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ON CONVERGENCE IN PROBABILITY TO BROWNIAN MOTION

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Several methods for embedding discrete martingales in Brownian motion by means of stopping times have been presented. Various conditions on the increments of the martingales are sufficient to insure that the trajectories of the embedded process and the Brownian motion are close. This paper will characterize all discrete stochastic processes, which can be constructed on some probability space supporting a Brownian motion, in such a way that the constructed process and the Brownian motion are close in probability, under suitable normalization. These are exactly the processes $\{S_j \mid j = 0, 1, \cdots\}$ such that, for any $\varepsilon > 0$ and $M = 0, 1, \cdots$ the conditional probability that the $(M + 1)$st change in size of at least 1 completed by the trajectory $0, S_1/n, S_2/n, \cdots$ is in $[1, 1 + \varepsilon]$ (or $[-(1 + \varepsilon), -1]$), given the process up to the Mth such change, converges in probability to $\frac{1}{2}$ as $n \to \infty$. It is not required that any moments exist, or even that $E(S_{j+1} \mid S_1, \cdots, S_j) = S_j$, a.s. In proving the main result, a new technique for constructing discrete processes from Brownian motion is presented.

1. Introduction. Several methods for embedding discrete martingales in Brownian motion by means of stopping times have been presented. (For example, see [2], [5], and [6].) Various conditions on the increments of the martingales are sufficient to insure that the trajectories of the embedded process and the Brownian motion are close ([3], [7], [8]). This paper will characterize all discrete stochastic processes, which can be constructed on some probability space supporting a Brownian motion, in such a way that the constructed process and the Brownian motion are close in probability, under suitable normalization. These are exactly the processes $\{S_j \mid j = 1, 2, \cdots\}$ such that

(1) for any $\varepsilon > 0$ and $M = 0, 1, 2, \cdots$, the conditional probability that the $(M + 1)$st change in size of at least 1 completed by the trajectory $0, S_1/n, S_2/n, \cdots$ is in $[1, 1 + \varepsilon]$ (or $[-(1 + \varepsilon), -1]$), given the process up to the Mth such change, converges in probability to $\frac{1}{2}$ as $n \to \infty$.

It is not required that any moments exist, or even that $E(S_{j+1} \mid S_1, \cdots, S_j) = S_j$.

In proving this result a new technique for constructing discrete processes from Brownian motion is presented. Roughly speaking, any process satisfying (1) may be approximated by a martingale whose increments have a 2 point, mean 0 distribution, conditionally upon the past. This martingale can easily be embedded in a Brownian motion by the usual hitting times. Then, a process with the same distribution as the given one is extrapolated from the embedded martingale, by

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auxiliary randomization. In general, this method will not construct each increment of the given process strictly in succession. However, it does construct the process in a forward manner, by constructing successive groups of increments in which the process does not change much. (See (4e) and (4f) for a formal statement.)

The main result in this paper is an analogue to (4) in [1], which deals with weak convergence. Here it was shown that, for any process $\{S_j \mid j = 1, 2, \cdots\}$

(2) there exists a time change $\{T_t \mid 0 \leqq t < \infty\}$ such that the distribution of $\{S_{T_{nt}}/n^{\frac{1}{2}} \mid 0 \leqq t \leqq 1\}$ converges weakly to Brownian motion (on [0, 1]) as $n \to \infty$

if and only if

(3) for any M, the distribution of the first M changes of at least 1 completed by the trajectory $0, S_1/n, S_2/n, \cdots$ converges weakly to the distribution of M i.i.d., fair, ± 1 random variables as $n \to \infty$.

It follows that, when $S_1, S_2 - S_1, \cdots$ are independent, weak convergence of the suitably normalized trajectories to Brownian motion is equivalent to their convergence in probability (possibly on another space).

Assuming $\{S_j \mid j = 1, 2, \cdots\}$ is a martingale, (15) specifies a Lindeberg type growth condition on the increments which is necessary and sufficient to insure that the suitably normalized trajectories of $\{S_j \mid j = 1, 2, \cdots\}$ converge in L_2-norm to a Brownian motion. In this case the process may be normalized in the usual way, involving conditional variances of increments given the past.

