) if and only if $d \leqq 2N$ ($d < 2N$).

0. Introduction. In this paper we study sample function properties of the N-parameter Wiener process $W^{(N)}$, and more generally of the N-parameter Wiener process $W^{(N,d)}$ with values in d-dimensional Euclidean space R^d. More precisely, we obtain information about the continuity and recurrence properties of the sample functions.

Our parameter space is R_+^N, that is the set of $t \in R^N$ with all components nonnegative. When dealing with a point t in the parameter space we sometimes find it convenient to write more explicitly $t = \langle t_1, t_2, \cdots, t_N \rangle$ or simply $\langle t_i \rangle$; this notation will be reserved for the parameter space. In case all $t_i = \lambda$ we write $t = \langle \lambda \rangle$.

$W^{(N)}$ is to be a separable real-valued Gaussian process on the parameter space R_+^N with mean zero and covariance

$$E(W_s^{(N)} W_t^{(N)}) = \prod_{i=1}^{N} s_i \wedge t_i\,, \qquad s = \langle s_i \rangle,\ t = \langle t_i \rangle\,.$$

We call $W^{(N)}$ the *N-parameter Wiener process*. $W^{(N,d)}$ is to be the process with values in R^d determined by making each component an N-parameter Wiener process, the components being independent. For $s = \langle s_i \rangle$, $t = \langle t_i \rangle$ two points of R_+^N with $s_i \leqq t_i$, we write

$$\Delta(s, t) \qquad \text{for} \qquad \times_{i=1}^{N} [s_i, t_i]$$

and in case $s = \langle 0 \rangle$ we write $\Delta(t)$ for $\Delta(s, t)$. By an *interval* we mean a subset of R_+^N which is the product of N one-dimensional intervals, not necessarily closed, possibly degenerate.

Let now, for brevity's sake, $X = W^{(N)}$. Let $t \in R_+^N$ and let f be the indicator

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function of $\Delta(t)$. We define

$$\int f(u)\,dX_u = X(\Delta(t)) = X_t\,.$$

If g is the indicator function of $\Delta(s)$, $s \in R_+{}^N$, one can verify

$$E[\int f(u)\,dX_u \int g(u)\,dX_u] = E(X_s X_t) = |\Delta(s) \cap \Delta(t)| = \int_{R_+{}^N} f(u)g(u)\,du \tag{0.1}$$

where $|\cdot|$ denotes the N-dimensional Lebesgue measure. Now one can extend the map $f \to \int f(u)\,dX_u$ to all of $L_2(R_+{}^N)$ so that the mapping is linear and the equality between the extremes in (0.1) is preserved. If f is the indicator function of a set F we write $X(F)$ for $\int f(u)\,dX_u$. For measurable subsets F and G of $R_+{}^N$ this gives

$$E[X(F)X(G)] = |F \cap G|\,. \tag{0.2}$$

Note that if S is the symmetric difference between F and G then the variance of $X(F) - X(G)$ is equal to that of $X(S)$, both quantities being equal to $|S|$. So it is natural for us to use the distance $\delta(F, G) = |S|$. We shall write $\delta(s, t)$ for $\delta(\Delta(s), \Delta(t))$, and $\delta(t)$ for $\delta(0, t)$. By (0.2), $E[X(F)X(G)] = 0$ when F and G are disjoint, and $E[\{X(F)\}^2]$ depends only on $|F|$; so in an obvious sense X has independent and stationary increments. Of course this refers to increments of the form $X(F)$. For increments of the form $X_t - X_s$ the corresponding assertions are completely false. In particular if $s = \langle s_i \rangle$, $t = \langle s_i + u_i \rangle$, the variance of $X_t - X_s$ will depend crucially on s.

The first introduction of $W^{(N)}$, as far as we know, occurred in Kitagawa [5] in connection with applications to statistical problems. Chentsov [1] showed that $W^{(N)}$ has a.s. continuous sample functions; see also Yeh [13]. For further results, consult also Yeh [14], [15], W. J. Park [9], C. Park [8], Kuelbs [7].

More recently there have appeared results on sample-function behavior. Laws of the iterated logarithm type, overlapping but not subsumed by our results here, are given in Zimmerman [16]; also [16] and Pyke [10] contain some information about moduli of continuity, but these are included in the results of this paper. Generalizations of Strassen's law of the iterated logarithm are given in Wichura [12]. An interesting study of continuity properties of $W^{(N)}(F)$, where F ranges over suitable classes of sets (not necessarily intervals) is contained in Dudley [4]. Some further references are given in [10].

Our principal results appear in Sections 2 and 3. Section 2 gives integral tests for upper and lower functions. Let $X = W^{(N,d)}$. Theorem 2.1 tells when a function $\varphi(\lambda)$ is such that a.s. $|X(\Delta)| \leqq |\Delta|^{\frac{1}{2}}\varphi(|\Delta|^{-1})$ for all intervals Δ included in $\Delta(\langle 0 \rangle, \langle 1 \rangle)$ which are sufficiently small; Theorem 2.2 answers the same question when the Δ are restricted to be of the form $\Delta(t)$. In Theorem 2.3 we find those φ such that, for s fixed and $\delta(s) > 0$, $|X_t - X_s| \leqq (\delta(s, t))^{\frac{1}{2}}\varphi((\delta(s, t))^{-1})$ provided $\delta(s, t)$ is sufficiently small. In the case $N = 1$, $d = 1$ Theorem 2.1 reduces to the Chung–Erdös–Sirao test [3], while Theorems 2.2 and 2.3 both reduce to the Kolmogorov integral test. For the uniform problem for $X_t - X_s$, with s and t

both varying we do not have a test; however the modulus of continuity is given in Theorem 2.4. In discussing tests for upper functions our arguments closely follow Sirao [11], who considered the problem for Lévy's N-parameter Brownian motion; however the nature of the process we are dealing with creates new difficulties which are made manifest particularly in the somewhat unusual grid we are forced to introduce in the proof of Theorem 2.1.

In the one-parameter case, $N = 1$, it is well known that the Wiener process is interval recurrent for $d = 1, 2$, i.e. every open sphere is entered after arbitrarily large time, while for $d \geqq 3$ $|W_t^{(1,d)}| \to \infty$ as $t \to \infty$. Also, the probability that any given $x \in R^d$ is hit by $W_t^{(1,d)}$ for some $t > 0$ is zero for $d \geqq 2$, one for $d = 1$. In Section 3 we prove analogues for $W^{(N,d)}$: with a suitable notion of interval recurrence $W^{(N,d)}$ is interval recurrent for $d \leqq 2N$, but not for $d > 2N$. For $x \in R^d$, $P[W_t^{(N,d)} = x$ for some t with $\delta(t) > 0] = 1$ (0) when $d < 2N$ $(d \geqq 2N)$, P of course being our probability measure. An open problem, for $d < 2N$, is whether the range of the process $\{W_t^{(N,d)} : t \in R_+^N\}$ is all of R^d. It is, of course, a consequence of Theorem 3.3 that the complement of the range has measure zero.

1. Preliminaries. In this section we shall prove several preliminary results that we need. Throughout this section, X will be used to denote the process $W^{(N,d)}$. First we state the very useful scaling property. If $r = \langle r_i \rangle \in R_+^N$ with $\delta(r) > 0$, then

$$Y_t = \{\delta(r)\}^{-\frac{1}{2}} X_{\langle r_i t_i \rangle}$$

and X_t have the same finite dimensional distributions.

We now make some remarks to show that we have appropriate analogues of the familiar zero-one laws. We introduce some notation for this argument only. Let $\mathscr{C}_n$ be the class of time intervals in R_+^N with vertices of the form $\langle k_i 2^{-n} \rangle$, k_i nonnegative integers, and having all sides of equal length, and for $n > 0$ each member of $\mathscr{C}_n$ is to be a subcube of one in $\mathscr{C}_0$. Let $\mathscr{C}_\infty = \bigcup_{n=0}^{\infty} \mathscr{C}_n$, and

$$\mathscr{F}_n = \mathscr{B}(X(\Delta) : \Delta \in \mathscr{C}_n)\,, \qquad \mathscr{F}_\infty = \bigvee_{n=0}^{\infty} \mathscr{F}_n\,,$$

the notation indicating that $\mathscr{F}_n$ is the Borel field generated by the indicated class of random variables, and $\mathscr{F}_\infty$ is the smallest Borel field including all $\mathscr{F}_n$. For a subset D of R_+^N we put

$$\mathscr{C}_n(D) = \{\Delta \in \mathscr{C}_n : \Delta \subseteqq D\}\,, \qquad \mathscr{F}_n(D) = \mathscr{B}\{X(\Delta) : \Delta \in \mathscr{C}_n(D)\}\,,$$
$$\mathscr{F}_\infty(D) = \bigvee_{n=0}^{\infty} \mathscr{F}_n(D)\,.$$

First we have the Kolmogorov Zero-One Law: *Let $D_m \subseteqq R_+^N$, $m = 1, 2, \cdots$, $D_m \downarrow \emptyset$; if $A \in \mathscr{F}_\infty(D_m)$ for every m then $P(A) \in \{0, 1\}$.* The proof is the same as in the usual setting: simply approximate A as in the proof of the Hewitt–Savage Zero-One Law below and use independence. *Furthermore, if the D_m also satisfy $\Delta(t) \subseteqq D_m$ for every $t \in D_m$ then $A \in \mathscr{B}(X_t : t \in D_m)$ for every m implies $P(A) \in \{0, 1\}$.* This is clear, since our hypotheses imply $A \in \mathscr{F}_\infty(D_m)$ for every m.

A version of the Hewitt–Savage Zero-One Law will also be required. A *permutation* is a 1–1 map σ of $\mathscr{C}_\infty$ onto itself such that (i) for every $\Delta \in \mathscr{C}_\infty$ there exists a vector $v(\Delta)$ in R^N such that $\sigma(\Delta)$ is Δ translated by $v(\Delta)$; (ii) $\Delta_1 \subseteqq \Delta_2$, $\Delta_1 \in \mathscr{C}_\infty$, $\Delta_2 \in \mathscr{C}_\infty$ implies that $v(\Delta_1) = v(\Delta_2)$. For every $A \in \mathscr{F}_\infty$ there exists a Borel function g on R^∞ such that the indicator function of A has the form

$$I_A = g(X(\Delta_1), X(\Delta_2), \cdots)$$

where $\Delta_1, \Delta_2, \cdots$ is an enumeration of $\mathscr{C}_\infty$. We define $\sigma(A)$ by

$$I_{\sigma(A)} = g(X(\sigma(\Delta_1)), X(\sigma(\Delta_2)), \cdots);$$

though the choice of g corresponding to A is not unique, the definition of $\sigma(A)$ does not in fact depend on the particular choice of g. If A and $\sigma(A)$ differ only by a set of probability zero A is *essentially invariant* under σ. Observe that for any Borel function g and permutation σ the random variables $g(X(\Delta_1), X(\Delta_2), \cdots)$ and $g(X(\sigma(\Delta_1)), X(\sigma(\Delta_2)), \cdots)$ have the same distribution; it is a consequence of this that $P(A) = P[\sigma(A)]$ for all $A \in \mathscr{F}_\infty$. We can therefore obtain a Hewitt–Savage Zero-One Law: *Let D_m be a subset of R_+^N, $m = 1, 2, \cdots, D_m \downarrow \emptyset$; if $A \in \mathscr{F}_\infty$ is such that for every m there exists a permutation σ such that A is essentially invariant under σ and σ maps $\mathscr{C}_\infty(D_m{}^c)$ into $\mathscr{C}_\infty(D_m)$, then $P(A) \in \{0, 1\}$.* The usual proof works: under the given hypotheses there exist $A_m \in \mathscr{F}_\infty(D_m{}^c)$ approximating A in the sense that both $P(A_m)$ and $P(A_m \cap A)$ converge to $P(A)$ as $m \to \infty$. Since A_m and $\sigma(A_m)$ are independent

$$P[A_m \cap \sigma(A_m)] = P[A_m] \cdot P[\sigma(A_m)] .$$

Now $\sigma(A_m)$ approximates $\sigma(A)$. Using the essential invariance of A under σ and letting $m \to \infty$ leads to $P(A) = (P(A))^2$, so that $P(A) \in \{0, 1\}$. We will require the corollary: *if $A \in \mathscr{B}(X_t : t = \langle t_i \rangle, t_i \geqq n, i = 1, 2, \cdots, N)$, $n = 1, 2, \cdots$, then $P(A) \in \{0, 1\}$.*

Next we need the asymptotic behavior of the distribution of the modulus of a standard normal random variable in R^d.

