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THE 1971 WALD MEMORIAL LECTURES[1]

DISTRIBUTION FUNCTION INEQUALITIES FOR MARTINGALES[2]

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This is a guide to some recent work in the theory of martingale inequalities. Methods are simplified; some new proofs are given. A number of new results are also included.

Let Uf and Vf be nonnegative random variables associated with a martingale f. In many interesting cases, the inequality

$$P(Vf > \lambda) \leqq cP(Uf > \lambda),$$

which usually does not hold for all $\lambda > 0$, does hold for *enough* λ so that

$$EVf \leqq cEUf$$

and more. The underlying theory, introduced in [6], has also proved fruitful in other probability applications; see [5] and [8]. For an entirely nonprobabilistic application to harmonic functions, see [7].

Our main object here is to simplify some of the ideas and methods of [6] and to illustrate their use by a number of applications, old and new.

This begins in Chapter II. In Chapter I, some earlier work is simplified; only a few elementary propositions from standard martingale theory are needed and these are listed.

Chapter I

A martingale identity

In this chapter, elementary proofs are given of some of the inequalities for the martingale square function established in [2]. The key lemma contains an

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[1] The Wald Memorial Lectures were established by the Council of the Institute of Mathematical Statistics in 1956 to honor Professor Abraham Wald, whose productive and influential career was prematurely ended in an airplane crash, December 13, 1950. (See "Abraham Wald" by J. Wolfowitz, *Ann. Math. Statist.* **23** (1952), 1-13.) The Wald Lectures given at an annual meeting of the IMS consist of a sequence of usually 3 or 4 one-hour talks, for the purpose of providing a specialist time to develop his material in sufficient detail to make it accessible to persons not particularly conversant in the subject. Professor Burkholder's Wald Memorial Lectures were presented at the 1971 Annual Meeting held at Colorado State University, Fort Collins, Colorado, under the title "Martingale theory and its applications."

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interesting martingale identity and a related inequality for nonnegative submartingales. This lemma also makes possible a simple proof of Gundy's decomposition theorem for L^1-bounded martingales [22] and is used again in Section 18.

1. Notation. Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\mathscr{A}_0, \mathscr{A}_1, \cdots$ a nondecreasing sequence of sub-σ-fields of $\mathscr{A}$. In this chapter, $f = (f_1, f_2, \cdots)$ is a martingale or a nonnegative submartingale relative to $\mathscr{A}_1, \mathscr{A}_2, \cdots$ and $d = (d_1, d_2, \cdots)$ is the difference sequence of f: $f_n = \sum_{k=1}^n d_k$, $n \geqq 1$. The square function of f is $S(f) = [\sum_{k=1}^\infty d_k^2]^{\frac{1}{2}}$ and the maximal function is $f^* = \sup_n |f_n|$. Let $S_n(f) = [\sum_{k=1}^n d_k^2]^{\frac{1}{2}}$, $f_n^* = \sup_{1 \leq k \leq n} |f_k|$, $n \geqq 1$, $S_\infty(f) = S(f)$, $S_0(f) = f_0 = 0$, and, if f converges almost everywhere, let f_∞ denote its limit function. For $0 < p < \infty$, let $||f||_p = \sup_p ||f_n||_p$ where $||f_n||_p = [E|f_n|^p]^{1/p}$; f is L^p-bounded if $||f||_p$ is finite and is L^∞-bounded if $||f||_\infty = \sup_{0<p<\infty} ||f||_p$ is finite.

We shall use the following facts from Doob [15]: If f is L^1-bounded and μ is a stopping time, then f converges almost everywhere and $||f_\mu||_1 \leqq ||f||_1$. (This is transparent for f of the form

$$(f_1, \cdots, f_{n-1}, f_n, f_n, f_n, \cdots) .$$

To prove the inequalities of Section 3, we need only consider such f.) If n is a positive integer and $\lambda > 0$, then

$$\lambda P(f_n^* > \lambda) \leqq \int_{\{f_n^* > \lambda\}} |f_n| \leqq ||f||_1 . \tag{1.1}$$

The following is classical: If X and Y are nonnegative random variables satisfying

$$\lambda P(Y > \beta\lambda) \leqq \alpha \int_{\{Y > \lambda\}} X \tag{1.2}$$

for all $\lambda > 0$ and some numbers $\alpha > 0$, $\beta \geqq 1$, then

$$||Y||_p \leqq \alpha\beta^p q ||X||_p , \tag{1.3}$$

$p^{-1} + q^{-1} = 1$, $1 < p < \infty$. (Doob gives the proof of this only for $\alpha = \beta = 1$ but the same proof, which rests on the formula

$$||Y||_p^p = \int_0^\infty p\lambda^{p-1} P(Y > \lambda)\, d\lambda$$

and Hölder's inequality, applies to the above case. If Y satisfies (1.2), then so does $Y \wedge n$, the minimum of Y and n, so (1.3) may be proved under the additional assumption that $||Y||_p$ is finite.)

Doob's inequality

$$||f||_p \leqq ||f^*||_p \leqq q||f||_p , \qquad 1 < p < \infty , \tag{1.4}$$

follows immediately from (1.1) and (1.2). An application of (1.1) to the submartingale $\{|f_n|^p, n \geqq 1\}$ gives

$$\lambda^p P(f^* > \lambda) \leqq ||f||_p^p , \qquad 1 \leqq p < \infty . \tag{1.5}$$

We shall need these two facts in later chapters.

2. An identity.

LEMMA 2.1. *Suppose that* $f = (f_1, f_2, \cdots)$ *is either a martingale or a nonnegative submartingale and is* L^1*-bounded. Let* μ *be the stopping time defined by*

$$\mu(\omega) = \inf\{n : |f_n(\omega)| > \lambda\}$$

where $\lambda > 0$. *Then*

$$\|S_{\mu-1}(f)\|_2^2 + \|f_{\mu-1}\|_2^2 \leqq 2Ef_\mu f_{\mu-1} \leqq 2\lambda\|f\|_1 \tag{2.1}$$

and equality holds on the left in the martingale case.

Recall that $\inf \phi = \infty$. Here $f_\infty = \lim_{n\to\infty} f_n$ exists almost everywhere so f_μ and $f_{\mu-1}$ are well defined.

PROOF. Notice that $|f_{\mu-1}| \leqq \lambda$, so $Ef_\mu f_{\mu-1} \leqq \lambda\|f_\mu\|_1 \leqq \lambda\|f\|_1$. By elementary algebra,

$$S_{n-1}^2(f) + f_{n-1}^2 = 2f_n f_{n-1} - 2g_n$$

where $g_n = \sum_{k=1}^n f_{k-1} d_k$, $n \geqq 1$. Let $\nu = \mu \wedge n$. Then

$$Eg_\nu = \sum_{k=1}^n E[f_{k-1} I(\mu \geqq k) E(d_k \mid \mathscr{F}_{k-1})] \geqq 0\,,$$

which implies

$$ES_{\nu-1}^2(f) + Ef_{\nu-1}^2 \leqq 2Ef_\nu f_{\nu-1} \tag{2.2}$$

with equality holding in the martingale case. Here we have used the fact that the indicator function $I(\mu \geqq k)$ is $\mathscr{F}_{k-1}$ measurable, as is f_{k-1}, that $f_{k-1}I(\mu \geqq k)$ is bounded, and that $E(d_k \mid \mathscr{F}_{k-1}) \geqq 0$, $k \geqq 2$, with equality holding in the martingale case. Since $\sup_n f_{\nu-1}^2 \leqq \lambda^2$ and $\sup_n |f_\nu f_{\nu-1}| \leqq \lambda^2 + \lambda|f_\mu|$ are integrable, we may take the limit of both sides of (2.2) to obtain the left side of (2.1).

3. Square function inequalities. We now use Lemma 2.1 to give elementary proofs of several of the main results of [2].

THEOREM 3.1. *If* $f = (f_1, f_2, \cdots)$ *is a martingale or a nonnegative submartingale, then*

$$\lambda P(S(f) > \lambda) \leqq 3\|f\|_1\,, \qquad \lambda > 0\,. \tag{3.1}$$

PROOF. We may assume that f is L^1-bounded. Fix λ and define μ as in Lemma 2.1. Since $S_{\mu-1}(f) = S(f)$ on the set $\{\mu = \infty\} = \{f^* \leqq \lambda\}$, we have

$$\begin{aligned} \lambda P(S(f) > \lambda, f^* \leqq \lambda) &\leqq \lambda P(S_{\mu-1}(f) > \lambda) \\ &\leqq \lambda\|S_{\mu-1}(f)\|_2^2/\lambda^2 \\ &\leqq 2\|f\|_1\,. \end{aligned} \tag{3.2}$$

Therefore,

$$\begin{aligned} \lambda P(S(f) > \lambda) &\leqq \lambda P(f^* > \lambda) + \lambda P(S(f) > \lambda, f^* \leqq \lambda) \\ &\leqq \|f\|_1 + 2\|f\|_1 = 3\|f\|_1\,. \end{aligned}$$

THEOREM 3.2. *Let $1 < p < \infty$. There are positive real numbers c_p and C_p such that if $f = (f_1, f_2, \cdots)$ is a martingale then*

$$c_p\|S(f)\|_p \leqq \|f\|_p \leqq C_p\|S(f)\|_p \,. \tag{3.3}$$

The proof shows the optimum choice of c_p satisfies $c_p^{-1} = O(p^{\frac{1}{2}}q)$; similarly $C_p = O(q^{\frac{1}{2}}p)$ where, as usual, $p^{-1} + q^{-1} = 1$.

