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A SIMPLE ANALYTIC PROOF OF THE POLLACZEK-WENDEL IDENTITY FOR ORDERED PARTIAL SUMS

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In this note we prove an identity due to Pollaczek (1952) and Wendel (1960). The identity describes the distributions of ordered partial sums of independent identically distributed random variables and thus generalizes Spitzer's identity. Our proof follows from a simple analytic argument applying a kind of Wiener-Hopf decomposition. We also give an extension of the Pollaczek-Wendel identity.

Let $\{X_n, n = 1, 2, \cdots\}$ be a sequence of independent identically distributed random variables with common distribution function $G(x) = P\{X_n < x\}$. The nth partial sum is denoted by $S_n = X_1 + X_2 + \cdots + X_n$ for $n = 1, 2, \cdots$; while $S_0 = 0$. If we have a sequence of real numbers $a_1, a_2, \cdots, a_n$; then by $\max^{(k)}_{1\leq i\leq n} a_i$ we denote the kth largest element of this sequence, so $\max^{(1)}_{1\leq i\leq n} a_i = \max_{1\leq i\leq n} a_i$ and $\max^{(n)}_{1\leq i\leq n} a_i = \min_{1\leq i\leq n} a_i$. We introduce the random variables $R_{n,k} = \max^{(k)}_{1\leq i\leq n} S_i$. The distributions of these variables were studied by Pollaczek (1952) and Wendel (1960), who both found a result which generalizes Spitzer's identity. In the latter identity only the variables $R_{n,1} = \max_{1\leq i\leq n} S_i$ occur. The treatment by Pollaczek requires very complicated analytical arguments, while Wendel applied a simpler algebraic method. Analytical methods are unpopular because of their apparent complexity. This note is intended to illustrate that properties of random walks having simple combinatorial and algebraic proofs may have even simpler analytical proofs. It also shows the power of the Wiener-Hopf decomposition. The present approach is an extension of a very elegant derivation of Spitzer's identity given by Cohen (1969). We define

$$S_{n,k} = \max^{(k)}_{1\leq i\leq n} (S_n - S_{n-i}), \qquad n = 1, 2, \cdots; k = 1, 2, \cdots, n;$$

and note that $S_{n,k}$ and $R_{n,k}$ are identically distributed. Moreover we see that the joint distributions of S_n and $S_{n,k}$ on the one hand and S_n and $R_{n,k}$ on the other hand are the same. We proceed by considering the variables $S_{n,k}$ instead of $R_{n,k}$; and for convenience we define $S_{n,k} = 0$ for $n = 0, 1, 2, \cdots$; $k = n + 1, n + 2, \cdots$. We shall use the notation $x^+ = \max(0, x)$ and $x^- = \min(0, x)$.

THEOREM 1.

$$(1) \quad S_{n+1,1} = S^+_{n,1} + X_{n+1}; \qquad n = 0, 1, 2, \cdots;$$
$$S_{n+1,k} = S^+_{n,k} + S^-_{n,k-1} + X_{n+1}; \qquad n = 1, 2, 3, \cdots; k = 2, 3, \cdots, n + 1.$$

PROOF.

$$S_{n+1,1} = \max(X_{n+1}, S_{n,1} + X_{n+1}) = S^+_{n,1} + X_{n+1}, \qquad n = 0, 1, 2, \cdots;$$
$$S_{n+1,k} = \max{}^{(k)}(X_{n+1}, S_{n,1} + X_{n+1}, \cdots, S_{n,k-1} + X_{n+1}, S_{n,k} + X_{n+1}),$$

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for $n = 1, 2, \cdots; k = 2, 3, \cdots, n + 1$; hence

$$\text{if} \quad S_{n,k} < 0 \quad \text{then} \quad S_{n+1,k} = \min(X_{n+1}, S_{n,k-1} + X_{n+1}) = S_{n,k-1}^{-} + X_{n+1},$$

$$\text{if} \quad S_{n,k} \geqq 0 \quad \text{then} \quad S_{n+1,k} = S_{n,k} + X_{n+1}.$$

Define for $|q| < 1$, $|r| < 1$, $\operatorname{Re} \rho_1 \geqq 0$, $\operatorname{Re} \rho_3 \geqq 0$,

$$\Phi(q, r, \rho_1, \rho_3) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} q^n r^{k-1} E\{\exp(-\rho_1 S_{n,k}^{+} - \rho_3 S_n)\},$$

and for $\operatorname{Re} \rho = 0$,

$$\Gamma(\rho) = \int_{-\infty}^{\infty} e^{-\rho x} dG(x) = E\{\exp(-\rho X_n)\}.$$

THEOREM 2. (The Pollaczek–Wendel identity.)

$$\Phi(q, r, \rho_1, \rho_3) \tag{2}$$
$$= \frac{1}{(1 - r)\{1 - q\Gamma(\rho_3)\}} \times \exp\left[\sum_{n=1}^{\infty} \frac{q^n}{n} (r^n - 1) E\{\exp(-\rho_3 S_n)(1 - \exp(-\rho_1 S_n^{+}))\}\right],$$

for $|q| < 1$, $|r| < 1$, $\operatorname{Re} \rho_1 \geqq 0$, $\operatorname{Re} \rho_3 = 0$.