LEMMA 1.1. *Let U be a normal random variable in R^d with mean 0 and identity covariance matrix. Then*

$$P[|U| \geqq \lambda] \sim c_d \lambda^{d-2} e^{-\lambda^2/2} .$$

PROOF. This follows easily from the well-known fact that $|U|^2$ has a χ^2 distribution with d degrees of freedom.

The next two lemmas give bounds for the distribution of the supremum of X taken over certain time sets. A lemma similar to Lemma 1.2 was obtained in [16].

LEMMA 1.2. *Let $S = \Delta(u, v)$ be a fixed time interval. Then*

$$P[\sup_{R \subset S} |X(R)| > \lambda] \leqq 4^N P[|X(S)| > \lambda] ,$$

where R ranges over intervals.

PROOF. This is proved in very much the same way as the classical result for

$N = 1$ except that a little care is needed to avoid dependence difficulties. Fix n and let

$$t_{ki} = u_i + (v_i - u_i)(k_i + 1)2^{-n}, \qquad i = 1, 2, \cdots, N,$$

where $k = 1 + k_1 + k_2 2^n + \cdots + k_N 2^{n(N-1)}$ and each k_i runs from 0 to $2^n - 1$. Now let

$$t_k = \langle t_{ki} \rangle, \qquad t_k^* = \langle v_1, \cdots, v_{N-1}, t_{kN} \rangle, \qquad t_k' = \langle t_{k1}, \cdots, t_{k,N-1}, v_N \rangle,$$

and define

$$E_k = [\sup_{s \in \Delta(u, t_k)} |X(\Delta(s, t_k))| > \lambda], \qquad F_k = E_k E_{k-1}^c \cdots E_1^c.$$

Now we use the usual argument. If we give ourselves $X(\Delta(s, t))$ for all $s, t \in \Delta(u, t_k^*)$, then we are given whether F_k occurs and if it does we can find an s so that $|X(\Delta(s, t_k))| > \lambda$. Since $\Delta(s, t_k')\backslash\Delta(s, t_k)$ is disjoint from $\Delta(u, t_k^*)$, we get that $X(\Delta(s, t_k')\backslash\Delta(s, t_k))$ is independent of the condition and so must conditionally have probability $\frac{1}{2}$ of being in a half-space which will make $|X(\Delta(s, t_k'))| > \lambda$. Thus

$$P(F_k) \leqq 2P[F_k; \sup_{s,t \in \Delta(u,v), t_N = v_N} |X(\Delta(s, t))| > \lambda].$$

Summing over k and letting $n \to \infty$, we obtain the bound

$$P[\sup_{R \subset S} |X(R)| > \lambda] \leqq 2P[\sup_{s,t \in \Delta(u,v), t_N = v_N} |X(\Delta(s, t))| > \lambda].$$

Now we do the same thing over except that we only need consider those k with $k_N = 2^n - 1$ and this time we will be able to increase $t_{k,N-1}$ to v_{N-1}. Continuing in this way we get

$$P[\sup_{R \subset S} |X(R)| > \lambda] \leqq 2^N P[\sup_{s \in \Delta(u,v)} |X(\Delta(s, v))| > \lambda].$$

Finally, we can repeat the same argument with s varying but with

$$s_{ki} = v_i - (v_i - u_i)(k_i + 1)2^{-n}$$

to get the ordering reversed.

LEMMA 1.3. *Let $S = \Delta(u, v)$ be a fixed time interval and $s = \langle s_i \rangle$ a fixed time point such that for each i, either $s_i \leqq u_i$ or $s_i \geqq v_i$. Let*

$$\gamma = 2 \max_{1 \leqq i \leqq N} \{\max(v_i - u_i, u_i - s_i, s_i - v_i)\},$$

$$\rho = \max_{1 \leqq i \leqq N} v_i, \qquad \text{and} \qquad w = \langle w_i \rangle \qquad \text{where}$$

$$\begin{aligned} w_i &= u_i \qquad \text{if} \quad s_i \geqq v_i \\ &= v_i \qquad \text{if} \quad s_i \leqq u_i . \end{aligned}$$

If $0 < \beta < \lambda$ and U is a normal random variable in R^d with mean 0 and covariance I, then

$$(1.1) \qquad P[\sup_{t \in S} |X_t - X_s| > \lambda] \leqq 2^N P[|X_w - X_s| > \lambda - \beta] + 4^N P[|U| > \beta\gamma^{-1}N^{-\frac{1}{2}}\rho^{1-N/2}].$$

REMARK. This lemma is more complicated than the last one because of independence difficulties. If we assumed that $s_i \leqq u_i$ for all i, these difficulties would not arise and we would only need the first term on the right of (1.1) and could let $\beta = 0$.

PROOF. We give the proof for $\rho = 1$; the general case follows from the scaling property. The proof is by induction of the dimension of S; the time dimension is to be N but S may have any dimension ν with $0 \leqq \nu \leqq N$. The induction hypothesis will be as in (1.1) but with 2^N, 4^N, and $N^{-\frac{3}{2}}$ replaced by 2^ν, 4^ν, and $N^{-\frac{1}{2}}\nu^{-1}$ respectively. The case $\nu = 0$ is clear; we assume the induction hypothesis proved for all S with dimension less than ν. We can also assume that for our rectangle S of dimension ν we have $u_i < v_i$ for $i = 1, \cdots, \nu$ and $u_i = v_i$ for $i = \nu + 1, \cdots, N$. There are two cases depending on whether $s_\nu \leqq u_\nu$ or $s_\nu \geqq v_\nu$; we consider the latter case which is the harder one. For a general time point $t = \langle t_i \rangle$ we introduce three related points:

$$t^1 = \langle u_1 \wedge s_1, \cdots, u_{\nu-1} \wedge s_{\nu-1}, t_\nu, u_{\nu+1} \wedge s_{\nu+1}, \cdots, u_N \wedge s_N \rangle$$
$$t^2 = \langle t_1, \cdots, t_{\nu-1}, u_\nu, u_{\nu+1}, \cdots, u_N \rangle$$
$$t^3 = \langle u_1 \wedge s_1, \cdots, u_{\nu-1} \wedge s_{\nu-1}, u_\nu, u_{\nu+1} \wedge s_{\nu+1}, \cdots, u_N \wedge s_N \rangle .$$

Now let $t_k = \langle t_{ki} \rangle$ as before where with n a fixed positive integer,

$$t_{ki} = v_i - (v_i - u_i)(k_i + 1)2^{-n} , \qquad i = 1, 2, \cdots, \nu ,$$
$$t_{ki} = u_i \qquad i = \nu + 1, \cdots, N ,$$

and $k = 1 + k_1 + k_2 2^n + \cdots + k_\nu 2^{n(\nu-1)}$ with $0 \leqq k_i < 2^n$. Set

$$E_k = [|X_{t_k} - X_s| > \lambda] \qquad \text{and} \qquad F_k = E_k E_{k-1}^c \cdots E_1^c .$$

If $j \leqq k$, then

$$\Delta(t_k^1)\backslash\Delta(t_k^3) \subset \Delta(t_k^1) \subset \Delta(t_j) \cap \Delta(s)$$

so that $X_{t_k^1} - X_{t_k^3}$ is independent of F_k. Thus, as in the proof of the last lemma,

$$P(F_k) \leqq 2P[F_k; |X_{t_k} - X_s - X_{t_k^1} + X_{t_k^3}| > \lambda] .$$

Summing over k and letting $n \to \infty$, we obtain the bound

$$\begin{aligned} P[\sup_{t \in S} |X_t - X_s| > \lambda] &\leqq 2P[\sup_{t \in S} |X_t - X_s - X_{t^1} + X_{t^3}| > \lambda] \\ &\leqq 2P[\sup_{t \in S} |X_t - X_{t^1} - X_{t^2} + X_{t^3}| > \beta\nu^{-1}] \\ &\quad + 2P[\sup_{t \in S} |X_{t^2} - X_s| > \lambda - \beta\nu^{-1}] . \end{aligned} \tag{1.2}$$

Now we need an estimate for the first term in the last expression. This is obtained as in the proof of Lemma 1.2 by discretizing t and first moving t_ν out to v_ν, then $t_{\nu-1}$ to $v_{\nu-1}$, etc. Since s is not involved here, there are no independence difficulties. We obtain the bound

$$\begin{aligned} P[\sup_{t \in S} |X_t - X_{t^1} - X_{t^2} + X_{t^3}| > \lambda] &\leqq 2^\nu P[|X_v - X_{v^1} - X_{v^2} + X_{v^3}| > \lambda] \\ &\leqq 2^\nu P[|U| > \lambda\gamma^{-1}N^{-\frac{1}{2}}] , \end{aligned}$$

where the last bound follows since each component of $X_v - X_{v1} - X_{v2} + X_{v3}$ has variance

$$(v_\nu - u_\nu)(\textstyle\prod_{i\neq\nu} v_i - \prod_{i\neq\nu} u_i \wedge s_i) \leqq N\gamma^2 .$$

(Recall that we assumed that $v_i \leqq 1$ for all i.) Now we use this bound in (1.2) and note that the induction hypothesis applies to the last term in (1.2) since when t varies over S, t^2 varies over a $\nu - 1$ dimensional interval. Furthermore, note that the point w determined by this new interval is the same as that determined by S. Thus

$$\begin{aligned} P[\sup_{t\in S} |X_t - X_s| > \lambda] \leqq\ & 2^{\nu+1} P[|U| > \beta\nu^{-1}\gamma^{-1}N^{-\frac{1}{2}}] \\ & + 2\{2^{\nu-1}P[|X_w - X_s| > \lambda - \beta\nu^{-1} - (\nu - 1)\beta\nu^{-1}] \\ & + 4^{\nu-1}P[|U| > (\nu - 1)\beta\nu^{-1}\gamma^{-1}N^{-\frac{1}{2}}(\nu - 1)^{-1}]\} . \end{aligned}$$

This has the right form since $2^{\nu+1} + 2 \cdot 4^{\nu-1} \leqq 4^\nu$ if $\nu \geqq 2$, while in the case $\nu = 1$, $2^{\nu+1} = 4^\nu$ and the last term does not enter from the induction hypothesis. We have not considered the case when $s_\nu \leqq u_\nu$ but this is easy because the independence difficulties are avoided. To check this case, discretize t so that the νth component increases. Then $X_{t_k} - X_s$ and $X_{t_k^*} - X_{t_k}$ are independent where t_k^* is the same as t_k but with $t_{k\nu}^* = v_\nu$. Thus

$$P[\sup_{t\in S} |X_t - X_s| > \lambda] \leqq 2P[\sup_{t\in S} |X_{t^*} - X_s| > \lambda]$$

in the usual way and now the induction hypothesis applies. Note also that if $s_i \leqq u_i$ for all i, this will prove (1.1) without the last term as claimed in the remark.

