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ON L_p CHEBYSHEV–CRAMÉR ASYMPTOTIC EXPANSIONS

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An L_1-smoothing lemma is used to prove an L_1 version of the Chebyshev–Cramér asymptotic expansion for independent (identically distributed) random variables. The conditions imposed are exactly those demanded for the L_∞ version.

1. Introduction. We have shown recently [2] that an L_1 version of the Berry–Esséen theorem can be proved exactly as the usual L_∞ version if one uses an appropriate L_1 smoothing lemma. We show here that the same holds for the Chebyshev–Cramér asymptotic expansions.

2. Notation and results. Throughout we let $X_1, X_2, \cdots$ be a sequence of independent random variables with $EX_i = 0$, $EX_i^2 = \sigma_i^2 < \infty$, all i, with $\sigma_i = 1$ in the identically distributed case. Let $S_n = X_1 + \cdots + X_n$, $s_n^2 = \sigma_1^2 + \cdots + \sigma_n^2$, $F_n(x) = P(S_n \leqq xs_n)$,

$$\mathfrak{N}(x) = \textstyle\int_{-\infty}^{x} \mathfrak{n}(y)\,dy\,, \qquad \mathfrak{n}(y) = (2\pi)^{-\frac{1}{2}} \exp(-y^2/2)\,, \qquad \varphi_k(t) = E\exp(itX_k)\,.$$

We wish to examine the L_p norm, with Lebesgue measure, of the expansion error

$$\varepsilon_{n,k}(x) = F_n(x) - \mathfrak{N}(x) - \mathfrak{n}(x) \textstyle\sum_1^k n^{-s/2} Q_s(x)$$

where the Q_s are appropriate polynomials. Write

$$\varepsilon_{n,k,p} = ||\varepsilon_{n,k}||_p\,.$$

Consider first the case where the X_i's are independent and identically distributed (i.i.d.). Under Cramér's condition

$$\text{(C)} \qquad \limsup_{|t|\to\infty} |\varphi_1(t)| < 1\,.$$

Feller ([3] page 541) shows that $\varepsilon_{n,k,\infty} = o(n^{-k/2})$ if the first $k+2$ moments of X_1 are finite. Ibragimov [5] extends this result by giving necessary and sufficient conditions for certain rates of convergence to zero of $\varepsilon_{n,k,\infty}$ (Theorems 1, 2 below with $p = \infty$). We extend this further to include the $\varepsilon_{n,k,p}$ case, $1 \leqq p \leqq \infty$.

To be more precise, let $X_1, X_2, \cdots$ be i.i.d. F with characteristic function φ, and let $\alpha_s = EX_1^s$, if this moment exists. Let $\{\beta_i\}_1^\infty$ be a sequence of reals and form the polynomials Q_s in the usual way using the sequence of β's. (See, for example, Cramér [1] page 70, ff. Ibragimov [5], or Feller's constructive approach [3], page 535, which we essentially follow below.)

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THEOREM 1. *Let* $X_1, X_2, \cdots$ *be* i.i.d. F. *In order that*

$$\varepsilon_{n,k,p} = o(n^{-k/2}) , \qquad 1 \leqq p \leqq \infty$$

it is necessary, and for distributions satisfying condition (C) *also sufficient, that*

(1) *the absolute moments of* F *up to order* $k+1$ *inclusive are finite, and* $\alpha_1 = \beta_1, \cdots, \alpha_{k+1} = \beta_{k+1}$,

(2) $$\int_{|x|>z} |x|^{k+1} F(dx) = o(z^{-1}) , \qquad z \to \infty ,$$

(3) $$\lim_{z\to\infty} \int_{-z}^{z} x^{k+2} F(dx) = \beta_{k+2} .$$

THEOREM 2. *Let* $X_1, X_2, \cdots$ *be* i.i.d. F. *Let* $0 < \delta \leqq 1$. *In order that*

$$\varepsilon_{n,k,p} = O(n^{-(k+\delta)/2}) , \qquad 1 \leqq p \leqq \infty$$

it is necessary, and for distributions satisfying condition (C) *also sufficient, that*