PROOF. For $n = 1, 2, \cdots; k = 2, 3, \cdots, n + 1$; we have

$$\begin{aligned}\exp(-\rho_1 S_{n+1,k}^{+}) &= \exp(-\rho_1 S_{n+1,k}) + 1 - \exp(-\rho_1 S_{n+1,k}^{-}) \\ &= \exp(-\rho_1 X_{n+1})\{\exp(-\rho_1 S_{n,k}^{+}) + \exp(-\rho_1 S_{n,k-1}^{-})\} \\ &\quad + 1 - \exp(-\rho_1 S_{n+1,k}^{-}),\end{aligned}$$

or upon multiplying with $\exp(-\rho_3 S_{n+1})$ and rearranging

$$\begin{aligned}&\exp(-\rho_1 S_{n+1,k}^{+} - \rho_3 S_{n+1}) - \exp(-(\rho_1 + \rho_3) X_{n+1}) \exp(-\rho_1 S_{n,k}^{+} - \rho_3 S_n) \\ &= \exp(-\rho_3 S_{n+1}) - \exp(-\rho_1 S_{n+1,k}^{-} - \rho_3 S_{n+1}) \\ &\quad - \exp(-(\rho_1 + \rho_3) X_{n+1})\{\exp(-\rho_3 S_n) - \exp(-\rho_1 S_{n,k-1}^{-} - \rho_3 S_n)\}.\end{aligned} \tag{3}$$

Taking expectations (3) yields for $\operatorname{Re} \rho_1 = 0$, $\operatorname{Re} \rho_3 = 0$, $n = 1, 2, \cdots; k = 2, 3, \cdots, n + 1$;

$$\begin{aligned}&E\{\exp(-\rho_1 S_{n+1,k}^{+} - \rho_3 S_{n+1})\} - \Gamma(\rho_1 + \rho_3) E\{\exp(-\rho_1 S_{n,1}^{+} - \rho_3 S_n)\} \\ &= E\{\exp(-\rho_3 S_{n+1}) - \exp(-\rho_1 S_{n+1,k}^{-} - \rho_3 S_{n+1})\} \\ &\quad - \Gamma(\rho_1 + \rho_3) E\{\exp(-\rho_3 S_n) - \exp(-\rho_1 S_{n,k-1}^{-} - \rho_3 S_n)\},\end{aligned}$$

and for $n = 0, 1, \cdots$; we have

$$\begin{aligned}&E\{\exp(-\rho_1 S_{n+1,1}^{+} - \rho_3 S_{n+1})\} - \Gamma(\rho_1 + \rho_3) E\{\exp(-\rho_1 S_{n,1}^{+} - \rho_3 S_n)\} \\ &= E\{\exp(-\rho_3 S_{n+1}) - \exp(-\rho_1 S_{n+1,1}^{-} - \rho_3 S_{n+1})\}.\end{aligned}$$

Combining the last two formulas it follows for $\operatorname{Re} \rho_1 = 0$, $\operatorname{Re} \rho_3 = 0$, $|q| < 1$, $|r| < 1$, that

$$\begin{aligned}&\Phi(q, r, \rho_1, \rho_3) \frac{1 - q\Gamma(\rho_1 + \rho_3)}{1 - qr\Gamma(\rho_1 + \rho_3)} \\ &= \frac{1}{(1 - r)\{1 - qr\Gamma(\rho_3)\}} + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} q^n r^{k-1} \\ &\quad \times E\{\exp(-\rho_3 S_n)(1 - \exp(-\rho_1 S_{n,k}^{-}))\}.\end{aligned} \tag{4}$$

Define for $\text{Re}\,\rho_3 = 0$, $|q| < 1$, $|r| < 1$,

$$K^+(q, r, \rho_1, \rho_3) = \exp\left[\sum_{n=1}^{\infty} \frac{q^n}{n}(r^n - 1)E\{\exp(-\rho_3 S_n)(\exp(-\rho_1 S_n^+) - 1)\}\right], \quad \text{Re}\,\rho_1 \geqq 0\,,$$

$$K^-(q, r, \rho_1, \rho_3) = \exp\left[\sum_{n=1}^{\infty} \frac{q^n}{n}(r^n - 1)E\{\exp(-\rho_1 S_n^- - \rho_3 S_n)\}\right], \quad \text{Re}\,\rho_1 \leqq 0\,,$$