LEMMA 3.1. *Let $f = (f_1, f_2, \cdots)$ be a nonnegative submartingale and n a positive integer. Then $Y = S_n(\theta f) \vee f_n^*$ satisfies*

$$\lambda P(Y > \beta\lambda) \leqq 3 \int_{\{Y>\lambda\}} f_n \,, \qquad \lambda > 0 \,, \tag{3.4}$$

where $\theta > 0$ and $\beta = (1 + 2\theta^2)^{\frac{1}{2}}$, and

$$\|S_n(f)\|_p \leqq 9p^{\frac{1}{2}}q\|f_n\|_p \,, \qquad 1 < p < \infty \,. \tag{3.5}$$

PROOF. First note that (3.4) and (1.3) give

$$\theta\|S_n(f)\|_p \leqq \|Y\|_p \leqq 3\beta^p q\|f_n\|_p \,.$$

If $\theta = p^{-\frac{1}{2}}$, then $\beta^p = (1 + 2/p)^{p/2} < e < 3$ and (3.5) follows.

Since $\beta > 1$, the left side of (3.4) is no greater than

$$\lambda P(f_n^* > \lambda) + \lambda P(S_n(\theta f) > \beta\lambda, f_n^* \leqq \lambda) \,.$$

The first term of this expression does not exceed

$$\int_{\{f_n^*>\lambda\}} f_n \leqq \int_{\{Y>\lambda\}} f_n$$

and we now show the second term does not exceed twice the last integral.

Let $g_k = I(S_k(\theta f) > \lambda)f_k = I_k f_k$. Then $g = (g_1, g_2, \cdots)$ is a nonnegative submartingale; notice $I_k \leqq I_{k+1}$ so

$$\begin{aligned} E(g_{k+1} \mid \mathscr{A}_k) &= E(I_{k+1}f_{k+1} \mid \mathscr{A}_k) \\ &\geqq I_k E(f_{k+1} \mid \mathscr{A}_k) \\ &\geqq I_k f_k = g_k \,. \end{aligned}$$

We shall show

$$\{S_n(\theta f) > \beta\lambda, f_n^* \leqq \lambda\} \subset \{S_n(g) > \lambda, g_n^* \leqq \lambda\} \,. \tag{3.6}$$

Using this and (3.2), we obtain

$$\begin{aligned} \lambda P(S_n(\theta f) > \beta\lambda, f_n^* \leqq \lambda\} &\leqq \lambda P(S_n(g) > \lambda, g_n^* \leqq \lambda\} \\ &\leqq 2\|g_n\|_1 \\ &\leqq 2 \int_{\{Y>\lambda\}} f_n \,, \end{aligned}$$

which is the desired inequality. To prove (3.6) we consider the stopping time $\tau = \inf\{k : S_k(\theta f) > \lambda\}$. On the set on the left in (3.6), $1 \leqq \tau \leqq n$, $g_n^* \leqq \lambda$, and

$$\begin{aligned} \beta^2\lambda^2 < S_n^2(\theta f) &= S_{\tau-1}^2(\theta f) + \theta^2 d_\tau^2 + \theta^2 \textstyle\sum_{\tau<k\leqq n} d_k^2 \\ &\leqq \lambda^2 + \theta^2\lambda^2 + \theta^2 S_n^2(g) \,, \end{aligned}$$

which implies $S_n(g) > \lambda$. For example, the nonnegativity of f implies that $|d_k| \leqq f_k \vee f_{k-1} \leqq f^*$ so that on the set in question $|d_\tau| \leqq f^* \leqq \lambda$. Therefore, (3.6) holds and the proof of the lemma is complete.

PROOF OF THEOREM 3.2. We fix n and define nonnegative martingales g and h by

$$g_k = E(f_n^+ \mid \mathscr{A}_k), \qquad h_k = E(f_n^- \mid \mathscr{A}_k), \qquad k \geqq 1 .$$

Then $g_n = f_n^+$, $h_n = f_n^-$, $f_k = g_k - h_k$, $1 \leqq k \leqq n$, and $S_n(f) \leqq S_n(g) + S_n(h)$. Therefore, by (3.5), we have

$$\begin{aligned} ||S_n(f)||_p &\leqq ||S_n(g)||_p + ||S_n(h)||_p \\ &\leqq 9p^{\frac{1}{2}}q(||g_n||_p + ||h_n||_p) \\ &\leqq 18p^{\frac{1}{2}}q||f_n||_p , \end{aligned} \tag{3.7}$$

which implies the left side of (3.3). The right side of (3.3) follows from the left side by a simple duality argument: We may suppose that $||S(f)||_p < \infty$. Since $|f_n| \leqq nd^*$ and $d^* \leqq S(f)$, this implies that f_n is in L^p. We may also suppose that $||f_n||_p > 0$. Then $||f_n||_p = Ef_n g_n$ where the function $g_n = \operatorname{sgn} f_n |f_n|^{p-1}/||f_n||_p^{p-1}$ satisfies $||g_n||_q = 1$. Let g be the martingale defined by $E(g_n \mid \mathscr{A}_k)$, $k \geqq 1$, and let $(e_1, e_2, \cdots)$ be its difference sequence. Then, by orthogonality, Schwarz's inequality, Hölder's inequality, and (3.7), we have

$$\begin{aligned} ||f_n||_p &= Ef_n g_n = E \textstyle\sum_{k=1}^n d_k e_k \leqq ES_n(f)S_n(g) \\ &\leqq ||S_n(f)||_p ||S_n(g)||_q \leqq 18q^{\frac{1}{2}}p||S_n(f)||_p . \end{aligned}$$

This establishes Theorem 3.2.

4. Decomposition of an L^1-bounded martingale. This is Gundy's decomposition [22] simplified; here only one stopping time is used.

THEOREM 4.1. *Suppose $f = (f_1, f_2, \cdots)$ is an L^1-bounded martingale and λ is a positive real number. Then there are martingales X, Y, Z, with corresponding difference sequences x, y, z, all relative to $\mathscr{A}_1, \mathscr{A}_2, \cdots$, such that*

$$f = X + Y + Z , \tag{4.1}$$

$$||X||_2^2 \leqq 2\lambda ||f||_1 , \tag{4.2}$$

$$||\textstyle\sum_{k=1}^\infty |y_k| ||_1 \leqq 4||f||_1 , \tag{4.3}$$

$$P(Z^* > 0) \leqq ||f||_1/\lambda . \tag{4.4}$$

PROOF. Let $\mu = \inf\{n : |f_n| > \lambda\}$, and define x, y, z by

$$x_1 = d_1 I(\mu > 1) , \qquad y_1 = d_1 I(\mu = 1) , \qquad z_1 = 0 ,$$

and, for $k \geqq 2$,

$$\begin{aligned} x_k &= d_k I(\mu > k) - E(d_k I(\mu > k) \mid \mathscr{A}_{k-1}) , \\ y_k &= d_k I(\mu = k) - E(d_k I(\mu = k) \mid \mathscr{A}_{k-1}) , \\ z_k &= d_k I(\mu < k) . \end{aligned}$$

Clearly (4.1) holds and X, Y, Z are martingales relative to $\mathscr{A}_1, \mathscr{A}_2, \cdots$. Note that $\|x_k\|_2 \leqq \|d_k I(\mu > k)\|_2 \leqq 2\lambda$, Therefore, by Lemma 2.1,

$$\|X_n\|_2^2 = \sum_{k=1}^n \|x_k\|_2^2 \leqq \sum_{k=1}^n \|d_k I(\mu > k)\|_2^2$$
$$\leqq \|S_{\mu-1}(f)\|_2^2 \leqq 2\lambda\|f\|_1 .$$

This gives (4.2). Also,

$$\|\sum_{k=1}^\infty |y_k|\|_1 \leqq 2 \sum_{k=1}^\infty \|d_k I(\mu = k)\|_1$$
$$= 2\|d_\mu I(\mu < \infty)\|_1 \leqq 4\|f_\mu I(\mu < \infty)\|_1$$
$$\leqq 4\|f\|_1 ,$$
$$P(Z^* > 0) \leqq P(\mu < \infty) = P(f^* > \lambda) \leqq \|f\|_1/\lambda .$$

5. Remarks. Austin [1] proved that if f is an L^1-bounded martingale, then the integral of $S^2(f)$ over the set $\{f^* \leqq \lambda\}$ is finite. Lemma 2.1 may be viewed as a refinement of this result.

A slight modification of the present proof of Theorem 3.1 gives $2^{\frac{3}{2}}$ rather than 3 in the right-hand side of (3.1); the best constant is not known. For other proofs of Theorem 3.1 (with less precise information about the best constant), see Gundy [22] and Rao [36].