The next lemma is the form of the extended Borel Cantelli lemma proved by Chung and Erdös in [2].

LEMMA 1.4. *Let $\{E_k\}$ be a sequence of events satisfying the following conditions:*

(i) $\sum_{k=1}^\infty P(E_k) = +\infty$.

(ii) *For every pair of positive integers h, n with $n \geqq h$, there exist $c(h) > 0$ and $H(n, h) > n$ such that for every $m \geqq H(n, h)$ we have*

$$P(E_m \mid E_h^c \cdots E_n^c) > c(h)P(E_m) .$$

(iii) *There exist two absolute constants c_1 and c_2 with the following property: to each E_j, there corresponds a set of events $E_{j_1}, \cdots, E_{j_s}$ belonging to $\{E_k\}$ such that*

(a) $\sum_{i=1}^s P(E_j E_{j_i}) < c_1 P(E_j)$

and if $k > j$ but E_k is not among the E_{j_i}, $1 \leqq i \leqq s$, then

(b) $P(E_j E_k) < c_2 P(E_j)P(E_k)$.

Then the probability that infinitely many events E_k occur is equal to one.

The final two lemmas generalize some estimates of Chung, Erdös, and Sirao [3] to the d-dimensional case. In the proofs of these lemmas and in later proofs as well, we shall use the convenient practice of letting c stand for unimportant positive constants which may even change from line to line.

LEMMA 1.5. *Let U and V be normal random variables in R^d with mean* 0 *and identity covariance matrix, and suppose that*

$$E(U_i V_j) = \rho\delta_{ij} .$$

Then there is a positive constant c, independent of ρ, such that if $|\rho| < 1/ab$,

$$P[|U| > a, |V| > b] \leqq cP[|U| > a]P[|V| > b] .$$

PROOF. This is essentially the proof in [3]. We can assume that $a \leqq b$ and also that a is large since the lemma is trivially true otherwise. Since

$$P[|U| > a, |V| > b] = P[|U| > a, b < |V| \leqq 2b] + P[|U| > a, |V| > 2b] ,$$

it suffices to obtain the appropriate bound for each term. The second term is easy since, by Lemma 1.1, for b large

$$\begin{aligned} P[|U| > a, |V| > 2b] &\leqq P[|V| > 2b] \leqq cb^{d-2}e^{-(2b)^2/2} \\ &\leqq c\{b^{d-2}e^{-b^2/2}\}^2 \\ &\leqq cP[|U| > b]P[|V| > b] \leqq cP[|U| > a]P[|V| > b] . \end{aligned}$$

For the first term, since $|(u, v)| \leqq |u|\,|v|$,

$$\begin{aligned} &P[|U| > a, b < |V| \leqq 2b] \\ &\quad\leqq c \iint_{\substack{|u|>a \\ b<|v|\leqq 2b}} \exp\left\{-\frac{1}{2(1-\rho^2)}(|u|^2 - 2\rho(u, v) + |v|^2)\right\} du\, dv \\ &\quad\leqq c \iint_{\substack{|u|>a \\ b<|v|\leqq 2b}} \exp\left\{-\frac{1}{2(1-\rho^2)}(|u| - |\rho|\,|v|)^2\right\} \exp\{-\tfrac{1}{2}|v|^2\}\, du\, dv . \end{aligned}$$

Now, on the range of integration

$$|u| - |\rho|\,|v| \geqq |u| - \frac{2}{a} \geqq |u|\left(1 - \frac{2}{a^2}\right),$$

and for $a > 2$, we obtain the bound

$$\begin{aligned} P[|U| > a, b < |V| \leqq 2b] &\leqq cP[|U| > (a - 2a^{-1})(1 - \rho^2)^{-\frac{1}{2}}]P[|V| > b] \\ &\leqq cP[|U| > a]P[|V| > b] , \end{aligned}$$

the last step being a consequence of Lemma 1.1.

LEMMA 1.6. *Let U and V be normal random variables in R^d with mean* 0 *and identity covariance matrix, and suppose that*

$$E(U_i V_j) = \rho\delta_{ij} .$$

Then there is a positive constant c, independent of ρ, such that

$$P[|U| > a, |V| > a] \leqq ce^{-(1-\rho^2)a^2/8}P[|U| > a] , \qquad \text{for all } a \geqq 0 .$$

PROOF. The case $d = 1$ is an easy consequence of Lemma 5 in [3] so we assume $d \geqq 2$. Also we may assume that $a^2(1 - \rho^2)$ is large since the lemma is clear otherwise. Now

$$P[|U| > a, |V| > a]$$
$$= c(1 - \rho^2)^{-d/2} \int\int_{a<|u|\leq|v|} \exp\left\{-\frac{1}{2(1-\rho^2)}(|u|^2 - 2\rho(u, v) + |v|^2)\right\} dv\, du .$$

For each fixed u, we make an orthogonal transformation on v given by $w = Av$ where the first row of A is given by $a_{1i} = u_i|u|^{-1}$. Then

$$(u, v) = \sum_{i=1}^d u_i v_i = |u| \sum_{i=1}^d a_{1i} v_i = |u| w_1$$

and so

$$P[|U| > a, |V| > a]$$
$$= c(1 - \rho^2)^{-d/2} \int\int_{a<|u|\leq|w|} \exp\left\{-\frac{1}{2(1-\rho^2)}(|u|^2 - 2\rho|u|w_1 + |w|^2)\right\} dw\, du$$
$$= c(1 - \rho^2)^{-d/2} \int\cdots\int_{a<|u|\leq r} \exp\left\{-\frac{1}{2(1-\rho^2)}\right\}(|u|^2 - 2\rho|u|r\cos\varphi_1 + r^2)\Big\}$$
$$J(r, \varphi_1, \cdots, \varphi_{d-1})\, d\varphi_1 \cdots d\varphi_{d-1}\, dr\, du ,$$

where we have changed w to spherical coordinates at the last step and $J(r, \varphi_1, \cdots, \varphi_{d-1})$ is the Jacobian. Since

$$|J(r, \varphi_1, \cdots, \varphi_{d-1})| \leq r^{d-1}|\sin\varphi_1|^{d-2} ,$$

we obtain the bound

$$c(1-\rho^2)^{-d/2} \int_{|u|>a} \int_{|u|}^\infty \int_0^\pi \exp\left\{-\frac{1}{2(1-\rho^2)}(|u|^2 - 2\rho|u|r\cos\varphi_1 + r^2)\right\}$$
$$\times\, r^{d-1}|\sin\varphi_1|^{d-2}\, d\varphi_1\, dr\, du .$$

This is an even function of ρ so we replace ρ by $|\rho|$ in the exponent and then the φ_1 integration over $[0, \frac{1}{2}\pi]$ will dominate the integration over $[\frac{1}{2}\pi, \pi]$ so we restrict the φ_1 integration to $[0, \frac{1}{2}\pi]$. On this range we use the inequalities $\cos\varphi_1 \leq 1 - \pi^{-1}\varphi_1^2$ and $|\sin\varphi_1| \leq \varphi_1$. Then

$$\int_0^{\frac{1}{2}\pi} \exp\left\{-\frac{|\rho|\,|u|r}{(1-\rho^2)\pi}\varphi_1^2\right\}\varphi_1^{d-2}\, d\varphi_1 \leq \left(\frac{(1-\rho^2)\pi}{|\rho|\,|u|r}\right)^{\frac{1}{2}(d-1)} \int_0^\infty e^{-y^2} y^{d-2}\, dy$$

and so we have the bound

$$c(1 - \rho^2)^{-\frac{1}{2}}|\rho|^{-(d-1)/2} \int\int_{a<|u|\leq r} \exp\left\{-\frac{1}{2(1-\rho^2)}(|u|^2 - 2|\rho|\,|u|r + r^2)\right\}$$
$$\times\, (r|u|^{-1})^{(d-1)/2}\, dr\, du .$$

The exponential may be written as

$$\exp\left\{-\frac{1}{2(1-\rho^2)}(r - |\rho|\,|u|)^2\right\}\exp\{-\tfrac{1}{2}|u|^2\} ,$$

and for the r integration, we obtain

$$\int_{|u|}^\infty \exp\left\{-\frac{(r - |\rho|\,|u|)^2}{2(1-\rho^2)}\right\} r^{(d-1)/2}\, dr \leq c\exp\left\{-\frac{|u|^2(1-|\rho|)^2}{2(1-\rho^2)}\right\}|u|^{(d-3)/2} .$$

Therefore, we have the bound

$$c(1-\rho^2)^{-\frac{1}{2}}|\rho|^{-(d-1)/2}\int_{|u|>a}\exp\left\{-\frac{|u|^2(1-\rho^2)}{2(1+|\rho|)^2}\right\}\exp\left\{-\frac{|u|^2}{2}\right\}|u|^{-1}\,du$$
$$\leqq c|\rho|^{-(d-1)/2}\exp\left\{-\frac{a^2(1-\rho^2)}{2(1+|\rho|)^2}\right\}P[|U|>a]\,.$$

This completes the proof for $|\rho| \geqq \varepsilon$. To handle the small values of $|\rho|$, we note that the lemma follows from Lemmas 1.5 and 1.1 if $|\rho| < a^{-2}$, while for $|\rho| \geqq a^{-2}$, we can replace $|\rho|^{-(d-1)/2}$ by a^{d-1} and then for $|\rho| < \varepsilon$, we have

$$a^{d-1}\exp\left\{-\frac{a^2(1-\rho^2)}{2(1+|\rho|)^2}\right\} \leqq a^{d-1}\exp\left\{-\frac{a^2(1-\rho^2)}{2(1+\varepsilon)^2}\right\} \leqq c\exp\left\{-\frac{a^2(1-\rho^2)}{8}\right\}$$

for large a.