then it is immediately seen that

$$\frac{1 - q\Gamma(\rho_1 + \rho_3)}{1 - qr\Gamma(\rho_1 + \rho_3)} = K^+(q, r, \rho_1, \rho_3)K^-(q, r, \rho_1, \rho_3)\,,$$

so that instead of (4) we can write for $\text{Re}\,\rho_1 = 0$, $\text{Re}\,\rho_3 = 0$, $|r| < 1$, $|q| < 1$,

$$\Phi(q, r, \rho_1, \rho_3)K^+(q, r, \rho_1, \rho_3) = F(q, r, \rho_1, \rho_3)\,, \tag{5}$$

where $F(q, r, \rho_1, \rho_3)$ is a function which for fixed values of q, r and ρ_3 is an analytic function of ρ_1 for $\text{Re}\,\rho_1 \leqq 0$. The lefthand member of (5) is analytic for $\text{Re}\,\rho_1 \geqq 0$ and both members are continuous for $\text{Re}\,\rho_1 = 0$. Hence both members are analytic continuations of each other (see Titchmarsh, 1944). We introduce the function $H(q, r, \rho_1, \rho_3)$ which equals the lefthand member for $\text{Re}\,\rho_1 \geqq 0$ and the righthand member for $\text{Re}\,\rho_1 \leqq 0$. It is easily seen that $H(q, r, \rho_1, \rho_3)$ is bounded for $|r| < 1$, $|q| < 1$, $\text{Re}\,\rho_3 = 0$ and every ρ_1. According to Liouville's theorem $H(q, r, \rho_1, \rho_3)$ is independent of ρ_1. Consequently from (5) we have

$$\Phi(q, r, \rho_1, \rho_3)K^+(q, r, \rho_1, \rho_3) = \Phi(q, r, 0, \rho_3)K^+(q, r, 0, \rho_3) = \frac{1}{(1 - r)\{1 - q\Gamma(\rho_3)\}}\,.$$

This proves the theorem.

From Theorem 2 we immediately find the distribution of $S_{n,k}^-$ by noting that

$$S_{n,k}^- = -[\max\nolimits_{1 \leqq i \leqq n}^{(n-k+1)} \{-(S_n - S_{n-i})\}]^+\,,$$

i.e. we take $-S_n$ instead of S_n in (2) and $\max^{(n-k+1)}$ instead of $\max^{(k)}$. The result is

$$\sum_{n=0}^{\infty} q^n \sum_{k=1}^{\infty} r^{k-1} E\{\exp(-\rho_2 S_{n,k}^- - \rho_3 S_n)\} = \frac{1}{\{1 - q\Gamma(\rho_3)\}(1 - r)} + \frac{1}{\{1 - qr\Gamma(\rho_3)\}(1 - r)} - \frac{1 - q\Gamma(\rho_2 + \rho_3)}{1 - qr\Gamma(\rho_2 + \rho_3)}\,\Phi(q, r, \rho_2, \rho_3)\,,$$

for $|q| < 1$, $|r| < 1$, $\text{Re}\,\rho_2 \leqq 0$, $\text{Re}\,\rho_3 = 0$. Using the identity: $\exp(-\rho_1 x^+ - \rho_2 x^-) = \exp(-\rho_1 x^+) + \exp(-\rho_2 x^-) - 1$; we can combine the above expression with Theorem 2, so that we find the following extension of the Pollaczek–Wendel identity.

THEOREM 3.

$$\sum_{n=0}^{\infty} q^n \sum_{k=1}^{\infty} r^{k-1} E\{\exp(-\rho_1 S_{n,k}^+ - \rho_2 S_{n,k}^- - \rho_3 S_n)\}$$
$$= \Phi(q, r, \rho_1, \rho_3) - \frac{1 - q\Gamma(\rho_2 + \rho_3)}{1 - qr\Gamma(\rho_2 + \rho_3)} \Phi(q, r, \rho_2, \rho_3)$$
$$= \frac{1}{\{1 - q\Gamma(\rho_3)\}(1 - r)}$$
$$\times \left\{\exp\left[\sum_{n=1}^{\infty} \frac{q^n}{n} (r^n - 1) E\{\exp(-\rho_3 S_n)(1 - \exp(-\rho_1 S_n^+))\}\right]\right.$$
$$\left. - \exp\left[\sum_{n=1}^{\infty} \frac{q^n}{n} (r^n - 1) E\{\exp(-\rho_2 S_n^- - \rho_3 S_n)\}\right]\right\},$$

for Re $\rho_1 \geqq 0$, Re $\rho_2 \leqq 0$, Re $\rho_3 = 0$, $|q| < 1$, $|r| < 1$. *This result was obtained in* 1963 *by Port, using combinatorial methods.*

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