If f is an L^∞-bounded martingale, then, by the results of Section 3,

$$\|S(f)\|_p = O(p^{\frac{1}{2}}) , \qquad p \to \infty . \tag{5.1}$$

From this easily follows

$$Ee^{tS^2(f)} < \infty \tag{5.2}$$

for all small positive t. (Expand the exponential function in its power series and take expectations term by term.) That is, $S(f)$ belongs to exp L^2 if f is L^∞-bounded. This was observed first by Sjölin [41] for the special case in which f is the sequence of 2^nth partial sums of a Walsh series.

Suppose that f is simple random walk stopped at ± 2:

$$f_n = \sum_{k=1}^n I(\tau \geqq k)x_k , \qquad n \geqq 1 ,$$

where $x_1, x_2, \cdots$ are independent with $P(x_k = -1) = P(x_k = 1) = \frac{1}{2}$ and

$$\tau = \inf\{n : |\sum_{k=1}^n x_k| = 2\} .$$

Then an elementary calculation shows

$$\liminf_{p\to\infty} \|S(f)\|_p/p^{\frac{1}{2}} > 0$$

so the order of magnitude in (5.1) cannot be improved.

For f the sequence of 2^nth partial sums of a Walsh series, (3.3) was proved by Paley [35]. His results are still fresh and interesting; for example, see Sjölin [41], Hunt [25], and earlier references given there. Marcinkiewicz and Zygmund [27] proved (3.3) in case $d = (d_1, d_2, \cdots)$, the difference sequence of f, is independent and $Ed_k = 0$, $k \geqq 1$. Their inequality, valid for $1 \leqq p < \infty$, is used frequently. Another proof of (3.3) has been obtained recently by Gordon [21].

Applications of (3.3) in a general martingale setting can be found in the work of Chow [9], [11], Millar [32], Doléans [14], Stein [42], etc. Also, see [3], page 1287, but note that the simpler inequality (3.1) can be used there in place of (3.3).

For applications of the martingale decomposition of Section 4, see Gundy [22], [23]; for related decompositions, see Chow [10] and Davis [12]; also see Meyer [29]. A more general decomposition is presented on page 293 of [6]. Theorem 4.1 remains true and is strengthened if (4.4) is replaced by

$$P(\sup_n E(|z_n| \mid \mathscr{A}_{n-1}) > 0) \leqq \|f\|_1/\lambda . \tag{4.4'}$$

See page 280 of [6].

Chapter II

Distribution function inequalities and general integral inequalities

It is helpful to begin this chapter with a particularly easy but important special case.

6. Brownian motion. Let $X = \{X(t), 0 \leqq t < \infty\}$ be one-dimensional Brownian motion: X is a real process with independent increments, continuous sample functions, and $X(t)$ is normally distributed with $EX(t) = 0$ and $EX^2(t) = t$. Let $\mathscr{B}(t)$ be the σ-field generated by $\{X(s), 0 \leqq s \leqq t\}$. A stopping time τ of X is a function from Ω into $[0, \infty]$ such that

$$\{\tau < t\} \in \mathscr{B}(t) , \qquad t > 0 .$$

If $0 \leqq b \leqq \infty$, let $X^*(b) = \sup_{t \geqq 0} |X(b \wedge t)|$.

Consider a non-decreasing continuous function Φ on $[0, \infty]$ with $\Phi(0) = 0$. Suppose that Φ satisfies the growth condition

$$\Phi(2\lambda) \leqq c\Phi(\lambda) , \qquad \lambda > 0 . \tag{6.1}$$

The letter c denotes a positive real number not necessarily the same number from one use to the next. For example, if $0 < p < \infty$, then $\Phi(\lambda) = \lambda^p$ defines such a function.

THEOREM 6.1. *If τ is a stopping time of X, then*

$$cE\Phi(\tau^{\frac{1}{2}}) \leqq E\Phi(X^*(\tau)) \leqq CE\Phi(\tau^{\frac{1}{2}}) . \tag{6.2}$$

The choice of c and C depends only on the growth constant $c_{(6.1)}$.

This is Theorem 7.2 of [6] with Φ more general. (The results of [6], [8], and [7] hold without the assumption made in those papers that Φ is the integral of a function that also satisfies a growth condition.)

The above two-sided Φ inequality can be deduced almost immediately, as we shall see, from the following more fundamental distribution function inequalities.

THEOREM 6.2. *Let $\beta > 1$ and $\delta > 0$. If τ is a stopping time of X, then*

$$P(\tau^{\frac{1}{2}} > \beta\lambda, X^*(\tau) \leqq \delta\lambda) \leqq \frac{\delta^2}{\beta^2 - 1} P(\tau^{\frac{1}{2}} > \lambda), \qquad \lambda > 0, \tag{6.3}$$

$$P(X^*(\tau) > \beta\lambda, \tau^{\frac{1}{2}} \leqq \delta\lambda) \leqq \frac{\delta^2}{(\beta - 1)^2} P(X^*(\tau) > \lambda), \qquad \lambda > 0. \tag{6.4}$$

The proofs of these two theorems will be given in Section 8.

7. Lemma for Φ inequalities. Assume, as above, that Φ is a non-decreasing continuous function on $[0, \infty]$ with $\Phi(0) = 0$ and that Φ satisfies the growth condition (6.1). Also, to eliminate the trivial case, suppose that Φ is not identically zero. Then $\Phi(\lambda)$ is finite and positive for finite and positive λ.

LEMMA 7.1. *Suppose that f and g are nonnegative measurable functions on a probability space $(\Omega, \mathscr{A}, P)$ and $\beta > 1$, $\delta > 0$, $\varepsilon > 0$ are real numbers such that*

$$P(g > \beta\lambda, f \leqq \delta\lambda) \leqq \varepsilon P(g > \lambda), \qquad \lambda > 0. \tag{7.1}$$

Let γ and η be real numbers satisfying

$$\Phi(\beta\lambda) \leqq \gamma\Phi(\lambda), \qquad \Phi(\delta^{-1}\lambda) \leqq \eta\Phi(\lambda), \qquad \lambda > 0. \tag{7.2}$$

Finally, suppose that $\gamma\varepsilon < 1$. Then

$$E\Phi(g) \leqq \frac{\gamma\eta}{1 - \gamma\varepsilon} E\Phi(f). \tag{7.3}$$

Note that the existence of γ and η satisfying (7.2) is assured by (6.1). For example, a possible choice for γ is c^k where k is the positive integer satisfying $2^{k-1} < \beta \leqq 2^k$ and $c = c_{(6.1)}$:

$$\Phi(\beta\lambda) \leqq \Phi(2^k\lambda) \leqq c^k\Phi(\lambda), \qquad \lambda > 0.$$

PROOF. We may assume in the proof that $E\Phi(g)$ is finite. For if g satisfies (7.1), then so does $g \wedge n$; if (7.3) holds for $g \wedge n$, $n \geqq 1$, then it holds for g.

Consider the Lebesgue-Stieltjes measure, also denoted by Φ, satisfying

$$\int_{[a,b)} d\Phi(\lambda) = \Phi(b) - \Phi(a), \qquad 0 \leqq a < b \leqq \infty.$$

This measure is positive and σ-finite on the σ-field of Borel subsets of $[0, \infty)$. If h is a nonnegative measurable function on $(\Omega, \mathscr{A}, P)$, then

$$\Phi(h) = \int_{[0,h)} d\Phi(\lambda) = \int_0^\infty I(h > \lambda)\, d\Phi(\lambda).$$

Using Fubini's theorem, we obtain

$$E\Phi(h) = \int_0^\infty P(h > \lambda)\, d\Phi(\lambda). \tag{7.4}$$

Condition (7.1) implies that

$$\begin{aligned} P(g > \beta\lambda) &= P(g > \beta\lambda, f \leqq \delta\lambda) + P(g > \beta\lambda, f > \delta\lambda) \\ &\leqq \varepsilon P(g > \lambda) + P(f > \delta\lambda), \qquad \lambda > 0. \end{aligned} \tag{7.5}$$

Therefore, by (7.4) and (7.2)

$$E\Phi(\beta^{-1}g) \leqq \varepsilon E\Phi(g) + \eta E\Phi(f). \tag{7.6}$$

But

$$E\Phi(g) = E\Phi(\beta\beta^{-1}g) \leqq \gamma E\Phi(\beta^{-1}g). \tag{7.7}$$

Combining (7.6) and (7.7) and using the finiteness of $E\Phi(g)$, we obtain (7.3).

Later, we shall need several additional properties of Φ:

$$\Phi(\lambda_1 \vee \lambda_2) \leqq \Phi(\lambda_1) + \Phi(\lambda_2), \tag{7.8}$$

$$\Phi(\lambda_1 + \lambda_2) \leqq \Phi(2\lambda_1) + \Phi(2\lambda_2) \leqq c[\Phi(\lambda_1) + \Phi(\lambda_2)]. \tag{7.9}$$

8. Proofs for Brownian motion. It is now clear how the two-sided Φ inequality of Theorem 6.1 follows from Theorem 6.2 and the above lemma. For example, to prove the left-hand side of (6.2), let $g = \tau^{\frac{1}{2}}$ and $f = X^*(\tau)$. Let $\beta > 1$ and select γ satisfying (7.2). Choose δ small enough so that

$$\varepsilon = \frac{\delta^2}{\beta^2 - 1} < \gamma^{-1}.$$

Then, by Theorem 6.2 and the lemma,

$$E\Phi(\tau^{\frac{1}{2}}) \leqq \frac{\gamma\eta}{1 - \gamma\varepsilon} E\Phi(X^*(\tau)).$$

PROOF OF (6.3). The proof rests on just one fact about Brownian motion beyond its sample function continuity: If τ is a bounded stopping time of X, then

$$EX^2(\tau) = E\tau. \tag{8.1}$$

This is well known and follows at once from Doob's optional sampling theorem applied to the martingale $\{X^2(t) - t,\ t \geqq 0\}$; see [15], page 380.