2. Continuity properties. In this section we shall derive sharp local and uniform continuity results for $W^{(N,d)}$. These will take the form of integral tests which generalize the results of Kolmogorov and Chung, Erdös, Sirao [3]. (Actually, we have stated our integral tests in a form slightly different from the usual one (c.f. [3] or [11]). Our form is easier to use for checking specific functions but the usual form has other advantages. It is easy to pass from one to the other.) With the lemmas already given we can follow the method of proof of [11] without too much difficulty. The main point that requires some care is the choice of the manner of discretizing the unit time interval for the various problems. For the first theorem we give a fair amount of detail in the proof; for the others we only mention a few relevant differences. Throughout the section we use X to denote the process $W^{(N,d)}$.

THEOREM 2.1. *(Uniform continuity for intervals) Let φ be a nonnegative, nondecreasing, continuous function defined for large arguments. Then for almost all ω there is an $\varepsilon(\omega)$ such that for all intervals $\Delta(s,t)$ with $\Delta(s,t) \subset \Delta(\langle 1\rangle)$ and $|\Delta(s,t)| < \varepsilon(\omega)$,*

$$|X(\Delta(s,t))| < |\Delta(s,t)|^{\frac{1}{2}}\varphi(|\Delta(s,t)|^{-1}) \tag{2.1}$$

if and only if

$$\int^\infty (\log\xi)^{3N+\frac{1}{2}d-2}e^{-\varphi^2(\xi)/2}\,d\xi \tag{2.2}$$

converges.

COROLLARY 2.1. *The function*

$$\varphi(\xi) = \left(\textstyle\sum_{k=1}^{n} a_k \log_k \xi\right)^{\frac{1}{2}}\,,$$

where $\log_k \xi$ is the logarithm function iterated k times, does not satisfy (2.1) *if*

$$a_1 = 2\,, \qquad a_2 = 6N+d-2\,, \qquad a_k = 2 \qquad \text{for} \quad k \geqq 3\,,$$

but it does if a_n is increased by η for any $\eta > 0$.

PROOF. The first step is to observe that it suffices to prove the theorem for φ satisfying

$$(\log \xi)^{\frac{1}{2}} \leqq \varphi(\xi) \leqq 2(\log \xi)^{\frac{1}{2}} . \tag{2.3}$$

The proof of this goes exactly as in [11] so we do not repeat it.

Now suppose that the integral converges. Let $i = (i_1, \cdots, i_{N-1})$, $m = (m_1, \cdots, m_{N-1})$, and define the events

$$E(i, j, k; m, p) = [\sup |X(\Delta(s, t))| \geqq A^{\frac{1}{2}}\varphi(B^{-1})]$$

where the supremum is taken over all s, t satisfying

$$\frac{i_n}{p} 2^{-m_n/p} \leqq s_n < \frac{i_n + 1}{p} 2^{-m_n/p} , \qquad n = 1, 2, \cdots, N-1 ,$$

$$\frac{i_n}{p} 2^{-m_n/p} + 2^{-(m_n+1)/p} \leqq t_n < \frac{i_n + 1}{p} 2^{-m_n/p} + 2^{-m_n/p} ,$$
$$n = 1, 2, \cdots, N-1 ;$$

$$\frac{j-1}{2^p} \leqq s_N < \frac{j}{2^p} , \qquad \frac{j+k}{2^p} \leqq t_n < \frac{j+k+1}{2^p} ,$$

and

$$A = A(k, m, p) = (2^{-1/p} - p^{-1})^{N-1} 2^{-(m_1 + \cdots + m_{N-1})/p} k 2^{-p}$$

is the infimum of the volumes of the intervals involved, and

$$B = B(k, m, p) = (1 + p^{-1})^{N-1} 2^{-(m_1 + \cdots + m_{N-1})/p} (k+2) 2^{-p}$$

is the corresponding supremum. The parameters will be restricted to the following ranges:

$$0 \leqq i_n \leqq p 2^{m_n/p} , \qquad 1 \leqq j \leqq 2^p , \qquad \tfrac{1}{4}p \leqq k \leqq p , \qquad 0 \leqq m_n < p^2 ,$$
$$p = 3, 4, \cdots .$$

It is easy to check that

$$0 \leqq 1 - AB^{-1} \leqq cp^{-1} \tag{2.4}$$

where c depends only on the time dimension N. By Lemmas 1.2 and 1.1 we obtain

$$\begin{aligned} P(E(i, j, k, m, p)) &\leqq 4^N P[|U| \geqq (AB^{-1})^{\frac{1}{2}} \varphi(B^{-1})] \\ &\leqq c\{\varphi(B^{-1})\}^{d-2} \exp[-\varphi^2(B^{-1})/2] \exp[\varphi^2(B^{-1})\{1 - AB^{-1}\}/2] \end{aligned}$$

where U is a normal random variable with mean 0 and covariance matrix I and c depends only on N and d. The last factor is bounded by (2.3) and (2.4) since

$$\varphi^2(B^{-1}) \leqq 4 \log B^{-1} \leqq 4 \log 2^{Np} = O(p) .$$

Using (2.3) again, we see that $p^{-1}\varphi^2(B^{-1})$ is also bounded below and so for large p

$$P(E(i, j, k, m, p)) \leqq cp^{d/2-1} \exp\{-\tfrac{1}{2}\varphi^2(p^{-1}2^{p+1+(m_1+\cdots+m_{N-1})/p})\} .$$

Now for $\nu_n p \leqq m_n < (\nu_n + 1)p$, we replace m_n by $\nu_n p$ and obtain the bound

$$\begin{aligned}&\textstyle\sum_{i,j,k,m,p} P(E(i,j,k,m,p)) \\ &\quad \leqq c \sum_{p=3}^{\infty} \sum_{\nu_1=0}^{p-1} \cdots \sum_{\nu_{N-1}=0}^{p-1} p^{2N+d/2-2} 2^{\nu_1+\cdots+\nu_{N-1}+p} \exp\{-\tfrac{1}{2}\varphi^2(p^{-1}2^{\nu_1+\cdots+\nu_{N-1}+p+1})\} \\ &\quad \leqq c \sum_{r=3}^{\infty} r^{3N+d/2-2} r^{-1} 2^r \exp\{-\tfrac{1}{2}\varphi^2(r^{-1}2^r)\}\end{aligned}$$

where we have let $r = \nu_1 + \cdots + \nu_{N-1} + p + 1$ at the last step, there being no more than r^{N-1} ways of choosing $\nu_1, \cdots, \nu_{N-1}$, and p to accomplish this. This sum is now seen to converge by comparison with the integral (2.2) and so a.s. only finitely many of the $E(i, j, k, m, p)$ occur. Then for a given sample path we can find a p_0 such that none of these occur for $p \geqq p_0$. Let $\Delta(s, t)$ be an interval with $t_N - s_N = \min\{t_n - s_n\}$ and $|\Delta(s, t)| \leqq p_0^N 2^{-Np_0}$. There is clearly no loss of generality in the first assumption; the second implies that $t_N - s_N \leqq p_0 2^{-p_0}$. Now we choose p so that

$$(p + 1)2^{-p-1} < t_N - s_N \leqq p2^{-p}$$

and then j, k, m, and i so that

$$\begin{aligned}&(j - 1)2^{-p} \leqq s_N < j2^{-p}, \qquad (j + k)2^{-p} \leqq t_N < (j + k + 1)2^{-p}, \\ &2^{-(m_n+1)/p} < t_n - s_n \leqq 2^{-m_n/p}, \qquad p^{-1}i_n 2^{-m_n/p} \leqq s_n < p^{-1}(i_n + 1)2^{-m_n/p}.\end{aligned}$$

It is now easy to check that if $\Delta(s, t) \subset \Delta(\langle 1\rangle)$ the restrictions on the indices are satisfied and $\Delta(s, t)$ is one of the intervals in the event $E(i, j, k, m, p)$. It follows that (2.1) is true.

Now suppose that the integral (2.2) diverges. Define the events

$$F(i, j, k, m, p) = [|X(\Delta(s, t))| \geqq |\Delta(s, t)|^{\frac{1}{2}}\varphi(|\Delta(s, t)|^{-1})],$$

where

$$\begin{aligned}&s_n = p^{-1}i_n 2^{-m_n/p}, \qquad t_n = s_n + 2^{-m_n/p}, \qquad n = 1, 2, \cdots, N - 1, \\ &s_N = j2^{-p}, \qquad t_N = (j + k)2^{-p}.\end{aligned}$$

The parameters will be restricted as follows:

$$0 \leqq i_n \leqq \tfrac{1}{2}p2^{m_n/p}, \qquad 0 \leqq j \leqq 2^{p-1}, \qquad \tfrac{3}{4}p \leqq k \leqq p, \qquad p \leqq m_n < p^2, \qquad p = 3, 4, \cdots.$$

These conditions ensure that all intervals under consideration are contained in the unit interval. Now by Lemma 1.1 and (2.3)

$$P(F(i, j, k, m, p)) \geqq cp^{d/2-1} \exp\{-\tfrac{1}{2}\varphi^2(p^{-1}2^{1+p+(m_1+\cdots+m_{N-1})/p})\}.$$

If $(\nu_n - 1)p \leqq m_n < \nu_n p$, we replace m_n by $\nu_n p$ and obtain the bound

$$\begin{aligned}&\textstyle\sum_{i,j,k,m,p} P(F(i,j,k,m,p)) \\ &\quad \geqq c \sum_{p=3}^{\infty} \sum_{\nu_1=2}^{p} \cdots \sum_{\nu_{N-1}=2}^{p} p^{2N+d/2-2} 2^{p+\nu_1+\cdots+\nu_{N-1}} \exp\{-\tfrac{1}{2}\varphi^2(p^{-1}2^{1+p+\nu_1+\cdots+\nu_{N-1}})\} \\ &\quad \geqq c \sum_{r=3+2N}^{\infty} r^{3N+d/2-2} r^{-1} 2^r \exp\{-\tfrac{1}{2}\varphi^2(r^{-1}2^r)\}\end{aligned}$$

where we have let $r = 2 + p + \nu_1 + \cdots + \nu_{N-1}$ at the last step and the constant has also been changed. To see that there are at least cr^{N-1} ways of choosing $\nu_1, \cdots, \nu_{N-1}$, and p, simply note that if

$$\frac{r}{2} \leqq p \leqq \frac{2r}{3}, \qquad \nu_n \leqq \frac{r}{4N} \qquad \text{for } n = 1, 2, \cdots, N-2,$$

then there will be an appropriate value of ν_{N-1} at least for large r. By comparison with the integral (2.2), we know the sum diverges. Thus we have verified condition (i) of Lemma 1.4; next we will check condition (iii). We order the events so that $|\Delta(s, t)|$ decreases. Now fix an event $F = F(i, j, k, m, p)$ and let $F' = F'(i', j', k', m', p')$ be an event which follows F in the order. We will also use $\Delta = \Delta(s, t)$ for the interval involved in F and $\Delta' = \Delta'(s', t')$ for the one involved in F'. Let ρ denote the correlation between the first component of $X(\Delta)$ and the first component of $X(\Delta')$. By Lemma 1.5, since

$$c_1(pp')^{\frac{1}{2}} \leqq \varphi(|\Delta|^{-1})\varphi(|\Delta'|^{-1}) \leqq c_2(pp')^{\frac{1}{2}}$$