2. Main result. Formally, let $\{S_j \mid j = 0, 1, 2, \cdots\}$ be a process on a probability space $(\Omega, \mathscr{A}, P)$. Always assume $S_0 \equiv 0$. Use $\to_{\mathrm{P}}$ to denote convergence in probability, and take 1_A to be the indicator of the set A. This paper will consider when

(4) there exists a probability space $(\Omega', \mathscr{A}', P')$ which supports processes $\{B_t \mid 0 \leqq t < \infty\}$, $\{S_j' \mid j = 0, 1, 2, \cdots\}$, $\{\tau_k \mid k = 1, 2, \cdots\}$, and $\{T_t \mid 0 \leqq t < \infty\}$, with an increasing family $\{\mathscr{A}_t' \mid 0 \leqq t \leqq \infty\}$ of sub-σ-fields of $\mathscr{A}'$ such that

(a) the P-distribution of $(S_0, S_1, \cdots)$ equals the P'-distribution of $(S_0', S_1', \cdots)$,

(b) $\{B_t \mid 0 \leqq t < \infty\}$ is Brownian motion, B_t is $\mathscr{A}_t'$-measurable, and $\{B_{t+s} - B_t \mid 0 \leqq s < \infty\}$ is independent of $\mathscr{A}_t'$ for $0 \leqq t < \infty$,

(c) $\tau_1 \leqq \tau_2 \leqq \tau_3 \leqq \cdots$ are stopping times with respect to $\{\mathscr{A}_t' \mid 0 \leqq t < \infty\}$,

(d) $T_0 \equiv 0$ and $\{T_t \mid 0 \leqq t < \infty\}$ is a right continuous, non-decreasing, unbounded step function, having jumps of size 1, P'-a.s.,

(e) for $k = 0, 1, \cdots, S_0', S_1', \cdots, S'_{T_{\tau_k}}$ is measurable with respect to $\mathscr{A}'_{\tau_k}$, where $\mathscr{A}'_{\tau_k} = \{A \in \mathscr{A}' \mid A \cap [\tau_k \leqq t] \in \mathscr{A}_t', 0 \leqq t < \infty\}$,

(f) $\sup_{k,j} \{|S_j' - S'_{T_{\tau_k}}|/n^{\frac{1}{2}} : T_{\tau_{k-1}} \leqq j \leqq T_{\tau_k}, \tau_k \leqq n\} \to_{\mathrm{P}} 0$, as $n \to \infty$, and

(g) $\sup_{0 \leqq t \leqq 1} |S'_{T_{nt}} - B_{nt}|/n^{\frac{1}{2}} \to_{\mathrm{P}} 0$, as $n \to \infty$.

To state the main result, let Γ be the set of functions $y\colon \{0, 1, \cdots\} \to (-\infty, \infty)$ for which $y(0) = 0$. For $y \in \Gamma$ and $j \geqq 0$, identify y_j with $y(j)$, and define

$\sigma(0, m, y) = 0$, let $\sigma(k, m, y)$ be the first time j after $\sigma(k-1, m, y)$ for which $|y_j - y_{\sigma(k-1,m,y)}| \geqq 2^{-m}$, and $\sigma(k, m, y) = \sigma(k-1, m, y)$ when no such j exists, for $k \geqq 1$. For unbounded y, think of $\sigma(k, m, y)$ as the first time y completes k changes in size of at least 2^{-m}. Abbreviate $y_{\sigma(k,m,y)}$ by $y_{\sigma(k,m)}$.

For each $n = 1, 2, \cdots$ and $j = 0, 1, \cdots$, define $Z_j^n = S_j/n^{\frac{1}{2}}$. Let $Z_\cdot^n$, or simply Z^n, denote the trajectory of $\{Z_j^n \mid j = 0, 1, \cdots\}$. For each $\omega \in \Omega$, $Z^n(\omega) \in \Gamma$. Hence, for each m, n, k $Z^n_{\sigma(k,m)}$ is a random variable whose value at ω is $Z^n_{\sigma(k,m,Z^n(\omega))}(\omega)$.

Denote by $\mathscr{F}_j$ the σ-field generated by $S_0, S_1, \cdots, S_j$. If τ is a stopping time, let $\mathscr{F}_\tau = \{A \in \mathscr{A} \mid A \cap [\tau \leqq j] \in \mathscr{F}_j, j = 0, 1, \cdots\}$. For any set $A \in \mathscr{A}$, let $P(A \mid \mathscr{F}_\tau)$ denote a version of the conditional probability of A given $\mathscr{F}_\tau$.

Say $\{S_j \mid j = 0, 1, \cdots\}$ is strongly asymptotically fair if

(5) for all $M \geqq 0$ and $\varepsilon > 0$, both

$$P(Z^n_{\sigma(M+1,0)} - Z^n_{\sigma(M,0)} \in [1, 1+\varepsilon] \mid \mathscr{F}_{\sigma(M,0,Z^n)}) \to_P \tfrac{1}{2} \quad \text{and}$$
$$P(Z^n_{\sigma(M+1,0)} - Z^n_{\sigma(M,0)} \in [-(1+\varepsilon), -1] \mid \mathscr{F}_{\sigma(M,0,Z^n)}) \to_P \tfrac{1}{2},$$

as $n \to \infty$.

The main result is

(6) THEOREM 1. (4) *and* (5) *are equivalent.*

The proof of (5) implies (4) will be given in Section 3, and that of (4) implies (5) in Section 4.