It is enough to prove (6.3) for τ bounded; if (6.3) holds with τ replaced by $\tau \wedge n$, $n \geqq 1$, then it holds for τ.

Let

$$\mu = \inf\{t : (\tau \wedge t)^{\frac{1}{2}} > \lambda\},$$
$$\nu = \inf\{t : (\tau \wedge t)^{\frac{1}{2}} > \beta\lambda\},$$
$$\sigma = \inf\{t : |X(\tau \wedge t)| > \delta\lambda\}.$$

These are stopping times of X. Clearly,

$$\begin{aligned} P(\tau^{\frac{1}{2}} > \beta\lambda, X^*(\tau) \leqq \delta\lambda) &= P(\mu \leqq \nu < \infty, \sigma = \infty) \\ &\leqq P(\tau \wedge \nu \wedge \sigma - \tau \wedge \mu \wedge \sigma \geqq \beta^2\lambda^2 - \lambda^2) \\ &\leqq \frac{1}{(\beta^2 - 1)\lambda^2} E[\tau \wedge \nu \wedge \sigma - \tau \wedge \mu \wedge \sigma]. \end{aligned}$$

By (8.1),

$$E[\tau \wedge \nu \wedge \sigma - \tau \wedge \mu \wedge \sigma] = E[X^2(\tau \wedge \nu \wedge \sigma) - X^2(\tau \wedge \mu \wedge \sigma)].$$

By the sample function continuity of X,

$$X^2(\tau \wedge \nu \wedge \sigma) - X^2(\tau \wedge \mu \wedge \sigma) \leqq X^2(\tau \wedge \nu \wedge \sigma) \leqq \delta^2\lambda^2,$$

and the left-hand side vanishes on the set

$$\{\mu = \nu\} = \{\mu = \infty\} = \{\tau^{\frac{1}{2}} \leqq \lambda\}.$$

Therefore,

$$E[X^2(\tau \wedge \nu \wedge \sigma) - X^2(\tau \wedge \mu \wedge \sigma)] \leqq \delta^2\lambda^2 P(\tau^{\frac{1}{2}} > \lambda).$$

Combining these estimates, we obtain (6.3).

PROOF OF (6.4). Here define μ and ν by

$$\mu = \inf\{t: |X(\tau \wedge t)| > \lambda\}$$
$$\nu = \inf\{t: |X(\tau \wedge t)| > \beta\lambda\},$$

and let $b = \delta^2\lambda^2$. On the set $\{X^*(\tau) > \beta\lambda\}$, these stopping times satisfy $\mu \leqq \nu < \infty$, $|X(\tau \wedge \nu)| = \beta\lambda$, $|X(\tau \wedge \mu)| = \lambda$, so that

$$\begin{aligned} P(X^*(\tau) > \beta\lambda, \tau^{\frac{1}{2}} \leqq \delta\lambda) &\leqq P(|X(\tau \wedge \nu) - X(\tau \wedge \mu)| \geqq \beta\lambda - \lambda, \tau \leqq b) \\ &\leqq P(|X(\tau \wedge \nu \wedge b) - X(\tau \wedge \mu \wedge b)| \geqq (\beta - 1)\lambda) \\ &\leqq \frac{1}{(\beta - 1)^2\lambda^2} E[X(\tau \wedge \nu \wedge b) - X(\tau \wedge \mu \wedge b)]^2. \end{aligned}$$

Now X is a martingale and Doob's optional sampling theorem gives here that $\{X(\tau \wedge \mu \wedge b), X(\tau \wedge \nu \wedge b)\}$ is a two-term martingale. It follows from the orthogonality of its increments and (8.1) that

$$\begin{aligned} E[X(\tau \wedge \nu \wedge b) - X(\tau \wedge \mu \wedge b)]^2 &= EX^2(\tau \wedge \nu \wedge b) - EX^2(\tau \wedge \mu \wedge b) \\ &= E[\tau \wedge \nu \wedge b - \tau \wedge \mu \wedge b] \\ &\leqq bP(\mu < \infty) \\ &= \delta^2\lambda^2 P(X^*(\tau) > \lambda). \end{aligned}$$

This completes the proof of (6.4).

9. Inequalities for the Itô integral and other continuous-sample-function martingales. Let X be Brownian motion as before and consider the Itô integral

$$Y(t) = \textstyle\int_0^t v(s)\, dX(s), \qquad t \geqq 0,$$

where $\{v(t), t \geqq 0\}$ is a nonanticipating Brownian functional satisfying

$$P(\textstyle\int_0^t v^2(s)\, ds < \infty, t \geqq 0) = 1.$$

(For a discussion of the Itô integral, see [28].) Let

$$Y^* = \sup_{t \geqq 0} |Y(t)|, \qquad S(Y) = [\textstyle\int_0^\infty v^2(t)\, dt]^{\frac{1}{2}}.$$

(Note that for the special case $v(t) = I(\tau \geqq t)$, these give $X^*(\tau)$ and $\tau^{\frac{1}{2}}$, respectively.) Then, for $\beta > 1$ and $\delta > 0$,

$$(9.1) \qquad P(S(Y) > \beta\lambda, Y^* \leqq \delta\lambda) \leqq \frac{\delta^2}{\beta^2 - 1} P(S(Y) > \lambda), \qquad \lambda > 0,$$

and the analogues of (6.4) and (6.2) hold.

This can be seen in two different ways. The proofs of (6.3) and (6.4) can be carried over to this case; only slight and obvious modifications are needed. Or a time substitution can be made to reduce the problem to that of the special case $v(t) = I(\tau \geqq t)$ already treated; see Section 2.5 of [28].

Distribution function inequalities and Φ inequalities between the maximal function and the square function of any local martingale with continuous sample functions follow similarly.

10. General martingales and the problem of jumps. The sample function continuity of both $t \to (\tau \wedge t)^{\frac{1}{2}}$ and $t \to |X(\tau \wedge t)|$ is used in an essential way in the proof of Theorem 6.2. Indeed, the analogue of Theorem 6.2 does not hold for general martingales. Consider the discrete case (see Section 1 for notation): If $0 < p < 1$, there is no real number c_p such that

$$||S(f)||_p \leqq c_p ||f^*||_p$$

for all martingales $f = (f_1, f_2, \cdots)$. The same is true if the maximal function and square function are interchanged. The problem is caused by big jumps; see Example 8.1 of [6]. Since no general Φ inequality holds, no distribution function inequality between f^* and $S(f)$ can hold.

However, there are substitute results, obtained by restricting f, as we shall see in the following two sections and in Section 18.

11. Transforms of regular martingales. Let $X = (X_1, X_2, \cdots)$ be a martingale (relative to $\mathscr{A}_1, \mathscr{A}_2, \cdots$) with difference sequence $x = (x_1, x_2, \cdots)$. Then $f = (f_1, f_2, \cdots)$ is a transform of X if

$$f_n = \textstyle\sum_{k=1}^n d_k = \sum_{k=1}^n v_k x_k \,, \qquad n \geqq 1 \,,$$

where v_k is a real $\mathscr{A}_{k-1}$-measurable function, $k \geqq 1$.

In this section, we assume that X satisfies

$$E(x_k^2 \mid \mathscr{A}_{k-1}) = 1 \,, \qquad k \geqq 1 \,, \tag{11.1}$$

$$E(|x_k| \mid \mathscr{A}_{k-1}) \geqq c \,, \qquad k \geqq 1 \,. \tag{11.2}$$

THEOREM 11.1 *Suppose that Φ satisfies the conditions of Section 7 and f is the transform of a martingale X satisfying* (11.1) *and* (11.2). *Then*

$$cE\Phi(S(f)) \leqq E\Phi(f^*) \leqq CE\Phi(S(f)) \,. \tag{11.3}$$

The choice of c and C depends only on $c_{(6.1)}$ and $c_{(11.2)}$.

This is Theorem 5.2 of [6]. A distribution function inequality holds for each side. For example, if $\beta > 1$ and $0 < \delta < \beta - 1$, then

$$P(f^* > \beta\lambda,\, S(f) \vee v^* \leqq \delta\lambda) \leqq \frac{c\delta^2}{(\beta - \delta - 1)^2} P(f^* > \lambda) \,, \qquad \lambda > 0 \,, \tag{11.4}$$

with the choice of c depending only on $c_{(11.2)}$. Therefore, by Lemma 7.1,

$$\begin{aligned} E\Phi(f^*) &\leqq cE\Phi(S(f) \vee v^*) \\ &\leqq c[E\Phi(S(f)) + E\Phi(v^*)] \,. \end{aligned}$$

But it follows easily from (11.1) and (11.2) that

$$P(v^* > \lambda) \leqq cP(cd^* > \lambda)\,, \qquad \lambda > 0$$

(Lemma 2.5 of [6]). Accordingly, $E\Phi(v^*) \leqq cE\Phi(cd^*) \leqq cE\Phi(d^*) \leqq cE\Phi(S(f))$ and this gives the right-hand side of (11.3).