we can assume that for an appropriate c

$$c(pp')^{-1} \leqq \rho^2 . \tag{2.5}$$

Now

$$\begin{aligned} \rho^2 = \frac{|\Delta \cap \Delta'|^2}{|\Delta|\,|\Delta'|} &\leqq \frac{\prod_{n=1}^{N}[(t_n - s_n) \wedge (t_n' - s_n')]^2}{\prod_{n=1}^{N}(t_n - s_n)\prod_{n=1}^{N}(t_n' - s_n')} \\ &\leqq \frac{(t_n - s_n) \wedge (t_n' - s_n')}{(t_n - s_n) \vee (t_n' - s_n')} \end{aligned} \tag{2.6}$$

for $n = 1, 2, \cdots, N$. Using (2.6) with $n = N$ in (2.5) leads easily to

$$p' - p = O(\log p) . \tag{2.7}$$

Furthermore, for every n, we have

$$m_n' \leqq p'^2 = O(p^2) ; \qquad k' \leqq p' = O(p) .$$

To count the number of events F' which give rise to values of ρ satisfying (2.5) it only remains to see how many possible values of i' and j' there are. We will get an estimate by counting all those where Δ' intersects Δ. Increasing i_n' by one moves Δ' a distance $p'^{-1}2^{-m_n'/p'}$; the furthest we can move it is $2^{-m_n/p} + 2^{-m_n'/p'}$. Thus the number of possible values of i_n' is at most

$$p'2^{m_n'/p' - m_n/p} + p' ;$$

since $2^{m_n'/p' - m_n/p} = O(pp')$ by (2.5) and (2.6), this number is $O(p^3)$ by (2.7). In a similar way one shows that there are $O(p^3)$ relevant values of j'. Putting these estimates together we have at most $O(p^{5N-1}\log p)$ intervals Δ' which satisfy (2.5). Now we assume, for the moment, that $\rho^2 \leqq 1 - p^{-\frac{1}{2}}$. Then by Lemma 1.6,

$$P(F \cap F') \leqq c' \exp\{-\tfrac{1}{8}(1 - \rho^2)\varphi^2(|\Delta|^{-1})\}P(F) \leqq c' \exp\{-cp^{\frac{1}{2}}\}P(F) ,$$

and since $p^{5N-1}\log p \exp\{-cp^{\frac{1}{2}}\} = O(1)$ we may go on to consider those Δ' for which

$$\rho^2 \geqq 1 - p^{-\frac{1}{2}} . \tag{2.8}$$

We will do this by subdividing further as follows:

$$1 - \frac{\mu}{p} \leqq \rho^2 < 1 - \frac{\mu - 1}{p} , \qquad \mu = 1, \cdots, p^{\frac{1}{2}} . \tag{2.9}$$

As before, we have by Lemma 1.6 that

$$P(F \cap F') \leqq c' e^{-c\mu} P(F) .$$

We will show that the number of intervals Δ' which yield values of ρ satisfying the first inequality in (2.9) is $O(\mu^{2N})$. Since

$$\sum_{\mu=1}^{p^{\frac{1}{2}}} \mu^{2N} e^{-c\mu} \leqq K$$

independent of p, this will suffice to prove (iii). Proceeding as before, even condition (2.8) implies that $p' = p$ for large p. Similarly, by (2.6) and (2.9),

$$1 - \frac{\mu}{p} \leqq \rho^2 \leqq 2^{-|m_n/p - m_n'/p|} \Rightarrow -\frac{|m_n - m_n'|}{p} \log 2 \geqq \log\left(1 - \frac{\mu}{p}\right) \sim -\frac{\mu}{p}$$

so that $m_n - m_n' = O(\mu)$. In much the same way one shows that $k' - k = O(\mu)$. We must still count the number of possible values for i' and j'. Consider the edges of Δ and Δ' which are parallel to the nth coordinate axis. Changing i_n' by one moves Δ' a distance $p^{-1} 2^{-m_n'/p}$. Thus there are

$$|2^{-m_n'/p} - 2^{-m_n/p}| p 2^{m_n'/p} \leqq p(2^{|m_n' - m_n|/p} - 1) \leqq p(2^{c\mu/p} - 1) = O(\mu)$$

values of i_n' which give rise to an edge of length $2^{-m_n/p} \wedge 2^{-m_n'/p}$ for the intersection. If we have $m_n' \leqq m_n$ and change i_n' by an additional ν then we must have

$$1 - \frac{\mu}{p} \leqq \rho^2 \leqq \frac{2^{-m_n/p} - \nu p^{-1} 2^{-m_n'/p}}{2^{-m_n/p}} \leqq 1 - \frac{\nu}{p}$$

so that $\nu \leqq \mu$. A similar argument works if $m_n' > m_n$. Thus the number of possible values for i_n' is $O(\mu)$. Finally one estimates the number of possible values of j' as $O(\mu)$ in much the same manner. This then gives the total number of F' with ρ satisfying (2.9) as $O(\mu^{2N})$ as promised. To complete the proof of the theorem it only remains to verify condition (ii) of Lemma 1.4. For this part, we follow Sirao ([11] page 147). He proves that if $\{U_i\}_{i=1}^{\infty}$ are jointly normal and one dimensional with mean 0, variance 1, and $\rho_{ij} = E[U_i U_j]$, and

$$\rho_m = \max_{1 \leqq i \leqq n} \rho_{im} \to 0 \qquad \text{as } m \to \infty ,$$

then the ratio of the joint density of $(U_1, \cdots, U_n, U_m)$ to the product of the joint density of $(U_1, \cdots, U_n)$ and the density of U_m converges to one uniformly as $m \to \infty$ provided that the first n arguments are restricted to a compact set and the last argument is restricted to the interval $[-\rho_m^{-\alpha}, \rho_m^{-\alpha}]$ for some $\alpha < 1$. (If the distribution of $(U_1, \cdots, U_n)$ is singular the densities must be with respect to Lebesgue measure on a lower dimensional subspace.) The generalization of this fact to d dimensions follows immediately since the joint densities are just products of d copies of the one dimensional ones. We want to compare

$$P(F_m F_h^c \cdots F_n^c) \qquad \text{with} \quad P(F_m)P(F_h^c \cdots F_n^c)$$

where F_n has the form $[|U_n| \geqq c_n]$. (We have now ordered the events $F(i, j, k, m, p)$ in terms of decreasing $|\Delta(s, t)|$ and are using a single subscript.) The events $F_h^c, \cdots, F_n^c$ are thus of the form $[|U_n| < c_n]$ and so $U_h, \cdots, U_n$ are already restricted to compact sets. We replace F_m by

$$G_m = [c_m \leqq |U_m| < 2c_m]$$

and use the fact that $P(F_m) \leqq 2P(G_m)$. Then the above convergence of the densities suffices provided we show that $\rho_m \to 0$ and $2c_m \leqq \rho_m^{-\alpha}$ for some $\alpha < 1$. Now if the p parameter corresponding to F_m is p' and the ones for $F_h, \cdots, F_n$ are all no larger than p, we have by (2.6)

$$\rho_m \leqq (p' 2^{p-p'})^{\frac{1}{2}}$$

so that $\rho_m \to 0$, while

$$2c_m = 2\varphi(|\Delta(s, t)|^{-1}) \leqq cp'^{\frac{1}{2}} \leqq \rho_m^{-\frac{1}{2}}$$

for large values of p'.

THEOREM 2.2. (*Local continuity for intervals; local continuity at the origin for points*) *Let φ be a nonnegative, non-decreasing, continuous function defined for large arguments and let $s \in \Delta(\langle 1 \rangle)$. Then for almost all ω there is an $\varepsilon(\omega)$ such that for all intervals $\Delta(s, t)$ with $|\Delta(s, t)| < \varepsilon(\omega)$ and $\Delta(s, t) \subset \Delta(\langle 1 \rangle)$,*

$$|X(\Delta(s, t))| \leqq |\Delta(s, t)|^{\frac{1}{2}}\varphi(|\Delta(s, t)|^{-1}) \tag{2.10}$$

if and only if

$$\int^\infty \xi^{-1}(\log \xi)^{N-1}(\log\log \xi)^{N+\frac{1}{2}d-1}e^{-\varphi^2(\xi)/2}\, d\xi$$

converges.

REMARK. By scaling, it is clear that s can be any element of R_+^N, but it is important that t be restricted to a compact set, the set depending on s. There exist (unbounded) sequences t_n such that $|\Delta(t_n)| \to 0$ and

$$\limsup_{n\to\infty} \frac{|X(\Delta(t_n))|}{\{|\Delta(t_n)| \log\log |\Delta(t_n)|^{-1}\}^{\frac{1}{2}}} = \infty \quad \text{a.s.}$$

COROLLARY 2.2. *The function*

$$\varphi(\xi) = \left(\textstyle\sum_{k=2}^n a_k \log_k \xi\right)^{\frac{1}{2}},$$

where $\log_k \xi$ is the logarithm function iterated k times, does not satisfy (2.10) *if*

$$a_2 = 2N, \qquad a_3 = 2N + d, \qquad a_k = 2 \qquad \text{for} \quad k \geqq 4,$$

but it does if a_n is increased by η for any $\eta > 0$.

PROOF. First one shows as in [11] that it suffices to prove the theorem for φ satisfying

$$(\log\log\xi)^{\frac{1}{2}} \leqq \varphi(\xi) \leqq 2N(\log\log\xi)^{\frac{1}{2}} .$$

Then for the convergent case, let

$$E(k, m, p) = [\sup |X(\Delta(s, t))| \geqq A^{\frac{1}{2}}\varphi(B^{-1})]$$

where the supremum is taken over all t satisfying

$$2^{-(m_n+1)/\log p} \leqq t_n - s_n < 2^{-m_n/\log p}, \qquad n = 1, 2, \cdots, N-1,$$
$$k2^{-p} \leqq t_N - s_N < (k+1)2^{-p},$$

and

$$A = A(k, m, p) = 2^{-(m_1+\cdots+m_{N-1}+N-1)/\log p}k2^{-p}$$

is the infimum of the volumes of the intervals involved and B is the corresponding supremum. The parameters are restricted to the ranges

$$\tfrac{1}{4}\log p \leqq k \leqq \log p, \qquad 0 \leqq m_n < p\log p, \qquad p = 3, 4, \cdots .$$

The only other point where there is a significant difference in the convergent case is that we may want to allow for the possibility that some $t_n - s_n$ may be negative. In other words we may want to let t approach s in an arbitrary way instead of requiring that $t_n \geqq s_n$ for all n. However, this does not change the probability estimates and we only get 2^N cases exactly like the one we have described. Thus the theorem is valid without regard to whether the manner in which t approaches s is restricted or not. In the divergent case, let

$$F(k, m, p) = [|X(\Delta(s, t))| \geqq |\Delta(s, t)|^{\frac{1}{2}}\varphi(|\Delta(s, t)|^{-1})],$$

where

$$t_n - s_n = 2^{-m_n/\log p}, \qquad n = 1, \cdots, N-1,\ t_N - s_N = k2^{-p}$$

with parameter restrictions

$$\tfrac{3}{4}\log p \leqq k \leqq \log p, \qquad \log p \leqq m_n < p\log p, \qquad p = 3, 4, \cdots .$$

Then the proof goes as before except that in many estimates p is replaced by $\log p$; this is done, for example, in (2.5), (2.8), and (2.9). Furthermore, in counting the number of values of m_n' which give rise to ρ values satisfying the analogue of (2.5), one must use (2.6) to get a bound of $O(\log p \log\log p)$. Note that we have ensured that $t_n - s_n > 0$ for all n so that this proof is valid even if the manner in which t approaches s is restricted.