(4) *the absolute moment of order* $k+2$ *be finite and* $\alpha_1 = \beta_1, \cdots, \alpha_{k+2} = \beta_{k+2}$,

(5) $$\int_{|x|>z} |x|^{k+2} F(dx) = O(z^{-\delta}) , \qquad z \to \infty ,$$

and for $\delta = 1$ *also*

(6) $$\int_{-z}^{z} x^{k+3} F(dx) = O(1) , \qquad z \to \infty .$$

It should be noted that the conditions (1) to (6) are independent of p, $1 \leqq p \leqq \infty$.

If $X_1, X_2, \cdots$ are i.i.d. F with characteristic function φ, Theorems 1 and 2 have an equivalent form in terms of φ. The numbers μ_s below are related to the β's in the same way that the cumulants (semi-invariants) λ's are related to the moments α_s. (This relation is derived in our proof of equations (8) and (8′) below.) Under condition (4), $\lambda_s = \mu_s$, $s = 1, \cdots, k+2$.

THEOREM 1′. *In order that*

$$\varepsilon_{n,k,p} = o(n^{-k/2}) , \qquad 1 \leqq p \leqq \infty ,$$

it is necessary and for distributions satisfying condition (C) *also sufficient that*

(7) $$\varphi(t) = \exp\{\textstyle\sum_2^{k+2} (it)^s \mu_s/s! + o(|t|^{k+2})\}$$

as $t \to 0$.

THEOREM 2′. *Let* δ *be such that* $0 < \delta \leqq 1$. *In order that*

$$\varepsilon_{n,k,p} = O(n^{-(k+\delta)/2})$$

it is necessary and for distributions satisfying condition (C) *also sufficient that*

(8) $$\varphi(t) = \exp\{\textstyle\sum_2^{k+2} (it)^s \mu_s/s! + O(|t|^{k+2+\delta})\}$$

as $t \to 0$.

In the case of non-identically distributed summands, Feller ([3) page 546, ff.) gives a hint at what can be proved. For example, if condition

($C_{\frac{1}{2}}$) $$|\varphi_1(t) \cdots \varphi_n(t)| = o(n^{-\frac{1}{2}}) \quad \text{uniformly for} \quad |t| > \delta > 0$$

is satisfied and if there exists constants $\infty > b > a > 0$, $M > 0$ such that $an < s_n^2 < bn$, $EX_n^2 < M$ for all n, then $\varepsilon_{n,1,\infty} = o(n^{-\frac{1}{2}})$ with

$$n^{-\frac{1}{2}}Q_1(x) = \sum_1^n (EX_k^3/6s_n^3)(1 - x^2) .$$

To get an L_1 rate we also need

$$(\mathrm{C}_{\frac{1}{2}}') \qquad \left|\frac{d}{dt}\varphi_1(t) \cdots \varphi_n(t)\right| = o(n^{-\frac{1}{2}}) \quad \text{uniformly for} \quad |t| > \delta > 0 .$$

THEOREM 3. *Suppose $X_1, X_2, \cdots$ are independent, that $EX_n^4 < M < \infty$, that $0 < a < s_n^2/n < b < \infty$ for all n, and that conditions* $(\mathrm{C}_{\frac{1}{2}})$ *and* $(\mathrm{C}_{\frac{1}{2}}')$ *hold. Then*

$$\varepsilon_{n,1,p} = o(n^{-\frac{1}{2}}) , \qquad 1 \leqq p \leqq \infty .$$

We leave it to the reader to formulate the L_p conditions for other expansions whose L_∞ version is known. See Cramér [1] and Feller [3].