To prove (11.4) we need the following lemma (a special case of Theorems 2.1 and 2.2 of [6]), which is of interest in its own right. It shows how the jumps of the transform of a regular martingale can be controlled. The regularity conditions (11.1) and (11.2) cannot be relaxed substantially; see Example 8.3 of [6].

LEMMA 11.1. *Suppose that f is the transform of a martingale X satisfying* (11.1) *and* (11.2) *with $v = (v_1, v_2, \cdots)$ uniformly bounded by a positive real number b. If*

$$\tau = \inf\{n: |f_n| > b\} \quad or \quad \tau = \inf\{n: S_n(f) > b\}$$

then

$$\|f_\tau^*\|_2^2 \leqq cb^2 P(v^* > 0) \leqq cb^2$$

with the choice of c depending only on $c_{(11.2)}$.

Note that $f_\tau^* = f_\infty^* = f^*$ on $\{\tau = \infty\}$.

PROOF OF (11.4). Let

$$\begin{aligned} \mu &= \inf\{n: |f_n| > \lambda\}\,, \\ \nu &= \inf\{n: |f_n| > \beta\lambda\}\,, \\ \sigma &= \inf\{n \geqq 0: S_n(f) > \delta\lambda \ \text{ or } \ |v_{n+1}| > \delta\lambda\}\,. \end{aligned}$$

These are all stopping times. Let h be f started at μ and stopped at $\nu \wedge \sigma$:

$$h_n = \textstyle\sum_{k=1}^n I(\mu < k \leqq \nu \wedge \sigma) v_k x_k = \sum_{k=1}^n w_k x_k\,.$$

Note that h is a transform of X with multiplier sequence $w = (w_1, w_2, \cdots)$ uniformly bounded by $\delta\lambda$. Let

$$\tau = \inf\{n: S_n(h) > \delta\lambda\}\,.$$

By Lemma 11.1,

$$\begin{aligned} \|h_\tau^*\|_2^2 &\leqq c\delta^2\lambda^2 P(w^* > 0) \\ &\leqq c\delta^2\lambda^2 P(\mu < \infty) = c\delta^2\lambda^2 P(f^* > \lambda)\,. \end{aligned}$$

Therefore, using $S(h) \leqq S(f)$ and $d^* \leqq S(f)$, we obtain

$$\begin{aligned} P(f^* > \beta\lambda,\, S(f) \vee v^* \leqq \delta\lambda) &\leqq P(h^* > (\beta - \delta - 1)\lambda,\, \tau = \infty) \\ &\leqq P(h_\tau^* > (\beta - \delta - 1)\lambda) \\ &\leqq \frac{1}{(\beta - \delta - 1)^2\lambda^2} \|h_\tau^*\|_2^2 \\ &\leqq \frac{c\delta^2}{(\beta - \delta - 1)^2} P(f^* > \lambda)\,. \end{aligned}$$

This completes the proof of (11.4). The dual inequality

$$P(S(f) > \beta\lambda, f^* \vee v^* \leqq \delta\lambda) \leqq \frac{c\delta^2}{\beta^2 - \delta^2 - 1} P(S(f) > \lambda) \tag{11.5}$$

is proved similarly.

12. Another way to control jumps. Consider a martingale $f = (f_1, f_2, \cdots)$ satisfying $|d_k| \leqq w_k$ with w_k measurable relative to $\mathscr{A}_{k-1}$, $k \geqq 1$. Let Φ be as in Section 7. Then

$$E\Phi(f^*) \leqq cE\Phi(S(f)) + cE\Phi(w^*)\,, \tag{12.1}$$

$$E\Phi(S(f)) \leqq cE\Phi(f^*) + cE\Phi(w^*)\,, \tag{12.2}$$

with the choice of c depending only on $c_{(6.1)}$. These follow from Theorem 3.1 of [5] but a direct proof can easily be given. The proof here illustrates simplifications possible in the methods of Section 3 of [5].

In view of Lemma 7.1, it suffices to prove the following distribution function inequality and its dual: If $\beta > 1$ and $0 < \delta < \beta - 1$, then

$$P(f^* > \beta\lambda,\, S(f) \vee w^* \leqq \delta\lambda) \leqq \frac{2\delta^2}{(\beta - \delta - 1)^2} P(f^* > \lambda)\,, \qquad \lambda > 0\,. \tag{12.3}$$

To prove (12.3), let

$$\mu = \inf\{n : |f_n| > \lambda\}\,,$$
$$\nu = \inf\{n : |f_n| > \beta\lambda\}\,,$$
$$\sigma = \inf\{n : S_n(f) > \delta\lambda \quad \text{or} \quad w_{n+1} > \delta\lambda\}\,.$$

Then h defined by

$$h_n = \textstyle\sum_{k=1}^n I(\mu < k \leqq \nu \wedge \sigma)\, d_k$$

is a martingale such that $S(h) = 0$ on $\{\mu = \infty\} = \{f^* \leqq \lambda\}$ and $S^2(h) \leqq 2\delta^2\lambda^2$ everywhere: on $\{0 < \sigma < \infty\}$,

$$S^2(h) \leqq S_\sigma^2(f) = S_{\sigma-1}^2(f) + d_\sigma^2 \leqq \delta^2\lambda^2 + w_\sigma^2 \leqq 2\delta^2\lambda^2$$

and the inequality holds trivially elsewhere. Therefore,

$$\|h\|_2^2 = \|S(h)\|_2^2 \leqq 2\delta^2\lambda^2 P(f^* > \lambda)$$

so that, by (1.5),

$$\begin{aligned} &P(f^* > \beta\lambda,\, S(f) \vee w^* \leqq \delta\lambda) \\ &\qquad \leqq P(h^* > (\beta - \delta - 1)\lambda) \\ &\qquad \leqq \frac{1}{(\beta - \delta - 1)^2\lambda^2} \|h\|_2^2 \leqq \frac{2\delta^2}{(\beta - \delta - 1)^2} P(f^* > \lambda)\,. \end{aligned}$$

The proof of the dual of (12.3) is similar.

13. Remarks. By using the results of [2] and an approximation argument, Millar [32] proved an inequality essentially equivalent to (6.2) for $\Phi(\lambda) = \lambda^p$

($1 < p < \infty$). Earlier, various students of Skorokhod embedding (including Skorokhod) had obtained special cases for $2 \leqq p < \infty$; for example, see Rosenkrantz [37] and Sawyer [40]. Working independently of [6], Novikov [34] extended Millar's result in several directions. His main tool is Itô's lemma. For a related approach to (6.2), still in the special case of Φ a power, see Getoor and Sharpe [20], who use some techniques of Garsia.

It is certainly possible to have the choice of c and C in (6.2) depend on the whole of Φ, not just on its growth constant $c_{(6.1)}$.

For the case $\Phi(\lambda) = \lambda^p$, choose $\beta = 1 + p^{-1}$ and $\delta = cp^{-\frac{1}{2}}$ to obtain

$$(13.1) \qquad \|\tau^{\frac{1}{2}}\|_p \leqq O(p^{\frac{1}{2}})\|X^*(\tau)\|_p, \qquad p \to \infty.$$

The order of magnitude is $O(p)$ if $\tau^{\frac{1}{2}}$ and $X^*(\tau)$ are interchanged reflecting the difference between $\beta^2 - 1$ in (6.3) and $(\beta - 1)^2$ in (6.4). The order $O(p^{\frac{1}{2}})$ in (13.1) can also be obtained by other methods; see [38] for example.

Here is another distribution function inequality that can be proved with the methods of Section 8. Let Z be complex Brownian motion: $Z = X + iY$ where X and Y are independent one-dimensional Brownian motions as in Section 6. Let u be harmonic in some connected open set D of the complex plane and v a conjugate in the sense that $F = u + iv$ is analytic in D. For simplicity, suppose that D contains the origin and $F(0) = 0$. Let τ be a stopping time of Z such that if $t < \tau$, then $Z(t) \in D$; one example is

$$\tau = \inf\{t: Z(t) \notin D\}.$$

Let $u^* = \sup_{0 \leqq t < \tau} |u(Z(t))|$ and v^* be defined similarly. Then, for $\beta > 1$ and $\delta > 0$,

$$(13.2) \qquad P(v^* > \beta\lambda, u^* \leqq \delta\lambda) \leqq \frac{\delta^2}{(\beta - 1)^2} P(v^* > \lambda), \qquad \lambda > 0.$$

In the proof, one may assume that F is analytic in an open set containing the closure of D and this closure is compact. Then results of Doob [16] are available: $\{u(Z(\tau \wedge t)), t \geqq 0\}$ is a martingale and, for μ another stopping time,

$$(13.3) \qquad Eu^2(Z(\tau \wedge \mu)) = Ev^2(Z(\tau \wedge \mu)).$$

With (13.3) taking the place of (8.1), the proof of (13.2) has the same pattern as the proof of (6.4). For further information and for applications of the Φ inequality that follows from (13.2), see [8]. The Paley-Zygmund inequality used there can be avoided as in Section 8 above.