Next we will discuss these problems where we consider $X_t - X_s$ instead of $X(\Delta(s, t))$. We will consider the local problem (i.e. with s fixed) first this time since our results are more complete in this case. Distances are to be measured in terms of the metric δ defined in the introduction. Note that if $s_i = 0$ for any i, then s is identified with the origin and, moreover, $X_t - X_s = X_t = X(\Delta(t))$ while $\delta(s, t) = |\Delta(t)|$. Thus Theorem 2.2 gives the correct integral test for this problem for an s of this type. However, the integral test is different for s with $\delta(s) > 0$.

THEOREM 2.3. (*Local continuity away from the origin for points*) *Let φ be a nonnegative, non-decreasing, continuous function defined for large arguments and let $s \in R_+^N$ with $\delta(s) > 0$. Then for almost all ω there is an $\varepsilon(\omega)$ such that for all t with $\delta(s, t) < \varepsilon(\omega)$,*

$$|X_t - X_s| \leqq \{\delta(s, t)\}^{\frac{1}{2}}\varphi(1/\delta(s, t)) \tag{2.11}$$

if and only if

$$\int^\infty \xi^{-1}(\log\log \xi)^{N+\frac{1}{2}d-1}e^{-\varphi^2(\xi)/2}\, d\xi < \infty .$$

COROLLARY 2.3. *The function*

$$\varphi(\xi) = (\textstyle\sum_{k=2}^n a_k \log_k \xi)^{\frac{1}{2}} ,$$

where $\log_k \xi$ is the logarithm function iterated k times, does not satisfy (2.11) *if*

$$a_2 = 2 , \qquad a_3 = 2N + d , \qquad a_k = 2 \qquad \text{for} \quad k \geqq 4 ,$$

but it does if a_n is increased by η for any $\eta > 0$.

PROOF. The first step in the proof is to show as in [11] that it suffices to prove the theorem for φ satisfying

$$(\log\log \xi)^{\frac{1}{2}} \leqq \varphi(\xi) \leqq 2(\log\log \xi)^{\frac{1}{2}} .$$

For the convergent case, let

$$E_{kp} = [\sup |X_t - X_s| > A^{\frac{1}{2}}\varphi(B^{-1})]$$

where the supremum is taken over all those t satisfying

$$k_n 2^{-p} \leqq t_n - s_n < (k_n + 1)2^{-p} , \qquad n = 1, 2, \cdots, N \tag{2.12}$$

and A is the infimum of the values of $\delta(s, t)$ when t varies over this set and B the corresponding supremum. The range of the parameters is

$$|k_n| \leqq |k_N| + 1 , \qquad n = 1, 2, \cdots, N - 1 ;$$
$$\tfrac{1}{4}\log p \leqq |k_N| \leqq \log p ; \qquad p = 3, 4, \cdots .$$

For given t, let $u_n = s_n \wedge t_n$, $n = 1, \cdots, N$, and let

$$I = \{n \leqq N : k_n < 0\} = \{n \leqq N : u_n = t_n\} ; \qquad J = \{n \leqq N : k_n \geqq 0\} .$$

Then $\Delta(s) \cap \Delta(t) = \Delta(u)$ and

$$\delta(s, t) = \textstyle\prod_{n\in J} s_n(\prod_{n\in I} s_n - \prod_{n\in I} t_n) + \prod_{n\in I} t_n(\prod_{n\in J} t_n - \prod_{n\in J} s_n) . \tag{2.13}$$

We take p large enough so that there is a $c > 0$ with $t_n \geqq c$ for all n and all t that appear in any E_{kp}. Then it is easy to check from (2.13) that

$$c_1 2^{-p}\log p \leqq A \leqq B \leqq c_2 2^{-p}\log p$$

and that $B - A = O(2^{-p})$. Now we use Lemma 1.3 to estimate $P(E_{kp})$. First we have $\gamma \leqq 2^{-p+1}\log p$ and if we take $\beta = 2^{-p/2}$, it is easy to check that the second term in the estimate is dominated by the first. Now by Lemma 1.1,

$$P(E_{kp}) \leqq c(\log p)^{d/2-1}\exp[-\tfrac{1}{2}\varphi^2(B^{-1})]\exp[\varphi^2(B^{-1})\{1 - AB^{-1}\}/2]\exp[\varphi(B^{-1})\beta B^{-\frac{1}{2}}]$$

and the last two factors are bounded due to the bounds we have given for A, B, and φ. Thus

$$\sum_{k,p} P(E_{kp}) \leqq c \sum_p (\log p)^{N+d/2-1} \exp[-\varphi^2(2^p/c_2 \log p)/2]$$

and the series converges by comparison with the given integral. This proves the result for all t such that

$$|t_N - s_N| = \max_{1\leqq n\leqq N} |t_n - s_n|$$

and this is clearly sufficient. For the divergent case, let

$$F_{kp} = [|X_t - X_s| > \{\delta(s, t)\}^{\frac{1}{2}}\varphi(1/\delta(s, t))]$$

where $t_n = s_n + k_n 2^{-p}$ and $\frac{3}{4} \log p \leqq k_n \leqq \log p$, $p = 3, 4, \cdots$. Then one shows that infinitely many of these events occur by using Lemma 1.4 very much as in the previous theorem.

For the uniform problem involving $X_t - X_s$, we have been unable to obtain the integral test. The integral

$$\int^\infty \xi^{N-1}(\log \xi)^{2N+\frac{1}{2}d-1}e^{-\varphi^2(\xi)/2}\, d\xi$$

seems to be the correct criterion but there is enough dependence in the events involved that it is impossible to check condition (iii) of Lemma 1.4. We have also been unable to find any other version of the Borel–Cantelli Lemma which will work. We have, however, obtained the modulus of continuity for this problem and we give this now.

THEOREM 2.4. (*Uniform modulus of continuity for points*) *Let* $h(\xi) = (2\xi \log 1/\xi)^{\frac{1}{2}}$. *Then*

$$\lim_{\varepsilon\to 0} \sup_{s,t\in\Delta(\langle 1\rangle),\delta(s,t)\leqq\varepsilon} \frac{|X_t - X_s|}{h(\delta(s, t))} = N^{\frac{1}{2}} \quad \text{a.s.}$$

PROOF. For notational simplicity only, we give the proof for $N = 2$. Let K be the symmetric difference between $\Delta(s)$ and $\Delta(t)$ and write $K = A \cup B$ where A and B are disjoint intervals. Then for $\varepsilon > 0$, we have by Theorem 2.1

$$|X_t - X_s| \leqq |X(A)| + |X(B)| \leqq (1 + \varepsilon)(h(|A|) + h(|B|))$$

for $|A|$, $|B|$ sufficiently small. For ξ small, $h(\xi)$ is concave so

$$h(|A|) + h(|B|) \leqq 2h\left(\frac{|A| + |B|}{2}\right) \sim 2^{\frac{1}{2}}h(|A| + |B|)$$

as $|A| + |B| \to 0$. Since $\delta(s, t) = |A| + |B|$, this proves the upper bound. For the opposite inequality, first note that it is sufficient to prove it for $d = 1$ since $|X_t - X_s|$ is larger than any of its components. Now let $\varepsilon > 0$, $\eta > 0$, and set

$$A_{mk} = \Delta\left(\left\langle \tfrac{1}{2} + \frac{k}{m}\eta, 0\right\rangle, \left\langle \tfrac{1}{2} + \frac{k+1}{m}\eta, \tfrac{1}{2}\right\rangle\right),$$
$$k = 0, \cdots, m - 1;\ m = 1, 2, \cdots.$$

Then Lévy's proof for the uniform modulus of continuity of Brownian motion shows that with probability one, for all but finitely many m there exists a k such that

$$X(A_{mk}) \geqq (1 - \varepsilon)h(|A_{mk}|) . \tag{2.14}$$

Similarly, for all but finitely many m there is a j such that

$$X(B_{mj}) \geqq (1 - \varepsilon)h(|B_{mj}|) , \tag{2.15}$$

where

$$B_{mj} = \Delta\left(\left\langle 0, \tfrac{1}{2} + \frac{j}{m}\eta\right\rangle, \left\langle \tfrac{1}{2}, \tfrac{1}{2} + \frac{j+1}{m}\eta\right\rangle\right),$$
$$j = 0, \cdots, m-1;\ m = 1, 2, \cdots .$$

Let $A = A_{mk}$ and $B = B_{mj}$ be such that (2.14) and (2.15) hold and let

$$s = \left\langle \tfrac{1}{2} + \frac{k}{m}\eta, \tfrac{1}{2} + \frac{j}{m}\eta\right\rangle, \qquad t = \left\langle \tfrac{1}{2} + \frac{k+1}{m}\eta, \tfrac{1}{2} + \frac{j+1}{m}\eta\right\rangle .$$

Then $X_t - X_s = X(K)$ where K is the symmetric difference of $\Delta(s)$ and $\Delta(t)$. If $C = K\backslash(A \cup B)$, then $C = C' \cup C''$ with C', C'' disjoint intervals. Then

$$\frac{X(K)}{h(|K|)} = \frac{X(A)}{h(|K|)} + \frac{X(B)}{h(|K|)} + \frac{X(C')}{h(|K|)} + \frac{X(C'')}{h(|K|)} . \tag{2.16}$$

Since

$$|K| = |A| + |B| + |C| = 2|A| + |C| \leqq |A|(2 + 4\eta) ,$$

we have $h(|K|) \leqq (2 + 4\eta)^{\frac{1}{2}}h(|A|)$. Thus the sum of the first two terms on the right in (2.16) is at least $2(1 - \varepsilon)(2 + 4\eta)^{-\frac{1}{2}}$. For the last two terms we know by the first part that

$$X(C') \geqq -2h(|C'|) \geqq -2h(2\eta|K|) \sim -2(2\eta)^{\frac{1}{2}}h(|K|)$$

and $X(C'')$ is handled similarly. By choosing η and ε appropriately we can thus make (2.16) as close to $2^{\frac{1}{2}}$ as we desire.

3. Recurrence properties. If $X = W^{(1,d)}$ is the ordinary one parameter Wiener process with values in R^d, it is well known that if $d = 1$ or 2 X is interval recurrent in the sense that any open ball is entered by X_t for arbitrarily large values of t with probability one. For $d > 2$ the probability that a given ball is entered for arbitrarily large values of t is zero. For $d = 1$ the process is also point recurrent in the sense that for each x, the probability that $X_t = x$ for arbitrarily large values of t is one; by contrast, when $d \geqq 2$, the probability that $X_t = x$ for some $t > 0$ is zero. This section is devoted to establishing correct analogues for $W^{(N,d)}$.