3. Proofs. Theorems 2 and 2′ are proved in the following way:

(a) conditions (4), (5) [and (6)] imply (8) and (8′) below for $0 < \delta < 1$ [for $\delta = 1$],

(b) conditions (8), (8′) and (C) imply

$$\varepsilon_{n,k,p} = O(n^{-(k+\delta)/2}) , \quad \text{all} \quad p \quad \text{in} \quad [1, \infty] ,$$

(c) if the rate given in (b) holds for some p in $[1, \infty]$ then conditions (4) and (5) [and (6)] hold for $0 < \delta < 1$ [for $\delta = 1$].

We now consider each of these implications.

(a) This is independent of p, and the proof of the implication of (8) is in Ibragimov [5]. We adapt his arguments to prove a statement concerning the logarithmic derivative of φ [see (8′)] and this in turn implies (8).

LEMMA. (i) *Suppose the random variable X_1 has a finite absolute moment of order $k \geqq 2$. Then there exist constants $r_0, r_1, \cdots, r_k$ such that as $t \to 0$*

$$\frac{d}{dt}\log\varphi(t) - \sum_0^k (it)^s r_s/s! = \int_{-\infty}^{\infty} (e^{itx} - \sum_0^{k-1} (itx)^s/s!)ixF(dx) + O(|t|^{k+1}) .$$

The semi-invariants are now defined by $i\lambda_s = r_{s-1}$, $s = 1, \cdots, k$.

(ii) *Conditions* (4) *and* (5) [*and* (6)] *imply*

$$(8') \qquad \frac{d}{dt}\log\varphi(t) = i\sum_2^{k+2} (it)^{s-1}\mu_s/(s-1)! + O(|t|^{k+1+\delta})$$

as $t \to 0$, for $0 < \delta < 1$ [for $\delta = 1$].

PROOF. $E|X_1|^k < \infty$ implies

$$\varphi(t) - \sum_0^k (it)^s\alpha_s/s! = \int_{-\infty}^{\infty} (e^{itx} - \sum_0^k (itx)^s/s!)F(dx) = o(|t|^k)$$

and

$$\varphi'(t) - i\sum_0^{k-1} (it)^s\alpha_{s+1}/s! = \int_{-\infty}^{\infty} (e^{itx} - \sum_0^{k-1} (itx)^s/s!)ixF(dx) = o(|t|^{k-1}) .$$

Choose $\eta > 0$ so that $|t| < \eta$ implies $|1 - \varphi(t)| \leqq \frac{1}{2}$. Then for $|t| < \eta$ we have

$$\begin{aligned}\frac{d}{dt}\log\varphi(t) &= \varphi'(t)/\varphi(t) = \varphi'(t)\textstyle\sum_0^\infty [1-\varphi(t)]^s \\ &= \varphi'(t)[1 + \textstyle\sum_1^\nu (-1)^s(\sum_2^k (it)^j\alpha_j/j!)^s + o(|t|^k) + O(|t|^{2\nu+2})] \\ &= \varphi'(t) - i\textstyle\sum_0^{k-1}(it)^s\alpha_{s+1}/s! \\ &\quad + [i\textstyle\sum_0^{k-1}(it)^s\alpha_{s+1}/s!][\sum_0^\nu(-1)^s(\sum_2^k (it)^j\alpha_j/j!)^s] \\ &\quad + o(|t|^{k+1}) + O(|t|^{2\nu+3}).\end{aligned}$$

Take ν as the smallest integer for which $2\nu + 3 \geqq k + 2$ and define $r_0, \cdots, r_k$ so that the product of the terms in the square brackets is $\sum_0^k (it)^s r_s/s! + O(|t|^{k+1})$. This gives (i). Integration of this shows that $i\lambda_s = r_{s-1}$, $s = 1, \cdots, k$, for the λ's are defined by the relation

$$\log\varphi(t) = \textstyle\sum_1^k (it)^j\lambda_j/j! + o(|t|^k).$$

To prove (ii) notice that (i) and (4) imply $\mu_s = \lambda_s$, $s = 1, \cdots, k+2$ and