If we take the limit of both sides of (13.2) as $\beta \to \infty$, we obtain

$$P(v^* = \infty, u^* \leqq \delta\lambda) = 0, \qquad \lambda > 0.$$

Therefore, $P(v^* = \infty, u^* < \infty) = 0$; that is v^* is finite almost everywhere on $\{u^* < \infty\}$. This illustrates the kind of local behavior that can be deduced from distribution function inequalities; see [6] for more examples.

Chapter III

Integral inequalities: The convex case

Here Φ not only satisfies, as always, the mild conditions of Section 7, but also is convex. In this case, the big jumps of a martingale can be isolated and controlled.

14. Davis's decomposition of a martingale. If $f = (f_1, f_2, \cdots)$ is a martingale, then

$$f = g + h \tag{14.1}$$

where g and h are the martingales defined by

$$g_n = \sum_{k=1}^n a_k = \sum_{k=1}^n [y_k - E(y_k \mid \mathscr{A}_{k-1})] ,$$
$$h_n = \sum_{k=1}^n b_k = \sum_{k=1}^n [z_k + E(y_k \mid \mathscr{A}_{k-1})] ,$$

with

$$y_k = d_k I(|d_k| \leqq 2d_{k-1}^*) ,$$
$$z_k = d_k I(|d_k| > 2d_{k-1}^*) ,$$

and $d_k^* = \sup_{0 \leqq j \leqq k} |d_j|$ with $d_0 = 0$. Davis introduced this decomposition in [13] and used it to prove

$$cES(f) \leqq Ef^* \leqq CES(f) . \tag{14.2}$$

Note that $|y_k| \leqq 2d_{k-1}^*$ so that

$$|a_k| \leqq 4d_{k-1}^* , \tag{14.3}$$

a bound which is $\mathscr{A}_{k-1}$-measurable. This gives control of g. The martingale h is controlled by

$$\sum_{k=1}^\infty |b_k| \leqq \sum_{k=1}^\infty |z_k| + \sum_{k=1}^\infty E(|z_k| \mid \mathscr{A}_{k-1}) , \tag{14.4}$$

which follows from the fact that $y_1 = 0$ and, for $k \geqq 2$, $E(y_k \mid \mathscr{A}_{k-1}) + E(z_k \mid \mathscr{A}_{k-1}) = E(d_k \mid \mathscr{A}_{k-1}) = 0$. On the set $\{|d_k| > 2d_{k-1}^*\}$,

$$|d_k| + 2d_{k-1}^* \leqq 2|d_k| \leqq 2d_k^* .$$

Therefore, $|z_k| \leqq 2(d_k^* - d_{k-1}^*)$ and

$$\sum_{k=1}^\infty |z_k| \leqq 2d^* . \tag{14.5}$$

15. Inequality for the square function in the convex case. The following two-sided inequality generalizes both (3.3) and (14.2) and is a typical result of [5].

THEOREM 15.1. *Let Φ be a convex function satisfying the conditions of Section 7. If $f = (f_1, f_2, \cdots)$ is a martingale, then*

$$cE\Phi(S(f)) \leqq E\Phi(f^*) \leqq CE\Phi(S(f)) . \tag{15.1}$$

The choice of c and C depends only on $c_{(6.1)}$.

PROOF. We shall prove only the right-hand side; the proof of the left-hand side has exactly the same pattern.

Write $f = g + h$ as in (14.1) and note that

$$f^* \leqq g^* + h^* \leqq g^* + \sum_{k=1}^{\infty} |b_k| , \tag{15.2}$$

$$S(g) \leqq S(f) + S(h) \leqq S(f) + \sum_{k=1}^{\infty} |b_k| . \tag{15.3}$$

By (15.2) and (7.9),

$$E\Phi(f^*) \leqq cE\Phi(g^*) + cE\Phi(\sum_{k=1}^{\infty} |b_k|) . \tag{15.4}$$

By the basic inequality (12.1) applied to g,

$$E\Phi(g^*) \leqq cE\Phi(S(g)) + cE\Phi(4d^*) . \tag{15.5}$$

Now substitute (15.5) into (15.4) and use (15.3) and $d^* \leqq S(f)$ to see that

$$E\Phi(f^*) \leqq cE\Phi(S(f)) + cE\Phi(\sum_{k=1}^{\infty} |b_k|) .$$

In view of (14.4) and (14.5), this gives

$$E\Phi(f^*) \leqq cE\Phi(S(f)) + cE\Phi(\sum_{k=1}^{\infty} E(|z_k| \mid \mathscr{A}_{k-1})) . \tag{15.6}$$

We have not yet used the convexity of Φ. Using Lemma 16.1, below, we obtain

$$E\Phi(\sum_{k=1}^{\infty} E(|z_k| \mid \mathscr{A}_{k-1})) \leqq cE\Phi(\sum_{k=1}^{\infty} |z_k|) . \tag{15.7}$$

Combining (15.6), (15.7), and (14.5), we get the right-hand side of (15.1).

16. A convexity lemma. This was introduced in [5].

LEMMA 16.1. *Let Φ be a convex function satisfying the conditions of Section* 7. *Let $z_1, z_2, \cdots$ be nonnegative measurable functions on $(\Omega, \mathscr{A}, P)$. Then*

$$E\Phi(\sum_{k=1}^{\infty} E(z_k \mid \mathscr{A}_{k-1})) \leqq cE\Phi(\sum_{k=1}^{\infty} z_k) .$$

The choice of c depends only on $c_{(6.1)}$.

Recently, Neveu and Garsia have given new, elegant proofs of this. Garsia's proof is contained in [19], Neveu's in [33] and [30]. Neveu shows that the pair

$$W = \sum_{k=1}^{\infty} E(z_k \mid \mathscr{A}_{k-1}) , \qquad Z = \sum_{k=1}^{\infty} z_k$$

satisfies

$$\int_{\{W>\lambda\}} (W - \lambda) \leqq \int_{\{W>\lambda\}} Z , \qquad \lambda > 0 ,$$

and that any pair W, Z of nonnegative random variables satisfying this inequality must also satisfy

$$E\Phi(W) \leqq cE\Phi(Z) .$$

17. Quadratic variation of right-continuous martingales. Is the generality of Theorem 15.1 ever really needed? Or is it enough to know (15.1) for convex powers $\Phi(\lambda) = \lambda^p$ ($p \geqq 1$) and possibly for functions like $\Phi(\lambda) = (\lambda + 1) \log (\lambda + 1)$? Consider the following application from [5].

Let $X = \{X(t), 0 \leqq t \leqq 1\}$ be a right-continuous martingale and define $S_j \geqq 0$ by the following approximation to the quadratic variation of X:

$$S_j^2 = X^2(t_{j1}) + \sum_{k=2}^{\infty} [X(t_{jk}) - X(t_{j,k-1})]^2$$

where $0 = t_{j1} \leqq t_{j2} \leqq \cdots \leqq 1, j \geqq 1$. That is, $S_j = S(f_j)$ where $f_j = (f_{j1}, f_{j2}, \cdots)$ is the martingale defined by $f_{jn} = X(t_{jn})$. Assume that $t_{jk} = 1$ for $k \geqq k_j$ and

$$\lim_{j\to\infty} \sup_{k\geqq 2} (t_{jk} - t_{j,k-1}) = 0 .$$

THEOREM 17.1. *The sequence $\{S_j\}$ converges in L^1 if and only if $X^* = \sup_{0\leqq t\leqq 1} |X(t)|$ is integrable.*

PROOF. If X^* is integrable, then there is a function Φ satisfying the conditions of Section 7 and the following further conditions: Φ is convex, $\lim_{\lambda\to\infty}\Phi(\lambda)/\lambda = \infty$, and $E\Phi(X^*) < \infty$ (Lemma 5.1 of [5]). Therefore, by Theorem 15.1,

$$E\Phi(S_j) \leqq cE\Phi(f_j^*) \leqq cE\Phi(X^*) < \infty .$$

Thus, $\{S_j\}$ is uniformly integrable. But, by a result of Doléans [14], $\{S_j\}$ converges in probability. Combining these two facts gives convergence in L^1.

On the other hand, if $\{S_j\}$ converges in L^1, then $\sup_j \|S_j\|_1 < \infty$. By right-continuity, $X^* = \lim_{j\to\infty} f_j^*$. Therefore, by the inequality of Davis,

$$EX^* \leqq \liminf_{j\to\infty} Ef_j^* \leqq c \sup_j ES_j < \infty .$$

Chapter IV

Further results and applications

We begin this chapter by investigating some of the consequences of nonnegativity.

18. Square function inequalities for nonnegative martingales. Let $f = (f_1, f_2, \cdots)$ be a martingale and μ the stopping time defined by

$$\mu = \inf\{n : |f_n| > \lambda\} . \tag{18.1}$$

As we have seen (Lemma 2.1), $S_{\mu-1}(f) \in L^2$ if f is L^1-bounded. Much more is true if f is also nonnegative: $S_{\mu-1}(f) \in \exp L^2$.

THEOREM 18.1. *If f is a nonnegative martingale, λ is a positive real number, and μ is the stopping time defined by* (18.1), *then*

$$E[\exp tS^2_{\mu-1}(f)] \leqq \frac{1}{1 - 3t\lambda^2}, \qquad 0 < t < 1/3\lambda^2 . \tag{18.2}$$

Closely related to this are the following results, which, for the nonnegative case, also strengthen earlier conclusions.