For $N \geqq 2$, $d \geqq 1$, consider the one parameter process

$$Y_\lambda = W^{(N,d)}_{\langle \lambda, \lambda^{-1}, 1, \cdots, 1\rangle} , \qquad \lambda > 0 ,$$

with values in R^d. This process has the same finite dimensional distributions as

$\lambda^{-1}W^{(1,d)}_{\lambda^2}$ and it is easy to check that Y_λ enters any open ball for arbitrarily large λ with probability one regardless of the value of d. This shows that if we do not wish to conclude that $W^{(N,d)}$ is interval recurrent, we must exercise care about the way the N-dimensional parameter t goes to infinity. In particular it will not be enough to require that $\delta(t) \to \infty$. For by considering a process similar to Y_λ we can find, for any $\rho > 0$ and any open ball B, a t such that $\delta(t) = \rho$ and $W_t^{(N,d)} \in B$. Thus we obtain $\{t_n\}$ with $\delta(t_n) \to \infty$ and $W_{t_n}^{(N,d)} \in B$.

In Theorem 3.1 we show that when $d > 2N$, $W^{(N,d)}$ is not interval recurrent in a sense made precise in the theorem. On the other hand, Theorem 3.2 shows that when $d \leqq 2N$ a strong notion of interval recurrence does hold. Point recurrence is shown to hold when $d < 2N$ in Theorem 3.3. Finally, it is shown that $W^{(N,d)}$ does not hit points when $d \geqq 2N$ in Theorem 3.4.

THEOREM 3.1. (*Transience*) *Let* $d > 2N$, $\rho > 0$. *Then, with probability one, there exist only finitely many N-tuples of positive integers* $j = (j_1, j_2, \cdots, j_N)$ *such that*

$$\inf\{|W_t^{(N,d)}| : j_i \leqq t_i < j_i + 1,\ i = 1, \cdots, N\} < \rho .$$

REMARK. This is equivalent to saying that if t is restricted to the domain $\{t : t_i \geqq 1, i = 1, \cdots, N\}$ then $|W_t^{(N,d)}| \to \infty$ as $\delta(t) \to \infty$.

PROOF. Let $X = (X^{(1)}, X^{(2)}, \cdots, X^{(d)}) = W^{(N,d)}$. Fix $j = \langle j_1, j_2, \cdots, j_N\rangle$ with each $j_i \geqq 1$, and let

$$\langle j; p\rangle = \langle j_i + p_i j_i/\delta(j)\rangle , \qquad p_i = 0, 1, \cdots, \delta(j)/j_i .$$

If $\Delta_{jp} = \Delta(\langle j; p\rangle, \langle j; p+1\rangle)$, then we have

$$\begin{aligned} &P[|X_t| \leqq \rho \ \text{ for some } \ t \in \Delta_{jp}] \\ &\quad \leqq P[|X_{\langle j;p\rangle}| \leqq 2\rho] + \textstyle\int_{|x|>2\rho} P[\sup_{t\in\Delta_{jp}} |X_t - X_{\langle j;p\rangle}| \geqq |x| - \rho] P[X_{\langle j;p\rangle} \in dx] \\ &\quad \leqq c\rho^d\{\delta(j)\}^{-d/2} + c\{\delta(j)\}^{-d/2} \textstyle\int_{|x|>2\rho} P[|X_{\langle j;p+1\rangle} - X_{\langle j;p\rangle}| \geqq \frac{1}{2}|x|]\, dx , \end{aligned}$$

where Lemma 1.3 and the remark following it have been used at the last step. Since the variance of $X^{(1)}_{\langle j;p+1\rangle} - X^{(1)}_{\langle j;p\rangle}$ is the volume of $\Delta(\langle j; p+1\rangle)$ minus the volume of $\Delta(\langle j; p\rangle)$, it is bounded by $N2^N$. The last integral above is then bounded by a constant depending only on N by Lemma 1.1. Summing over all p_i, we obtain the bound

$$P[|X_t| \leqq \rho \ \text{ for some } \ t \in \Delta(j, j+1)] \leqq c\{\delta(j)\}^{N-1-d/2} , \tag{3.1}$$

the constant depending on ρ, d, and N. Since the bound is summable over all j_i, an application of Borel–Cantelli completes the proof.

THEOREM 3.2. (*Interval Recurrence*) *For* $n \geqq 1$, $p = (p_2, \cdots, p_N)$, $1 \leqq p_i \leqq n$, *let*

$$\langle n; p\rangle = \langle n, 1 + p_2/n, 1 + p_3/n, \cdots, 1 + p_N/n\rangle$$

and

$$A_{np} = [|W^{(N,d)}_{\langle n;p\rangle}| \leqq 1] .$$

Then for $d \leqq 2N$, *infinitely many of the events* A_{np} *occur* a.s.

REMARK. The same proof shows that for any open set B in R^d infinitely many of the events $W^{(N,d)}_{\langle n;p\rangle} \in B$ occur with probability one.

PROOF. Suppose $d = 2N$; it suffices (by a projection argument) to treat this case. Since

$$P(A_{np}) \geqq cn^{-d/2} = cn^{-N}$$

uniformly for p in the range in question,

$$\sum_{n=1}^{M} \sum_{p} P(A_{np}) \geqq c \sum_{n=1}^{M} n^{-1} \sim c \log M \tag{3.2}$$

as $M \to \infty$. Since the events under consideration are not independent, we need a refined version of Borel–Cantelli to finish the proof. A convenient one here is that given by Kochen and Stone [6]; it requires that

$$\liminf_{M\to\infty} \frac{\sum_{n,n'=1}^{M} \sum_{p,p'} P(A_{np} \cap A_{n'p'})}{\{\sum_{n=1}^{M} \sum_{p} P(A_{np})\}^2} < \infty .$$

According to (3.2) the denominator is at least $c \log^2 M$. A rather more tedious calculation bounds the numerator by a constant times $\log^2 M$. Thus we may conclude that there is positive probability that infinitely many A_{np} occur; by the zero-one law this probability must be one.

THEOREM 3.3. (*Point Recurrence*) *Let* $d < 2N$, $X_t = W_t^{(N,d)}$. *Then*

(i) $\{X_t \colon t \in \Delta(\langle 1 \rangle)\}$ *has positive d-dimensional volume* a.s.;
(ii) *for each* $x \in R^d$ *and* $\lambda > 0$, $P[X_t = x$ *for some* $t > \langle \lambda \rangle] = 1$.

PROOF. Let $d = 2N - 1$; it is enough to consider this case. Partition the cube $\Delta(\langle \frac{1}{2} \rangle, \langle 1 \rangle)$ into m^N little cubes, to be referred to as *cubicles*. Each cubicle has sides of length $1/2m$. For $a > 0$ let $\mathscr{C}(a)$ be the class of cubes in R^d with edges parallel to the coordinate axes, sides of length a, and vertices of the form $(ak_1, \cdots, ak_d)$ with $k_1, \cdots, k_d$ integers. Let

$$V = \{X_t \colon t \in \Delta(\langle \tfrac{1}{2} \rangle, \langle 1 \rangle)\}$$

and let $N(a)$ be the number of cubes of $\mathscr{C}(a)$ intersected by V. Each of the cubicles has a *least vertex*, i.e. closest to $\langle 0 \rangle$. Let $N(a, m)$ be the number of distinct cubes of $\mathscr{C}(a)$ into which these least vertices are mapped by X. Any pair of distinct least vertices mapping into the same cube of $\mathscr{C}(a)$ will be said to give rise to a coïncidence. Let $Q(a, m)$ denote the total number of coincidences. We need an estimate for $E[Q(a, m)]$. To this end, consider two least vertices:

$$s = \langle (m + j_i)/2m \rangle \quad \text{and} \quad t = \langle (m + k_i)/2m \rangle .$$

Note that

$$E[(W_t^{(N,1)} - W_s^{(N,1)})^2] \geqq cm^{-1} \sum_{i=1}^{N} |k_i - j_i| .$$

Thus we obtain

$$\begin{aligned} P[s \text{ and } t \text{ give rise to a coincidence}] &\leqq P[|X_s - X_t| < ad^{\frac{1}{2}}] \\ &\leqq ca^d m^{d/2} (\textstyle\sum_{i=1}^{N} |k_i - j_i|)^{-d/2} . \end{aligned}$$

Keeping t fixed and summing over s we obtain a bound for the expected number of coincidences involving t:

$$ca^d m^{d/2} \sum_{p_N=1}^m \sum_{p_{N-1}=0}^m \cdots \sum_{p_1=0}^m (p_1 + \cdots + p_N)^{-d/2} \leqq ca^d m^{d/2} \sum_{p_N=1}^m p_N^{-\frac{1}{2}} \leqq 2ca^d m^N .$$

Now summing over all t, we have the needed estimate

$$E[Q(a, m)] \leqq ca^d m^{2N} . \tag{3.3}$$

To utilize this result we conceive of our set up as an urn model. The cubes of $\mathscr{C}(a)$ are urns, the least vertices of the cubicles are balls, and the mapping X distributes these balls into a number of distinct urns. Note that the number of occupied urns is $N(a, m)$. It is easy to see that for given m and $N(a, m)$ the number of coincidences is minimized if the m^N balls are distributed as evenly as possible between the $N(a, m)$ urns. Keeping a fixed, we shall let m tend to infinity. Since $N(a, m) \leqq N(a)$, with probability one,

$$Q(a, m) \geqq \frac{1}{2} \frac{m^N}{N(a, m)} \left(\frac{m^N}{N(a, m)} - 1 \right) N(a, m) \sim \frac{m^{2N}}{2N(a, m)} \tag{3.4}$$

as m approaches infinity. Since Q is nonnegative, we have for $\varepsilon > 0$

$$P[Q > \varepsilon^{-1} E(Q)] < \varepsilon$$

and so it follows from (3.3) and (3.4) that for m sufficiently large

$$N(a) \geqq N(a, m) \geqq cm^{2N} Q^{-1} \geqq c\varepsilon m^{2N} [E(Q)]^{-1} > c' a^{-d}$$

with probability exceeding $1 - \varepsilon$. Now $a^d N(a)$ is the value of a Riemann sum approximating the volume of V; it is indeed an upper Darboux sum. Since V is the image of a compact set under the continuous map X, it is compact, and as a consequence the upper Darboux sums will converge to the volume of V. So V has positive volume with probability exceeding $1 - \varepsilon$; since ε is arbitrary, (i) is established.

We turn to the proof of (ii). We will write $\langle s_i \rangle > \langle t_i \rangle$ for $s_i > t_i$, $i = 1, 2, \cdots, N$. Clearly the proof of (i) can be used to show that for any non-degenerate interval Δ, $\{X_t : t \in \Delta\}$ has positive d-dimensional volume. In particular, $\{X_t : t > \langle 1 \rangle\}$ has positive d-dimensional volume. An application of Fubini's Theorem allows us to conclude that

$$\{x : P[X_t = x \text{ for some } t > \langle 1 \rangle] > 0\} \tag{3.5}$$

has positive volume.