$$\begin{aligned}&\left|\frac{d}{dt}\log\varphi(t) - i\textstyle\sum_2^{k+2}(it)^{s-1}\mu_s/(s-1)!\right| \\ &\quad= |\textstyle\int_{-\infty}^\infty (e^{itx} - \sum_0^{k+1}(itx)^s/s!)ixF(dx)| + O(|t|^{k+2}) \\ &\quad\leqq \frac{2|t|^{k+1}}{(k+1)!}\int_{|xt|>1}|x|^{k+2}F(dx) + \left|\int_{|tx|\leq 1}\frac{t^{k+2}x^{k+3}}{(k+2)!}F(dx)\right| \\ &\qquad + \frac{|t|^{k+3}}{(k+3)!}\int_{|tx|\leq 1}|x|^{k+4}F(dx) + O(|t|^{k+2}).\end{aligned}$$

Now argue exactly as in Ibragimov ([5] page 462): By (5) the first term in the right-hand side is $O(|t|^{k+1+\delta})$, $0 < \delta \leqq 1$. If $\delta = 1$, the second term is also $O(|t|^{k+1+\delta})$ by (6). If $\delta < 1$ the second term may be handled by introducing $R(u) = \int_{|x|>u}|x|^{k+2}\,dF(x)$. Then

$$\begin{aligned}|t|^{k+2}\int_{|tx|\leq 1}|x|^{k+3}\,dF(x) &= -|t|^{k+2}\textstyle\int_0^{1/|t|} x\,dR(x) \\ &\leqq |t|^{k+1}R(1/|t|) + |t|^{k+2}\textstyle\int_0^{1/|t|} R(x)\,dx = O(|t|^{k+1+\delta}).\end{aligned}$$

Notice this argument fails for $\delta = 1$. The third term is handled in the same way since, for $0 < \delta \leqq 1$,

$$-|t|^{k+3}\textstyle\int_0^{1/|t|} x^2\,dR(x) = O(|t|^{k+1+\delta}).$$

Thus (4), (5) [and (6)] imply (8′) which in turn implies (8).

(c) We must now show that if $\varepsilon_{n,k,p} = O(n^{-(k+\delta)/2})$ for some p in $[1, \infty]$, then (4) and (5) [and (6)] hold when $0 < \delta < 1$ [when $\delta = 1$]. The proof of this given by Ibragimov for the case $p = \infty$ was designed to work for the general case. The proof is by induction on k. In [4] the first step is proved for $k = 0$ and p in $[1, \infty]$. In [5] (page 463 ff.) the induction step is proved, but only for $p = \infty$. Here a certain function $A^\sim$ with $||A^\sim||_1 < \infty$, $||A^\sim||_\infty < \infty$ is introduced and one considers

$$\textstyle\int_{-\infty}^\infty |\varepsilon_{n,k+1}(x)A^\sim(x)|\,dx\,.$$

Using Hölder's inequality we see this is $O(n^{-(k+1+\delta)/2})$ if $\varepsilon_{n,k+1,p} = O(n^{-(k+1+\delta)/2})$, any p in $[1, \infty]$. The rest of Ibragimov's argument is independent of p, and we will not reproduce it as it is rather intricate. This completes the proof of (c).

(b) Since $||\cdot||_p{}^p \leqq ||\cdot||_\infty{}^{p-1}||\cdot||_1$, we need prove (b) only for $p = 1$, the $p = \infty$ case being given in [5]. We argue as in Feller [3] but replace the L_∞ smoothing lemma used there by the following

L_1-SMOOTHING LEMMA. *Let H be a (probability) distribution function, let G be a function of bounded variation and let $H^\wedge$ and $G^\wedge$ be their Fourier-Stieltjes transforms. If $G(-\infty) = 0$, $G(+\infty) = 1$ and $||H - G||_1 < \infty$, then for all $T > 0$*

$$||H - G||_1 \leqq 4\pi(1 + \text{Var } G)/T + (\tfrac{1}{2} + 4/T^2)^{\frac{1}{2}}\varepsilon + \delta_1 + \delta_2$$

where

$$\varepsilon^2 = \int_{-T}^{T} |H^\wedge(t) - G^\wedge(t)|^2 t^{-2}\, dt$$
$$\delta_1{}^2 = \int_{-T}^{T} |H^\wedge(t) - G^\wedge(t)|^2 t^{-4}\, dt$$
$$\delta_2{}^2 = \int_{-T}^{T} \left|\frac{d}{dt}(H^\wedge(t) - G^\wedge(t))\right|^2 t^{-2}\, dt\,.$$

This lemma is due to Esséen and is proved in [6] page 25. (Var G = total variation of G.)