THEOREM 18.2. *Let $\beta > 1$ and $0 < \delta < (\beta^2 - 1)^{1/2}$. If f is a nonnegative martingale, then*

$$P(S(f) > \beta\lambda, f^* \leqq \delta\lambda) \leqq \frac{2\delta^2}{\beta^2 - \delta^2 - 1} P(S(f) > \lambda) , \qquad \lambda > 0 .$$

By Lemma 7.1, this implies the following general Φ inequality.

THEOREM 18.3. *If Φ is a function satisfying the conditions of Section 7 and f is a nonnegative martingale, then*

$$E\Phi(S(f)) \leqq cE\Phi(f^*) . \tag{18.3}$$

The choice of c depends only on $c_{(6.1)}$.

To prove Theorem 18.1, we need the following elementary lemma.

LEMMA 18.1. *Suppose that g is a nonnegative measurable function on a probability space $(\Omega, \mathscr{A}, P)$ and α is a positive number such that*

$$\int_a^\infty P(g > \lambda)\, d\lambda \leqq \alpha P(g > a) , \qquad a > 0 . \tag{18.4}$$

Then $g \in \exp L$; in fact,

$$Ee^{tg} \leqq \frac{1}{1 - \alpha t} , \qquad 0 < t < \alpha^{-1} .$$

PROOF. Multiply the right-hand side of (18.4) by pa^{p-1} and integrate to obtain αEg^p. Do the same for the left-hand side and use Fubini's theorem to obtain $Eg^{p+1}/(p + 1)$. Therefore,

$$Eg^{p+1} \leqq \alpha(p + 1)Eg^p , \qquad p > 0 .$$

Let $a \to 0$ in (18.4) to get $Eg \leqq \alpha$. By induction,

$$Eg^k \leqq \alpha^k k! , \qquad k = 0, 1, 2, \cdots .$$

Therefore,

$$\begin{aligned} Ee^{tg} &= \sum_{k=0}^\infty t^k Eg^k/k! \\ &\leqq \sum_{k=0}^\infty (\alpha t)^k = (1 - \alpha t)^{-1} . \end{aligned}$$

PROOF OF THEOREM 18.1. Note that

$$S_{\mu-1}^2(f) = \sum_{k=1}^\infty I(\mu > k)\, d_k^2$$

where $I(\mu > k)\, d_k^2 = I(f_k^* \leqq \lambda)\, d_k^2 \leqq \lambda^2$. Let ν be the stopping time defined by

$$\nu = \inf \{n : \textstyle\sum_{k=1}^n I(\mu > k)\, d_k^2 > a\}$$

where $a > 0$. Then, for $b > 0$,

$$P(S_{\mu-1}^2(f) > a + b + \lambda^2) \leqq \sum_{n=1}^\infty P(\textstyle\sum_{k=n+1}^\infty I(\mu > k)\, d_k^2 > b, \nu = n) .$$

Integrating with respect to b gives

$$\int_{a+\lambda^2}^\infty P(S_{\mu-1}^2(f) > s)\, ds \leqq \sum_{n=1}^\infty E[\textstyle\sum_{k=n+1}^\infty I(\mu > k, \nu = n)\, d_k^2] .$$

For fixed n consider the martingale g defined by

$$g_k = I(\mu > n, \nu = n) f_{n+k-1} , \qquad k \geqq 1 .$$

Then g is a nonnegative martingale relative to $\{\mathscr{A}_{n+k-1}, k \geqq 1\}$. Let $(e_1, e_2, \cdots)$ denote its difference sequence and use Lemma 2.1 to obtain

$$\begin{aligned} E[\textstyle\sum_{k=n+1}^\infty I(\mu > k, \nu = n)\, d_k^2] &\leqq E[\textstyle\sum_{k=1}^\infty I(g_k^* \leqq \lambda) e_k^2] \\ &\leqq 2\lambda \|g\|_1 = 2\lambda E g_1 \leqq 2\lambda^2 P(\nu = n) . \end{aligned}$$

Therefore,

(18.5) $$\int_{a+\lambda^2}^{\infty} P(S_{\mu-1}^2(f) > s)\, ds \leqq 2\lambda^2 \sum_{n=1}^{\infty} P(\nu = n) = 2\lambda^2 P(\nu < \infty) = 2\lambda^2 P(S_{\mu-1}^2(f) > a)\,.$$

Since

$$\int_a^{a+\lambda^2} P(S_{\mu-1}^2(f) > s)\, ds \leqq \lambda^2 P(S_{\mu-1}^2(f) > a)\,,$$

addition gives

$$\int_a^{\infty} P(S_{\mu-1}^2(f) > s)\, ds \leqq 3\lambda^2 P(S_{\mu-1}^2(f) > a)\,, \qquad a > 0\,.$$

Theorem 18.1 now follows from Lemma 18.1.

PROOF OF THEOREM 18.2. Replace λ by $\delta\lambda$ in (18.1) and (18.5). Then

$$P(S(f) > \beta\lambda, f^* \leqq \delta\lambda) \leqq P(S_{\mu-1}(f) > \beta\lambda)$$

and, for $a = \lambda^2$ in (18.5),

$$\begin{aligned}(\beta^2 - \delta^2 - 1)\lambda^2 P(S_{\mu-1}(f) > \beta\lambda) &\leqq \int_{\lambda^2+\delta^2\lambda^2}^{\beta^2\lambda^2} P(S_{\mu-1}^2(f) > s)\, ds \\ &\leqq 2\delta^2\lambda^2 P(S_{\mu-1}^2(f) > \lambda^2) \\ &\leqq 2\delta^2\lambda^2 P(S(f) > \lambda)\,.\end{aligned}$$

For the case $\Phi(\lambda) = \lambda^p$, choose $\beta = 1 + p^{-1}$ and $\delta = cp^{-\frac{1}{2}}$ in Theorem 18.2 and Lemma 7.1 to obtain

$$\|S(f)\|_p \leqq cp^{\frac{1}{2}}\|f^*\|_p \leqq cp^{\frac{1}{2}}q\|f\|_p\,, \qquad 1 < p < \infty\,.$$

This gives another proof of part of Lemma 3.1.

19. A square function inequality for martingales of bounded mean oscillation. As we have seen in Chapter II, an L^∞-bounded martingale f satisfies $S(f) \in \exp L^2$. This is also an easy consequence of Theorem 18.1. Here is a more general result.

THEOREM 19.1. *Let f be a martingale with difference sequence d satisfying*

(19.1) $$E[\textstyle\sum_{k=n}^{\infty} d_k^2 \mid \mathscr{A}_n] \leqq 1\,, \qquad n \geqq 1\,.$$

Then

(19.2) $$E[\exp tS^2(f)] \leqq (1 - t)^{-1}\,, \qquad 0 < t < 1\,.$$

Garsia has also noticed this and has a different proof [18].

PROOF. For $a > 0$, $b > 0$, and $\tau = \inf\{n : S_n^2(f) > a\}$, we have

$$P(S^2(f) > a + b) \leqq \sum_{n=1}^{\infty} P(\textstyle\sum_{k=n}^{\infty} d_k^2 > b, \tau = n)\,.$$

Integrating with respect to b over $(0, \infty)$ gives

$$\begin{aligned}\int_a^{\infty} P(S^2(f) > \lambda)\, d\lambda &\leqq \sum_{n=1}^{\infty} E[I(\tau = n) \textstyle\sum_{k=n}^{\infty} d_k^2] \\ &= \sum_{n=1}^{\infty} E[I(\tau = n) E(\textstyle\sum_{k=n}^{\infty} d_k^2 \mid \mathscr{A}_n)] \\ &\leqq \sum_{n=1}^{\infty} E[I(\tau = n)] \\ &= P(\tau < \infty) = P(S^2(f) > a)\,.\end{aligned}$$

The theorem now follows from Lemma 18.1.

Let BMO be the linear span of all martingales satisfying(19.1). Suppose that f is L^∞-bounded. Then

$$E[\sum_{k=n}^\infty d_k^2 \mid \mathscr{A}_n] = E[(f_\infty - f_{n-1})^2 \mid \mathscr{A}_n] \leqq 4\|f\|_\infty^2$$

so that $f \in BMO$.

John and Nirenberg defined bounded mean oscillation (for functions) in [26]. This notion plays a fundamental role in the recent work of Fefferman and Stein [17], who prove, among other things, that BMO is the dual of H^1. Also see [18], [24], [20], and [31].

20. Concave Φ. Assume here that Φ not only satisfies the conditions of Section 7 but also is concave. Concavity implies $\Phi(2\lambda) \leqq 2\Phi(\lambda)$ so the growth condition is automatically satisfied. The following result is the concave version of the convexity lemma of Section 16.