Let $h(x, t) = P[X_s = x$ for some $s > t]$. Then $h(x, \langle \rho \rangle)$ is non-increasing as a function of ρ; letting ρ approach infinity we obtain a limit function $h(x)$. The scaling property implies

$$h(x, t) = h(\lambda^{N/2} x, \lambda t) , \qquad \lambda > 0 . \tag{3.6}$$

It follows that $h(x) = h(\lambda x)$, $\lambda > 0$. On the other hand, symmetry considerations make it apparent that $h(x) = h(y)$ if $|x| = |y|$. So there is a constant c_0 such that $h(x) = c_0$ for all $x \neq 0$. The zero-one law implies that $c_0 = 0$ or $c_0 = 1$; we wish to establish the second alternative. For $\rho \geqq 1$,

$$(3.7) \qquad h(x, \langle\rho\rangle) = \int \varphi(x-y)P[X_t - X_{\langle 1\rangle} = y \text{ for some } t > \langle\rho\rangle]\, dy$$

where φ is the standard normal density in R^d. Therefore $h(x, \langle\rho\rangle)$, $\rho \geqq 1$, is an equicontinuous family of functions. So the limit $h(x)$ will be continuous in x, i.e. $h(x) \equiv c_0$ including $x = 0$. Using (3.6), we obtain

$$(3.8) \qquad h(0, t) = h(0, \lambda t) = h(0) = c_0 .$$

According to (3.5), there exists an x making $h(x, \langle 1\rangle) > 0$. By (3.7) with $\rho = 1$,

$$\{y : P[X_t - X_{\langle 1\rangle} = y \text{ for some } t > \langle 1\rangle] > 0\}$$

has positive volume. Using (3.7) again with $x = 0$, $\rho = 1$, we see that $h(0, \langle 1\rangle) > 0$ and so by (3.8) $c_0 > 0$. This means $c_0 = 1$ and establishes (ii).

THEOREM 3.4. (*Non-recurrence for points*) *Let* $d \geqq 2N$, $X_t = W_t^{(N,d)}$. *Then*

(i) $\{X_t : t \in R_+^N\}$ *has zero d-dimesional volume* a.s.;
(ii) *for each* $x \in R^d$, $P[X_t = x$ *for some* t *with* $\delta(t) > 0] = 0$.

PROOF. It is enough to consider the case $d = 2N$. Because of scaling, in order to prove (i) it is sufficient to show that

$$V = \{X_t : t \in \Delta(\langle\tfrac{1}{2}\rangle, \langle 1\rangle)\}$$

has zero volume. We partition $\Delta = \Delta(\langle\frac{1}{2}\rangle, \langle 1\rangle)$ into m^N cubicles of side $1/2m$ and consider the class of cubes $\mathscr{C}(m^{-\frac{1}{2}})$ in R^d defined in the proof of Theorem 3.3. Let $N(m)$ be the number of these cubes intersected by V. Since

$$|V| \leqq N(m)(m^{-\frac{1}{2}})^d = N(m)m^{-N} ,$$

it will suffice to prove that $E\{N(m)\} = o(m^N)$. Fix m and consider the order on R_+^N given by letting s precede t if and only if for some j, $s_j < t_j$ and $s_n = t_n$ for $j < n \leqq N$. Let t^k, $k = 1, 2, \cdots, m^N$, be the least vertices of the cubicles arranged in increasing order, S_k the cubicle corresponding to t^k, and C_k the (random) element of $\mathscr{C}(m^{-\frac{1}{2}})$ which contains X_{t^k}. Define the events

$$F_k = [C_k \cap C_j = \phi \text{ for } j = 1, 2, \cdots, k-1] ;$$

F_k is the event that a new cube is hit at time t^k if we only observe X at the vertices. We are now ready to obtain an upper estimate for $N(m)$. If a new cube is hit at time t^k we count in a *central block* of $(2M+1)^d$ cubes centered at C_k, where M is a large integer to be chosen soon. If the cube is not new, this central block will have been covered previously. In addition, whether the cube is new or not, we add any cubes outside this central block which are intersected by $\{X_s : s \in S_k\}$. Let N_k denote the number of cubes added outside the central block. Then we have

$$(3.9) \qquad N(m) \leqq \textstyle\sum_{k=1}^{m^N} (2M+1)^d I(F_k) + \sum_{k=1}^{m^N} N_k .$$

Now

$$\begin{aligned} E(N_k) &= \textstyle\sum_{n \geqq M} E[N_k;\, nm^{-\frac{1}{2}} \leqq \sup_{s \in S_k} |X_s - X_{t^k}| < (n+1)m^{-\frac{1}{2}}] \\ &\leqq \textstyle\sum_{n \geqq M} (2n+3)^d P[\sup_{s \in S_k} |X_s - X_{t^k}| \geqq nm^{-\frac{1}{2}}] \\ &\leqq c' \textstyle\sum_{n \geqq M} (2n+3)^d n^{d-2} e^{-cn^2} \end{aligned}$$

by Lemmas 1.1 and 1.3 since the δ distance between the least and largest vertices of S_k is at most $N/2m$. Therefore, given $\varepsilon > 0$, we can choose M so large that $E(N_k) \leqq \varepsilon$, independent of k and m. Then by (3.9) we have

(3.10) $$E\{N(m)\} \leqq (2M+1)^d \textstyle\sum_{k=1}^{m^N} P(F_k) + \varepsilon m^N .$$

The remainder of the proof is devoted to showing that the sum is of smaller order then m^N. Let $h(x) = 2(x \log\log 1/x)^{\frac{1}{2}}$ and

$$\alpha_m = \sup_{t \in \Delta} P[|X_s - X_t| \geqq h(\delta(s, t)) \text{ for some } s \text{ with } \delta(s, t) \leqq m^{-\frac{1}{2}}] .$$

If it were not for the sup over t, it would be a consequence of Theorem 2.3 that $\alpha_m \to 0$. In fact, it is easy to see by looking at the proof of Theorem 2.3 that all the estimates are independent of t for $t \in \Delta$ so that $\alpha_m \to 0$. Now let

$$J_k = [|X_s - X_{t^k}| < h(\delta(s, t^k)) \text{ for all } s \text{ with } s_N \leqq t_N{}^k \text{ and } \delta(s, t^k) \leqq m^{-\frac{1}{2}}] ;$$

we know that $P(J_k{}^c) \leqq \alpha_m$ for all k. We let T_k be the sojourn time in C_k up to time $\langle 2 \rangle$, i.e.

$$T_k = \textstyle\int_{\Delta(\langle 2 \rangle)} I[X_t \in C_k]\, dt .$$

Note that

(3.11) $$\textstyle\sum_{k=1}^{m^N} I(F_k) T_k \leqq 2^N .$$

We introduce the time sets

$$\begin{aligned} H_k = \{t\colon 0 \leqq t_n - t_n{}^k \leqq (t_N - t_N{}^k)/\log\log m,\ 1 \leqq n < N, \\ m^{-1} \leqq t_N - t_N{}^k \leqq m^{-\frac{1}{2}} N^{-1} 2^{-N}\} , \end{aligned}$$

where we assume that m is large enough so that H_k is not empty. Now

(3.12) $$E(T_k;\, F_k) \geqq E(T_k;\, F_k J_k) \geqq E(\textstyle\int_{H_k} I[X_t \in C_k]\, dt;\, F_k J_k) .$$

For any $t \in H_k$, let

$$t' = \langle t_1, \cdots, t_{N-1}, t_N{}^k \rangle .$$

We want a lower estimate for

(3.13) $$P[X_t \in C_k \mid X_s\colon s_N \leqq t_N{}^k] = P[X_t - X_{t'} \in C_k - X_{t'} \mid X_s\colon s_N \leqq t_N{}^k] .$$

$X_t - X_{t'}$ is independent of the conditioning σ-field, J_k, F_k are in it, and $X_{t'}$ is measurable with respect to it. Since $X_{t^k} \in C_k$, $C_k - X_{t'}$ is a cube of side $m^{-\frac{1}{2}}$ which intersects the sphere of radius $|X_{t^k} - X_{t'}|$ centered at the origin. Furthermore, since

$$\delta(t^k, t') \leqq 2^N \textstyle\sum_{n=1}^{N-1} (t_n - t_n{}^k) \leqq 2^N N(t_N - t_N{}^k)/\log\log m \leqq m^{-\frac{1}{2}} ,$$

we have, on J_k

$$|X_{t^k} - X_{t'}| \leqq h(\delta(t', t^k)) \leqq c(t_N - t_N{}^k)^{\frac{1}{2}} ,$$

so that the cube $C_k - X_{t'}$ is contained in a sphere of radius $c(t_N - t_N^k)^{\frac{1}{2}}$. We can now estimate the conditional probability in (3.13) by scaling since

$$2^{-N}(t_N - t_N^k) \leqq \prod_{n=1}^{N-1} t_n(t_N - t_N^k) = \delta(t, t') \leqq 2^N(t_N - t_N^k) .$$

Because the normal density is bounded below on compact sets (the cube will be in a sphere of fixed radius after scaling) the lower estimate is just a constant times the volume of the rescaled cube. Thus, on J_k

$$P[X_t \in C_k \mid X_s : s_N \leqq t_N^k] \geqq c\{m^{-\frac{1}{2}}(t_N - t_N^k)^{-\frac{1}{2}}\}^d = cm^{-N}(t_N - t_N^k)^{-N} .$$

Inserting this bound in (3.12), we have

$$\text{(3.14)} \qquad \begin{aligned} E(T_k; F_k) &\geqq cE(\textstyle\int_{H_k} m^{-N}(t_N - t_N^k)^{-N}\, dt; F_k J_k) \\ &\geqq cm^{-N}(\log\log m)^{-N+1} \log m\, P(F_k J_k) . \end{aligned}$$

If $P(F_k) \geqq 2\alpha_m$, then, since $P(J_k) \geqq 1 - \alpha_m$, it follows that

$$\text{(3.15)} \qquad P(F_k) \leqq 2P(F_k J_k) .$$

Now, by (3.15), (3.14), and (3.11),

$$\begin{aligned} \textstyle\sum_{k=1}^{m^N} P(F_k) &\leqq 2\alpha_m m^N + \textstyle\sum_{\{k: P(F_k) \geqq 2\alpha_m\}} P(F_k) \\ &\leqq 2\alpha_m m^N + 2 \textstyle\sum_k P(F_k J_k) \\ &\leqq 2\alpha_m m^N + cm^N \frac{(\log\log m)^{N-1}}{\log m} \textstyle\sum_k E(T_k; F_k) \\ &\leqq 2\alpha_m m^N + cm^N \frac{(\log\log m)^{N-1}}{\log m} . \end{aligned}$$

By (3.10) this completes the proof of (i).

To prove (ii), define

$$h(x, t) = P[X_s = x \text{ for some } s \text{ with } s_n > t_n, n = 1, 2, \cdots, N]$$

as in the proof of the last theorem. By Fubini we know from (i) that $h(x, \langle 0 \rangle) = 0$ for almost all x. Since $h(x, \langle \rho \rangle) \leqq h(x, \langle 0 \rangle)$, we also have $h(x, \langle \rho \rangle) = 0$ for almost all x. But $h(x, \langle \rho \rangle)$ is continuous for fixed ρ by (3.7) so that $h(x, \langle \rho \rangle) \equiv 0$. Finally $h(x, \langle 0 \rangle) = \lim_{\rho \to 0} h(x, \langle \rho \rangle)$.

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