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WHICH FUNCTIONS OF STOPPING TIMES ARE STOPPING TIMES?[1]

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Some functions of stopping times are necessarily stopping times, but others need not be. For example, the sum $\tau_1 + \tau_2$ of two stopping times is, while for stochastic processes in continuous time, the product $\tau_1 \cdot \tau_2$ need not be. Determined here for each positive integer n are those functions ϕ for which $\phi(\tau)$ is a stopping time for all n-triples of stopping times $\tau = \tau_1, \cdots, \tau_n$.

1. Introduction. If τ is a stop rule and ϕ maps the set T of positive reals into itself, then $\phi(\tau)$ may or may not be a stop rule. Which are the (measurable) ϕ that transform all stop rules into stop rules? Among them are certainly those for which $\phi(t) \geqq t$ for all t. The only others are those which, for some c, satisfy

$$(1) \qquad \begin{aligned} \phi(t) &\geqq t \quad \text{for} \quad t \leqq c\,, \\ &= c \quad \text{for} \quad t > c\,. \end{aligned}$$

With the convention that c can be ∞, (1) characterizes the (measurable) ϕ that carry stop rules into stop rules. What positive-valued (measurable) functions ψ of two positive variables carry each pair of stop rules into stop rules? Just those ψ for which $\psi(s, t)$ satisfies Condition (1) in s for each t and in t for each s. The principle extends immediately from two to any finite or infinite number of arguments.

This characterization of the ϕ's that transform stop rules for discrete-time stochastic processes into stop rules was jointly reported in [1] (Section 2.9). This note reports that the same characterization holds for strict stop rules when the processes have a time-parameter that is continuous. Slight variants of the characterization hold when wide-sense stop rules are admitted.

2. Notation. $\mathscr{F}^* = [\mathscr{F}_t, 0 \leqq t < \infty]$ is an increasing family of σ-fields of subsets of a set U, and τ is a *stop rule* for $\mathscr{F}^*$, that is, a map of U into the non-negative reals such that $(\tau \leqq t) \in \mathscr{F}_t$ for each $t \geqq 0$.

LEMMA 1. *For each* $t \geqq 0$, *each Borel subset* B *of the closed interval* $[0, t]$, *and each stop rule* τ *for* $[\mathscr{F}_t, 0 \leqq t < \infty]$,

$$(2) \qquad \tau^{-1}(B) \in \mathscr{F}_t\,.$$

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PROOF. Verify: (i) If $B = [0, s]$ for an $s \leqq t$, then (2) holds; (ii) The set of B for which (2) holds is closed under countable unions; and (iii) If (2) holds for $B \subset [0, t]$, then it holds for the complement of B in [0, t]. This implies that (2) holds for all Borel $B \subset [0, t]$.

THEOREM 1. *A necessary and sufficient condition on a Borel-measurable, real-valued function* ϕ *of a nonnegative real variable that, for every increasing family of sigma fields,* $\mathscr{F}^* = [\mathscr{F}_t : 0 \leqq t < \infty]$, *and every stop rule* τ *for* $\mathscr{F}^*$, $\phi \circ \tau$ *is a stop rule for* $\mathscr{F}^*$ *is that, for some positive c, possibly infinite,* ϕ *satisfies Condition* (1).

PROOF. For the sufficiency, what must be verified is that, for each t,

$$(\phi \circ \tau \leqq t) \in \mathscr{F}_t . \tag{3}$$

The left-hand side of (3) is $\tau^{-1}(B)$ where $B = \phi^{-1}[0, t]$ is plainly a Borel set. Suppose first that $t < c$. Then $B \subset [0, t]$, and Lemma 1 yields the desired conclusion. If $t \geqq c$, then $B \supset (c, \infty)$, which implies that

$$B = B^* \cup (c, \infty) , \tag{4}$$

where B^*, being $B \cap [0, c]$, satisfies

$$B^* \subset [0, t] . \tag{5}$$

In view of (4), $\tau^{-1}(B)$ is the union of $\tau^{-1}(B^*)$ with $(\tau > c)$. Since $\tau^{-1}(B^*)$ is in $\mathscr{F}_t$, according to (5) and Lemma 1, and $(\tau > c)$, being in $\mathscr{F}_c$, is certainly in $\mathscr{F}_t$, so is their union, namely $\tau^{-1}(B)$.

If ϕ does not satisfy the condition, then there is a stop rule τ for a $\mathscr{F}^* = [\mathscr{F}_t : 0 \leqq t < \infty]$ where each $\mathscr{F}_t$ is either the trivial field or else the field of all subsets of a two-element set U, that is, a set of cardinality 2. For if ϕ does not satisfy the condition, then there exist t_0, t_1 such that $\phi(t_0) < t_0$, $\phi(t_1) \neq \phi(t_0)$ and $t_1 > \phi(t_0)$. For U, one may take, for example, two paths w_0 and w_1 which agree until time $\phi(t_0)$, but not thereafter, and for $\mathscr{F}_t$ the trivial, or the universal, field according as $t \leqq \phi(t_0)$ or $t > \phi(t_0)$. As is easily verified, if $\tau(w_i) = t_i$, then τ is a stop rule for $\mathscr{F}^*$, but $\phi(\tau)$ is not.

Turn now to the problem of determining those positive-value measurable functions φ of two positive variables that carry each pair of stop rules into a stop rule. For a necessary condition on φ let $\tau_1 \equiv s$ be a constant stop rule. Then $\varphi(s, \tau)$ must be a stop rule for each stop rule τ. Hence, according to Theorem (1), for each s, $\varphi(s, t)$ must satisfy (1) in t. Similarly, for each t, $\varphi(s, t)$ satisfies (1) in s. To see that this condition is sufficient, two preliminary lemmas are needed.