To apply this lemma, take $H = F_n$ and $G = G_{n,k} = F_n - \varepsilon_{n,k} = \mathfrak{N} + \mathfrak{n}\sum_1^k n^{-s/2}Q_s$. H and G meet all requirements: notice that Var $G_{nk} \leqq$ Var $G_{1k} < \infty$, and that $||H - \mathfrak{N}||_1 < \infty$ by Chebyshev's inequality.

The random variables $X_1, X_2, \cdots$ are i.i.d. with distribution function F and $\varphi = F^\wedge$. By equation (8)

$$\psi(t) = \log \varphi(t) + t^2/2 = \psi_k(t) + O(|t|^{k+2+\delta})\,,$$

where

$$\psi_k(t) = \sum_3^{k+2} (it)\mu_s/s!\,.$$

Define

$$\alpha = \alpha(t) = n\psi(t/n^{\frac{1}{2}}) \qquad \text{and}$$
$$\beta = \beta(t) = n\psi_k(t/n^{\frac{1}{2}})$$

and notice that if $\gamma > |\alpha|$, $\gamma > |\beta|$ then

$$|e^\alpha - \sum_0^k \beta^s/s!| \leqq e^\gamma(|\alpha - \beta| + |\beta|^{k+1}/(k+1)!)$$

and, writing $'$ for differentiation,

$$\left|\frac{d}{dt}(e^\alpha - \sum_0^k \beta^s/s!)\right| \leqq |e^\alpha(\alpha' - \beta')| + |\beta'(e^\alpha - \sum_0^{k-1} \beta^s/s!)|$$
$$\leqq e^\gamma\{|\alpha' - \beta'| + |\beta'|\,|\alpha - \beta| + |\beta'|\,|\beta|^k/k!\}\,.$$

By the relation between the sequences $\{\beta_j\}$ and $\{\mu_j\}$ we know that

$$H^\wedge(t) - G^\wedge(t) = e^{-t^2/2}(e^\alpha - \sum_0^k \beta^s/s!)$$

(see Feller [3] page 535) and thus

$$\frac{d}{dt}(H^\wedge(t) - G^\wedge(t)) = -t(H^\wedge(t) - G^\wedge(t)) + e^{-t^2/2}\frac{d}{dt}(e^\alpha - \sum_0^k \beta^s/s!)$$

with α and β as defined above. From (8) and (8′) it follows that there exists $t_0 > 0$, $0 < K < \infty$ and ρ, $0 < \rho < t_0$ such that $|t| < \rho n^{\frac{1}{2}}$ implies

$$|\alpha(t)| \leqq t^2/4\,, \qquad |\beta(t)| \leqq a|t|^3 n^{-\frac{1}{2}} \leqq t^2/4\,, \qquad a = 1 + |\mu_3|\,,$$
$$|\alpha(t) - \beta(t)| \leqq K|t|^{k+2+\delta} n^{-(k+\delta)/2}$$
$$|\alpha'(t) - \beta'(t)| \leqq (k + 2 + \delta)K|t|^{k+1+\delta} n^{-(k+\delta)/2}$$
$$|\beta'(t)| \leqq at^2 n^{-\frac{1}{2}}\,.$$

