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A NOTE ON CONTINUOUS PARAMETER ZERO-TWO LAW[1]

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Let $\{X_t\}$, $0 \leqq t < \infty$, be a Markov process with state space $(E, \mathscr{E})$. Let m be a σ-finite measure on $(E, \mathscr{E})$ and let the $L_\infty(E, \mathscr{E}, m)$ operator induced by the transition probability $P_t(x, A)$, $x \in E$, $A \in \mathscr{E}$, be conservative and ergodic for all $t > 0$. Let (m) abbreviate m modulo 0. For fixed $\alpha > 0$, set $h^\alpha(x) = \lim_{t\to\infty} ||P_t(x, \cdot) - P_{t+\alpha}(x, \cdot)||$, where $||\cdot||$ is the total variation.

THEOREM. *Either* $h^\alpha(x) = 0\ (m)$ *for* a.e. $\alpha \in \mathbb{R}_+$ *or* $h^\alpha(x) = 2\ (m)$ *for* a.e. $\alpha \in \mathbb{R}_+$. *In particular, if* $\{X_t\}$, $0 \leqq t < \infty$, *is a Markov process satisfying a Harris type recurrence condition, then* $h^\alpha(x) = 0\ (m)$ *for* a.e. $\alpha \in \mathbb{R}_+$.

1. Introduction. In a recent paper Ornstein and Sucheston (1970) proved the following: Let $P(x, A)$ be a Markov transition probability, and assume that there exists a σ-finite measure m such that $m(A) = 0$ implies $P(x, A) = 0$ m-a.e. and $m(A) > 0$ implies $\sum_{k=0}^\infty p^k(x, A) = \infty$ m-a.e. Then the total variation of the measure $P^n(x, \cdot) - P^{(n+1)}(x, \cdot)$ is either m-a.e. 2 for all n or it converges m-a.e. to 0 as $n \to \infty$. Here we obtain an analogous result for continuous parameter Markov processes.

Let $(\mathbb{R}, \mathscr{R}, \mu)$ be the real line with Lebesgue measure. Let $(E, \mathscr{E}, m)$ be a σ-finite measure space. Let $\{X_t\}$, $0 \leqq t < \infty$, be a Markov process on a measure space $(\Omega, \mathscr{F})$ with state space E and let $P_t(x, A)$, $x \in E$, $A \in \mathscr{E}$, be the transition probabilities associated with $\{X_t\}$. Let the notation (m) abbreviate m modulo 0. Assume that for each t, $m(A) = 0$ implies $P_t(x, A) = 0\ (m)$, then the functions $P_t(\cdot, \cdot)$ define positive linear contractions Q_t on $L_1 = L_1(E, \mathscr{E}, m)$ and P_t on $L_\infty = L_\infty(E, \mathscr{E}, m)$. Identifying under the Radon–Nikodym isomorphism L_1 with the space of m-continuous finite signed measures φ on $\mathscr{E}$, we define Q_t and P_t by:

$$Q_t\varphi(A) = \int \varphi(dx)P_t(x, A) \qquad \varphi \in L_1; \tag{1}$$

$$P_t h(x) = \int P_t(x, dy)h(y) \qquad h \in L_\infty. \tag{2}$$

An operator Q on L_1 is called *conservative* and *ergodic* if for each $0 \not\equiv f \in L_1^+$, $\sum_{i=0}^\infty Q^i f = \infty\ (m)$. In a similar manner, an operator P on L_∞ is called *conservative* and *ergodic* if for each $0 \not\equiv h \in L_\infty^+$, $\sum_{i=0}^\infty P^i h = \infty\ (m)$. $P = Q^*$, the adjoint of an L_1-operator Q, is conservative and ergodic if and only if Q is conservative and ergodic (see Ornstein and Sucheston (1970) page 1633). We assume that P_t is conservative and ergodic for all $t > 0$.