LEMMA 2. *Let* $\tau = (\tau_1, \tau_2)$ *be a pair of stop rules for an increasing family of sigma fields,* $[\mathscr{F}_s : 0 \leqq s < \infty]$, *t a positive number, I the closed interval* $[0, t]$, *and B a Borel subset of the square* $I \times I$. *Then*

$$\tau^{-1}(B) \in \mathscr{F}_t . \tag{6}$$

PROOF. According to Lemma 1, if B is the Cartesian product of two Borel subsets B_i of I, then

$$\tau^{-1}(B) = \tau_1^{-1}(B_1) \cap \tau_2^{-1}(B_2) \in \mathscr{F}_t ; \tag{7}$$

and since the set of B for which (6) holds is a σ-field, the proof is evident.

LEMMA 3. *Let φ satisfy* (1) *in s for each t and in t for each s, and let r be a positive number. Let $A = [(s, t) : \varphi(s, t) \leqq r]$ and*

$$\begin{aligned} A_1 &= A \cap [s \leqq r \text{ and } t \leqq r]; & A_2 &= A \cap [s \leqq r < t]; \\ A_3 &= A \cap [t \leqq r < s]; & A_4 &= A \cap [r < s \text{ and } r < t]. \end{aligned} \tag{8}$$

Then $A = A_1 \cup A_2 \cup A_3 \cup A_4$, and: (i) *$A_1$ is a subset of $I \times I$ where $I = [0, r]$;* (ii) *$A_2 = \alpha \times (r, \infty)$ for some subset α of $[0, r]$;* (iii) *$A_3 = (r, \infty) \times \alpha$ for some subset α of $[0, r]$;* (iv) *A_4 is either empty or equal to $(r, \infty) \times (r, \infty)$. Moreover, if φ is Borel, so is each A_i and α.*

PROOF. Plainly, (i) is trivial. Suppose that (s, t) is in A_2. To verify (ii), it is only necessary to check that for the same s and $t' > r$, (s, t') is in A_2. By hypothesis, $\varphi(s, t) \leqq r$, $s \leqq r < t$. Since for this s, φ satisfies (1) in t, $\varphi(s, t) = c = \varphi(s, t')$ for all $t' > c$, and *a fortiori*, for all $t' > r$. Hence, $\varphi(s, t') = c \leqq r$ for all $t' > r$, which implies that each such (s, t') is in A_2. The proof of (iii) is obtained from the proof of (ii) by interchanging the roles of s and t. Since the proof of (iv) is quite similar, it need not be given. It is trivial that each A_i is Borel if φ is Borel. To see that each α is Borel, recall that the measurability of a rectangle implies the measurability of each of its sides (e.g., see [2] Section 32, Problem 4).

LEMMA 4. *The condition that φ satisfy* (1) *in s for each t and in t for each s is sufficient for $\varphi \circ \tau$ to be a stop rule whenever $\tau = (\tau_1, \tau_2)$ is a pair of stop rules.*

PROOF. What must be seen is that for each positive r, $[\varphi \circ \tau \leqq r] = \tau^{-1}(A) \in \mathscr{F}_r$, where A is as in Lemma 3. In the notation of Lemma 3, $\tau^{-1}(A) = \bigcup \tau^{-1}(A_i)$. Hence, it is only necessary to show that for each i, $\tau^{-1}(A_i) \in \mathscr{F}_r$. Since, A_1 is a Borel subset of the square $[0, r] \times [0, r]$, Lemma (2) applies. Since, by Lemma 3, $A_2 = \alpha \times (r, \infty)$, where α is a Borel subset of $[0, r]$, $\tau^{-1}(A_2) = \tau_1^{-1}(\alpha) \cap \tau_2^{-1}(r, \infty)$. As Lemma (1) implies that $\tau_1^{-1}(\alpha) \in \mathscr{F}_r$, and since $\tau_2^{-1}(r, \infty) \in \mathscr{F}_r$ for any stop rule τ_2, their intersection, namely $\tau^{-1}(A_2)$, is in $\mathscr{F}_r$. The proofs that $\tau^{-1}(A_3)$ and $\tau^{-1}(A_4)$ are in $\mathscr{F}_r$ are quite analogous.

As is easily checked, the argument given for functions φ of two variables applies to functions of any finite or denumerable number of variables.

3. A variation. A very similar query to the one answered by Theorem 1 is to ask which Borel ϕ transform all wide-sense stop rules into (strict sense) stop rules? As usual, τ is a wide sense stop rule for $[\mathscr{F}_t : 0 \leqq t < \infty]$ if, for each t, $(\tau \leqq t)$ is in $\mathscr{F}_{t+}$ where $\mathscr{F}_{t+} = \bigcap_{s>t} \mathscr{F}_s$. Among these ϕ are certainly those for which $\phi(t) > t$ for all t. Now the only others are those which, for some c, satisfy

$$\phi(t) > t \quad \text{for} \quad t < c \qquad (9)$$
$$= c \quad \text{for} \quad t \geqq c\,.$$

Plainly, if there is a critical finite c, then $\phi(c)$ must be c. This contrasts with Condition (1) where $\phi(c)$ could exceed c.

As before, a Borel function of n variables, where n may be the smallest infinite cardinal, transforms every n-tuple of wide-sense stop rules into a stop rule if, and only if, φ satisfies Condition (9) in each variable separately. There is no need to give the arguments, for they are but slight variants of those given above.

REFERENCES

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