THEOREM 20.1. *Suppose that Φ is concave as above. Let $z_1, z_2, \cdots$ be nonnegative measurable functions on $(\Omega, \mathscr{A}, P)$. Then*

$$E\Phi(\sum_{k=1}^\infty z_k) \leqq 2E\Phi(\sum_{k=1}^\infty E(z_k \mid \mathscr{A}_{k-1})) . \tag{20.1}$$

PROOF. Let $Z_n = \sum_{k=1}^n z_k$, $W_n = \sum_{k=1}^n E(z_k \mid \mathscr{A}_{k-1})$, $0 \leqq n \leqq \infty$, $Z = Z_\infty$, and $W = W_\infty$. Then

$$E(Z \wedge \lambda) \leqq 2E(W \wedge \lambda) , \qquad \lambda > 0 . \tag{20.2}$$

To see this, let $\tau = \inf\{n \geqq 0 \colon W_{n+1} > \lambda\}$ and notice that $W_\tau \leqq W \wedge \lambda$. In view of

$$Z \wedge \lambda \leqq Z_\tau + \lambda I(\tau < \infty) ,$$

(20.2) follows from

$$\begin{aligned} EZ_\tau &= E \sum_{k=1}^\infty I(\tau \geqq k) z_k \\ &= E \sum_{k=1}^\infty I(\tau \geqq k) E(z_k \mid \mathscr{A}_{k-1}) \\ &= EW_\tau \leqq E(W \wedge \lambda) \end{aligned}$$

and

$$E[\lambda I(\tau < \infty)] = \lambda P(W > \lambda) \leqq E(W \wedge \lambda) .$$

Inequality (20.2) is a special case of (20.1) but actually implies (20.1). The concavity of Φ implies there is a nonnegative, non-increasing function φ on $(0, \infty)$ such that

$$\Phi(b) = \int_0^b \varphi(\lambda)\, d\lambda = \int_0^b [\varphi(\lambda) - \varphi(\infty)]\, d\lambda + b\varphi(\infty) , \qquad 0 \leqq b \leqq \infty ,$$

where $\varphi(\infty) = \lim_{\lambda\to\infty} \varphi(\lambda)$ (see [43], page 24). To prove (20.1), we may assume that $\varphi(\infty) = 0$. Integrating by parts and using $\varphi(\infty) = 0$ and $a\varphi(a) \leqq \Phi(a)$. we get

$$\begin{aligned} \Phi(t) &= \lim_{a\to 0, b\to\infty} \int_a^b \varphi(\lambda)\, d(t \wedge \lambda) \\ &= \lim_{a\to 0, b\to\infty} [\varphi(b)(t \wedge b) - \varphi(a)(t \wedge a) - \int_a^b (t \wedge \lambda)\, d\varphi(\lambda)] \\ &= -\int_0^\infty (t \wedge \lambda)\, d\varphi(\lambda) , \qquad 0 < t < \infty . \end{aligned}$$

Note that $-d\varphi(\lambda)$ defines a positive measure and the formula holds also for $t = 0$ and $t = \infty$. Therefore, by Fubini's theorem,

$$E\Phi(Z) = -\int_0^\infty E(Z \wedge \lambda)\, d\varphi(\lambda)$$
$$\leqq -2 \int_0^\infty E(W \wedge \lambda)\, d\varphi(\lambda) = 2E\Phi(W) .$$

The above proof can easily be adapted to give other results including the following theorem. Consider the operator

$$s(f) = [\textstyle\sum_{k=1}^\infty E(d_k^2 \mid \mathscr{A}_{k-1})]^{\frac{1}{2}} .$$

THEOREM 20.2. *Suppose that Φ is concave as above. If $f = (f_1, f_2, \cdots)$ is a martingale, then*

$$E\Phi((f^*)^2) \leqq 5E\Phi(s^2(f)) . \tag{20.3}$$

For the special case of a concave power, $\Phi(\lambda) = \lambda^p\ (0 < p \leqq 1)$, this was proved in [6] by another method. The operator $f \to f^*$ could be replaced here by the more general operators of Remark 2.2 in [6].

PROOF. Let $s_n^2(f) = \sum_{k=1}^n E(d_k^2 \mid \mathscr{A}_{k-1}),\ 0 \leqq n \leqq \infty,$

$$\tau = \inf\{n \geqq 0 : s_{n+1}^2(f) > \lambda\} ,$$

and denote by f^τ the martingale f stopped at τ: $(f^\tau)_n = \sum_{k=1}^n I(\tau \geqq k)\, d_k$. Then

$$E[(f^*)^2 \wedge \lambda] \leqq 5E[s^2(f) \wedge \lambda]$$

since $(f^*)^2 \wedge \lambda \leqq [(f^\tau)^*]^2 + \lambda I(\tau < \infty)$, where $||(f^\tau)^*||_2 \leqq 2||f^\tau||_2 = 2||s(f^\tau)||_2 = 2||s_\tau(f)||_2$, $s_\tau^2(f) \leqq s^2(f) \wedge \lambda$, and

$$E[\lambda I(\tau < \infty)] = \lambda P(s^2(f) > \lambda) \leqq E[s^2(f) \wedge \lambda] .$$

As in the last proof, (20.3) follows.

21. More about $s(f)$. It is easy to see that an inequality like (20.3) cannot hold for general Φ; see Example 8.2 of [6]. However, the following is true.

THEOREM 21.1. *If Φ is a function satisfying the conditions of Section 7 and f is a martingale then*

$$E\Phi(f^*) \leqq cE\Phi(s(f)) + cE\Phi(d^*) . \tag{21.1}$$

The choice of c depends only on $c_{(6.1)}$.

This follows from Theorem 3.1 of [5]; see the remark on the top of page 553 in [4]. Alternatively, the theorem follows at once, by Lemma 7.1, from the easy distribution function inequality

$$P(f^* > \beta\lambda,\ s(f) \vee d^* \leqq \delta\lambda) \leqq \frac{\delta^2}{(\beta - \delta - 1)^2} P(f^* > \lambda) , \qquad \lambda > 0 , \tag{21.2}$$

in which $\beta > 1$ and $0 < \delta < \beta - 1$.

To prove (21.2), let $\mu = \inf\{n : |f_n| > \lambda\}$, $\nu = \inf\{n : |f_n| > \beta\lambda\}$,

$$\sigma = \inf\{n \geqq 0 : |d_n| > \delta\lambda \ \text{ or } \ s_{n+1}(f) > \delta\lambda\} ,$$

and let h be the martingale f started at μ and stopped at $\nu \wedge \sigma$:

$$h_n = \sum_{k=1}^{n} I(\mu < k \leqq \nu \wedge \sigma)\, d_k .$$

Then the left-hand side of (21.2) is no greater than

$$P(\mu \leqq \nu < \infty, \sigma = \infty) \leqq P(h^* > (\beta - \delta - 1)\lambda)$$
$$\leqq \frac{1}{(\beta - \delta - 1)^2 \lambda^2} \|h\|_2^2 .$$

But $\|h\|_2 = \|s(h)\|_2$ where $s^2(h) \leqq s_\sigma^2(f) I(\mu < \infty) \leqq \delta^2 \lambda^2 I(f^* > \lambda)$. These facts imply (21.2).

Using the notation of Section 20, we can obtain

$$\text{(21.3)} \qquad P(Z > \beta\lambda, W \vee z^* \leqq \delta\lambda) \leqq \frac{\delta}{\beta - \delta - 1} P(Z > \lambda) , \qquad \lambda > 0 ,$$

by similar reasoning. This gives another proof of

$$E\Phi(Z) \leqq cE\Phi(W) + cE\Phi(z^*)$$

for general Φ; see [4].

Consider the following inequality of Rosenthal [39]: If $d = (d_1, d_2, \cdots)$ is an independent sequence of random variables, each with expectation zero, and $f_n = \sum_{k=1}^{n} d_k$, $n \geqq 1$, then

$$\text{(21.4)} \qquad \|f\|_p^p \leqq c_p (\sum_{k=1}^{\infty} E\, d_k^2)^{p/2} + c_p \sum_{k=1}^{\infty} E|d_k|^p , \qquad 2 \leqq p < \infty ,$$

and the reverse inequality is also true. Using $\Phi(d^*) \leqq \sum_{k=1}^{\infty} \Phi(|d_k|)$ and Theorem 21.1, we have

$$\text{(21.5)} \qquad E\Phi(f^*) \leqq cE\Phi(s(f)) + c \sum_{k=1}^{\infty} E\Phi(|d_k|)$$

for all martingales f and all Φ satisfying the condition of Section 7. In the independent case, $s^2(f) = \sum_{k=1}^{\infty} E(d_k^2 \mid \mathscr{A}_{k-1}) = \sum_{k=1}^{\infty} E\, d_k^2$ and (21.4) follows. The reverse of (21.4) follows from $\sum_{k=1}^{\infty} |d_k|^p \leqq S^p(f)$, $\|s(f)\|_p \leqq c_p \|S(f)\|_p$, which is a consequence of Lemma 16.1 (also, see [6]), and Theorem 3.2. Note that only the reverse inequality requires any restriction of Φ.

22. Concluding remarks. The methods of Chapter II can be used to give simpler derivations of the general operator inequalities of [5] and [6].

Many further applications could be discussed, some rather obvious: to Skorokhod embedding, square-root boundary problems in random walk, etc. Applications related to these are described in [6].

Finally, much of the work reviewed here has been joint work with R. F. Gundy; some also has been joint with B. J. Davis and M. L. Silverstein. To these three, and to many others who have taken an enthusiastic interest, I express my deep appreciation.

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