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Let $\alpha > 0$ be fixed. P_α is said to satisfy the *Harris Condition* if there exists a σ-finite measure π_α on $(E, \mathscr{E})$ such that for all $A \in \mathscr{E}$, $\pi_\alpha(A) > 0$ implies

(3) $$P_x[\sum_{k=0}^{\infty} 1_{\{X_{k\alpha} \in A\}} = \infty] = 1 \qquad \text{for all} \quad x \in E$$

(see e.g. Harris (1956), Jain (1966)). The probabilistic meaning of the Harris Condition is that starting from any point $x \in E$, with probability one the process $\{X_{k\alpha}\}$, $k = 1, 2, \cdots$, visits an arbitrary set A of positive π_α-measure infinitely many times.

For $t > 0$, $\alpha + t > 0$, define

$$h_t^\alpha(x) = ||P_t(x, \cdot) - P_{t+\alpha}(x, \cdot)|| .$$

Here $|| \ ||$ is the total variation. We show that for each

$$\alpha > 0 \ \lim_{t\to\infty} h_t^\alpha(x) =_{\text{def}} h^\alpha(x)$$

exists for m-a.e. x. Assume that for all $f \in L_\infty$ $P_t f(x)$ is bimeasurable with respect to $(\mathbb{R} \times E, \mathbb{R} \times \mathscr{E}, \mu \times m)$ (here we assume that $\mathbb{R} \times \mathscr{E}$ is complete with respect to $\mu \times m$).

2. Preliminary results and main theorem. Here we prove the following:

THEOREM 1. *Either $h^\alpha = 0$ (m) for almost every $\alpha > 0$ or $h^\alpha = 2$ (m) for almost every $\alpha > 0$.*

LEMMA 1. *For fixed $\alpha > 0$, h_t^α, $0 \leqq t < \infty$, satisfies:*

(a) $0 \leqq h_t^\alpha \leqq 2$ (m) *for all* $t > 0$,
(b) $P_r h_s^\alpha \geqq h_t^\alpha$ (m) *where* $r + s = t$,
(c) $h_t^\alpha \geqq h_s^\alpha$ (m) *where* $t < s$,
(d) $\lim_{t\to\infty} h_t^\alpha = h^\alpha$ (m), *constant*,
(e) $h^\alpha = h^{-\alpha}$ (m) .

PROOF. The proofs (a)—(d) do not differ substantially from the discrete parameter case (see Ornstein and Scheston (1970) and Foguel (1971) page 275).

It is easy to see that $h_t^\alpha(x)$ may be also defined as

$$\sup \{P_t g(x) - P_{t+\alpha} g(x) : -1 \leqq g \leqq 1, g \in L_\infty\}$$

where the supremum is in the L_∞ sense (see e.g. Foguel (1971) page 279).

Now we prove (e). We have

$$\begin{aligned} h_t^\alpha(x) &= ||P_t(x, \cdot) - P_{t+\alpha}(x, \cdot)|| \\ &= ||P_{t+\alpha}(x, \cdot) - P_{t+\alpha-\alpha}(x, \cdot)|| = h_{t+\alpha}^{-\alpha} . \end{aligned}$$

Taking the limit as $t \to \infty$ we obtain $h^\alpha = h^{-\alpha}$ (m).

LEMMA 2. (i) $h^{\alpha+\beta} \leqq h^\alpha + h^\beta$ (m). (ii) $h^{\alpha-\beta} \leqq h^\alpha + h^\beta$ (m).

PROOF OF (i).

$$\begin{aligned} ||P_t(x, \cdot) - P_{t+\alpha+\beta}(x, \cdot)|| \leqq\ & ||P_t(x, \cdot) - P_{t+\alpha}(x, \cdot)|| \\ & + ||P_{t+\alpha}(x, \cdot) - P_{t+\alpha+\beta}(x, \cdot)|| \end{aligned}$$

implies $h_t^{\alpha+\beta}(x) \leqq h_t^\alpha(x) + h_{t+\alpha}^\beta(x)$ for all $x \in E$. Taking the limit as $t \to \infty$ we have $h^{\alpha+\beta} \leqq h^\alpha + h^\beta \ (m)$.

(ii) follows immediately from (i) and Lemma 1 (e) since $h^{\alpha-\beta} \leqq h^\alpha + h^{-\beta} = h^\alpha + h^\beta(m)$.

LEMMA 3. *For each* $\alpha > 0$, $h^\alpha = 0 \ (m)$ *or* $h^\alpha = 2 \ (m)$.