This entails, for $|t| < \rho n^{\frac{1}{2}}$,

$$|H^{\wedge}(t) - G^{\wedge}(t)| \leqq e^{-t^2/4}(K|t|^{k+2+\delta} n^{-(k+\delta)/2} + a^{k+1} t^{3k+3} n^{-(k+1)/2}/(k+1)!)$$

and, with $K^* = (k + 2 + \delta)K$,

$$\left| e^{-t^2/2} \frac{d}{dt} (e^{\alpha} - \textstyle\sum_0^k \beta^s/s!) \right|$$
$$\leqq e^{-t^2/4}\{K^*|t|^{k+1+\delta} n^{-(k+\delta)/2} + aK|t|^{k+4+\delta} n^{-(k+2+\delta)/2} + a^{k+1}|t|^{3k+2} n^{-(k+1)/2}/k!\}\,.$$

Define $T = t_0 n^{(k+\delta)/2}$, $T_\rho = \rho n^{\frac{1}{2}}$. The above reasoning shows that the contribution to ε, δ_1 and δ_2 by the interval $|t| < T_\rho$ is $O(n^{-(k+\delta)/2})$. Considering δ_2 for $T_\rho \leqq |t| \leqq T$ we have the bound

$$\int_{T_\rho \leqq |t| \leqq T} n^{\frac{1}{2}} |\varphi^{n-1}(t/n^{\frac{1}{2}})\varphi'(t/n^{\frac{1}{2}})|\, dt + \int_{T_\rho \leqq |t| \leqq T} \left| \frac{d}{dt} G^{\wedge}(t) \right| dt\,.$$

But condition (C) implies $\max_{|t|>1} |\varphi(t)| = \theta < 1$ and $EX_1^2 = 1$ implies $|\varphi'(t)| \leqq 1$, and hence the first term is $O(\theta^n n^{(k+1+\delta)/2})$. This goes to zero faster than any power of n. Now $G^{\wedge} = e^{-t^2/2}P(t)$, $P(t)$ a polynomial in t, so that the second term also goes to zero faster than any power of n. The contributions to ε and δ_1 over $T_\rho < t \leqq T$ are treated similarly. This completes the proof of Theorems 2, 2′.

Theorems 1 and 1′ are proved in the same way and will not be discussed here.

Theorem 3 is easily proved: Set $v_n(t) = n^{-1} \sum_1^n \log \varphi_k(t)$,

$$\alpha(t) = nv_n(t/s_n) + t^2/2\,,$$
$$\beta(t) = nv_n'''(0)t^3/6s_n^3\,,$$
$$T = as_n,\ T_\rho = \rho s_n,\ a \quad \text{and} \quad 1/\rho \quad \text{sufficiently large.}$$

Now use the L_1 smoothing lemma and the argument similar to those for the proof of Theorems 2, 2′.

The uniform bound on fourth moments implies that v_n''' is uniformly continuous near zero, and this guarantees appropriate bounds for $\alpha(t) - \beta(t)$ and

$$\alpha'(t) - \beta'(t) = \frac{n}{s_n}[v_n'(t/s_n) - v_n'(0) - v_n''(0)t/s_n - v_n'''(0)t^2/2s_n^2]\,.$$

The $(C_{\frac{1}{2}}')$ condition states that $|(d/dt)\varphi_1(t) \cdots \varphi_{n+1}(t)| = o(n^{-\frac{1}{2}})$ uniformly for $|t| > \delta > 0$, which is all that is needed for the rate corresponding to the contributions to δ_2 given by the interval $T_\rho < t < T$. The $(C_{\frac{1}{2}})$ condition gives correct rates for the outer portions of ε and δ_1. This completes the proof of Theorem 3.

REMARK. Note that if $X_1, X_2, \cdots$ are i.i.d., if condition (C) and (4) and (5) [and (6)] hold for $X_1, X_2, \cdots$, and $EX_1^j = E\mathfrak{N}^j, j = 1, \cdots, k+2$, then $Q_s \equiv 0$, $s = 1, \cdots, k$, and a better rate holds for the Berry–Esséen theorem:

$$||F_n - \mathfrak{N}||_p = O(n^{-(k+\delta)/2}), \qquad 1 \leqq p \leqq \infty,$$

$0 < \delta \leqq 1$ [$\delta = 1$].

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