PROOF. Assume that $h^\alpha < 2 \ (m)$. Then $2 > h^\alpha = \lim_{t\to\infty} h_t^\alpha = \lim_{n\to\infty} h_{n\alpha}^\alpha = 0$ (m) by the discrete parameter zero-two law.

For fixed $\alpha > 0$ we observe from Lemma 2 that $h^{n\alpha} \leqq nh^\alpha \ (m)$ for all positive integers n. Then, using Lemma 3, we have that if $h^\sigma = 2 \ (m)$, then $h^{\sigma/n} = 2 \ (m)$ for all positive integers n, and also if $h^\sigma = 0 \ (m)$, then $h^{k\sigma} = 0 \ (m)$ for all positive integers k.

LEMMA 4. *Assume that for every* $f \in L_\infty P_t f(x)$ *is bimeasurable in the product* $(\mathbb{R} \times E, \mathbb{R} \times \mathscr{E}, \mu \times m)$. *Then for* m-a.e. x, $h^\alpha(x)$ *is measurable in* α.

PROOF. For fixed $\alpha, t > 0$ we have

$$h_t^\alpha(x) = \sup\{P_t g(x) - P_{t+\alpha} g(x) : -1 \leqq g \leqq 1, g \in L_\infty\}$$

where the supremum is in the L_∞ sense, $h_t^\alpha(x)$ is measurable in x because we can assume that the supremum is taken over a countable number of g(see e.g. Dunford and Schwartz (1958) page 336). For fixed δ, $t > 0$ we can find a sequence $g_k^{\delta,t}(x)$, $-1 \leqq g_k^{\delta,t} \leqq 1$, such that if

$$f_k^{\delta,t}(\alpha, x) =_{\text{def}} (P_t - P_{t+\alpha}) g_k^{\delta,t}(x) , \qquad \text{then}$$
$$f_k^{\delta,t}(\delta, x) \nearrow h_t^\delta(x) \ (m) \qquad \text{as } k \to \infty .$$

For fixed $\delta, t, k > 0, f_k^{\delta,t}(\alpha, x)$ is bimeasurable in (α, x). Set $e_k^t(\alpha, x) = \sup_\delta f_k^{\delta,t}(\alpha, x)$. Again we may assume that the supremum is over countably many δ, which implies that $e_k^t(\alpha, x)$ is (α, x)-bimeasurable for each integer $k > 0$ and real $t > 0$. For fixed $k, \beta, t > 0$,

$$\begin{aligned} f_k^{\beta,t}(\beta, x) \leqq e_k^t(\beta, x) &= \sup_\delta f_k^{\delta,t}(\beta, x) \\ &= \sup_\delta (P_t - P_{t+\beta}) g_k^{\delta,t}(x) \leqq h_t^\beta(x) \ (m) . \end{aligned}$$

This implies that for fixed $\beta, t > 0$ $e_k^t(\beta, x) \to h_t^\beta(x) \ (m)$ as $k \to \infty$. Hence, for fixed $t > 0$, $h_t(\alpha, x) =_{\text{def}} \lim_{k\to\infty} e_k^t(\alpha, x)$ exists and is bimeasurable in (α, x). Since for every $\alpha, t > 0$ $h_t(\alpha, x) = h_t^\alpha(x) \ (m)$, we may take $h_t(\alpha, x)$ as our version of $h_t^\alpha(x)$. For $s, t > 0$,

$$(P_{t+s} - P_{t+s+\alpha}) g(x) = (P_t - P_{t+\alpha})(P_s g(x)) \leqq h_t^\alpha(x)$$

$\mu \times m$-a.e. since $-1 \leqq P_s g \leqq 1$ if $-1 \leqq g \leqq 1$. Taking the supremum of the left-hand side we have that $h_{t+s}^\alpha(x) \leqq h_t^\alpha(x)$ $\mu \times m$-a.e. Because $h_t^\alpha(x)$ is decreasing in t and bounded below by 0, we have that $h^\alpha(x) = \lim_{t\to\infty} h_t^\alpha(x)$ is bimeasurable in (α, x), hence, for m-a.e. x, $h^\alpha(x)$ is measurable in α.

REMARK. Assume that $\{X_t\}$, $0 \leqq t < \infty$, takes values in a topological measure space $(E, \mathscr{E}', \mathscr{E}, m)$ (i.e. $(E, \mathscr{E}')$ is a Hausdorff space, $\mathscr{E}$ is the σ-algebra generated by $\mathscr{E}'$, and m is a σ-finite measure on $\mathscr{E}$). Also assume that $\{X_t\}$, $0 \leqq t < \infty$, has right continuous paths (i.e. $X_t(\omega)$ is right continuous in t for all $\omega \in \Omega$). We then have that $X_t(\omega)$ is bimeasurable in (t, ω) (see e.g. Meyer (1966) page 70). Then, for $f \in L_\infty$, we have that $f[X_t(\omega)]$ is bimeasurable in (t, ω) which in turn yields that $P_t f(x) = E_x[f[X_t(\omega)]]$ is bimeasurable in (t, x). To be precise, $P_t f(x)$ is bimeasurable in (t, x) with respect to $(\mathbb{R} \times E, \mathscr{R} \times \mathscr{E}, \mu \times m)$ (see e.g. Blumenthal and Getoor (1968) page 41).

Now we prove a continuous parameter version of the zero-two law.

THEOREM 1. *Either* $h^\alpha = 0\ (m)$ *for* μ-a.e. $\alpha > 0$ *or* $h^\alpha = 2\ (m)$ *for* μ-a.e. $\alpha > 0$.

PROOF. Let $A = \{(\alpha, x) : h^\alpha(x) \neq 0 \text{ and } h^\alpha(x) \neq 2\}$. For arbitrary $\alpha > 0$, set $A^\alpha = \{x : (\alpha, x) \in A\}$, and for arbitrary $x \in E$, set $A_x = \{\alpha : (\alpha, x) \in A\}$. From Lemma 3 we have that $m(A^\alpha) = 0$ for all $\alpha > 0$, and hence, by Tonelli's Theorem (see Dunford and Schwartz (1958) page 194), we have $\mu \times m(A) = 0$ and for m-a.e. $x \in E$, $\mu(A_x) = 0$. Thus, for the purpose of our discussion, we may assume that for each fixed $\alpha > 0$, $h^\alpha(x) = 0$ for *all* $x \in E$ or $h^\alpha(x) = 2$ for *all* $x \in E$. Let $B_1 = \{\alpha : h^\alpha = 2\}$ and $B_2 = \{\alpha : h^\alpha = 0\}$. Assume that $\mu(B_1) > 0$ and $\mu(B_2) > 0$. By Lemma 2 $h^{\alpha-\beta} \leqq h^\alpha + h^\beta$ for $\alpha, \beta \in B_2$. Then $\delta \in B_2 - B_2 = \{\alpha - \beta : \alpha, \beta \in B_2\}$ implies $h^\delta = 0$. But by a standard fact of measure theory (see e.g. Hewitt and Stromberg (1965) page 143) $B_2 - B_2$ contains an open interval around the origin, hence there exists an interval $I = (0, c)$ such that $\alpha \in I$ implies $h^\alpha = 0$. By the remark following Lemma 3, we have $B_2 = \mathbb{R}_+$ which is a contradiction. In fact, we proved that $h^\alpha(x) = 0$ $\mu \times m$-a.e. or $h^\alpha(x) = 2$ $\mu \times m$-a.e.

COROLLARY 1. *Let* $\{X_t\}$, $0 \leqq t < \infty$, *satisfy the hypotheses of the theorem and in addition, for each* $\alpha > 0$ *in a set of positive measure, let* $\{X_{k\alpha}\}$, $k = 1, 2, \cdots$, *be recurrent in the sense of Harris. Then* $h^\alpha = 0$ *for* μ-a.e. $\alpha > 0$.

PROOF. From Theorem 1 either $h^\alpha = 0$ or $h^\alpha = 2$ for μ-a.e. $\alpha > 0$. Using the aperiodicity of $\{X_{k\alpha}\}$, $k = 1, 2, \cdots$, and the results of Ornstein and Sucheston ((1970) page 1638), we have $h^\alpha = \lim_{k\to\infty} h^\alpha_{k\alpha} = 0$ for μ-a.e. $\alpha > 0$.

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