3. The construction. Suppose (5) holds, we prove (4). Let $(\Omega', \mathscr{A}', P')$ be a probability space which supports a Brownian motion $\{B_t \mid 0 \leqq t < \infty\}$, and a sequence of random variables $\xi_1, \xi_2, \cdots$ which are i.i.d. uniform on $[0, 1]$. $\{S_j' \mid j = 0, 1, \cdots\}$ will be constructed on this space.

It will be convenient to look at $\{Z_k^{4^n} \mid k = 0, 1, \cdots\}$. Accordingly, in order to simplify notation, define $Y_k^n = Z_k^{4^n} = S_k/2^n$, for $k, n = 0, 1, \cdots$.

Choose a sequence of positive numbers $\{\varepsilon_m \mid m = 0, 1, \cdots\}$ such that $4^m \varepsilon_m \to 0$ as $m \to \infty$. By (5), we may choose a sequence of positive integers $\{n(m) \mid m = 0, 1, \cdots\}$ such that

(7) $0 = n(0)$, $n(m) < n(m+1)$, $n(m) \to \infty$ as $m \to \infty$, and for each $m \geqq 1$

$$P[\textstyle\sum_{M=0}^{4^m} |P[Y^n_{\sigma(M+1,0)} - Y^n_{\sigma(M,0)} \in [1, 1+\varepsilon_m] \mid \mathscr{F}_{\sigma(M,0,Y^n)}] - \tfrac{1}{2}| \leqq \varepsilon_m] \geqq 1 - \varepsilon_m$$

and similarly for $[-(1+\varepsilon_m), -1]$, whenever $n \geqq n(m)$. From now on, unless otherwise stated, we assume m and n satisfy $n(m) \leqq n < n(m+1)$. So, m will be understood as a function of n.

Define $T(0) = \sigma(1, 0, Y^0) = \sigma(1, 0, S)$. For $n = 1, 2, \cdots$ let

$$T(n) = \sigma(4^m, m, Y^n) \vee T(n-1),$$

where $a \vee b$ and $a \wedge b$ denote the maximum and minimum of a and b. Observe that $T(n) \leqq T(n+1)$. As a consequence of (7), it turns out that $P[T(n) = \sigma(4^m, m, Y^n)] \to 1$ as $n \to \infty$. Think of $T(n)$ in these terms.

For each $n = 1, 2, \cdots$ define $M(n) = \sup\{k \mid \sigma(k, m, Y^n) \leqq T(n-1)\}$, the number of changes in size of at least 2^{-m} completed by $\{Y_j{}^n \mid j = 0, 1, \cdots\}$ up to time $T(n-1)$. Observe that for each n, $M(n)$ is $\mathscr{F}_{T(n-1)}$-measurable. For $k = 0, 1, \cdots, 4^m$ define $T(k, \text{n})$ as follows. Set $T(0, 0) = 0$, $T(1, 0) = \sigma(1, 0, Y^0) = T(0)$, and $T(k, n) = \sigma(k, m, Y^n) \vee T(n-1)$ for $n \geqq 1$. Check that

$$T(n-1) = T(0, n) \leqq T(k, n) \leqq T(k+1, n) \leqq T(4^m, n) = T(n) \tag{8}$$

for any n and $k < 4^m$, and

$$\sigma(k, m, Y^n) = T(k, n) \qquad \text{if} \quad M(n) + 1 \leqq k \leqq 4^m .$$

We now motivate the definition of a process $\{\tilde{S}_j \mid j = 0, 1, \cdots\}$, which will approximate $\{S_j \mid j = 0, 1, \cdots\}$ in a nice way. Recall that $M(n)$ is $\mathscr{F}_{T(k,n)}$-measurable, for $k = 0, 1, \cdots, 4^m$, by (8). First, notice

$$\max_{T(k-1,n) \leqq j \leqq T(k,n)} |S_j - S_{T(k-1,n)}| \leqq |S_{T(k,n)} - S_{T(k-1,n)}| .$$

Secondly, observe that the regular conditional P-distribution of the random variable $(S_{T(k,n)} - S_{T(k-1,n)})/2^n = Y^n_{T(k,n)} - Y^n_{T(k-1,n)}$, given $\mathscr{F}_{T(k-1,n)}$, is concentrated on 0 if $1 \leqq k \leqq M(n) \wedge 4^m$, and is nearly concentrated on $\pm 2^{-m}$ with equal probabilities if $M(n) + 2 \leqq k \leqq 4^m$, for large n, by (7), (The case when $k = M(n) + 1 \leqq 4^m$ is awkward, but irrelevant when n is large.) So, for each n, $\{S_{T(k,n)} \mid k = 0, 1, \cdots, 4^m\}$ is almost a nice martingale. This suggests defining $\{\tilde{S}_j \mid j = 0, 1, \cdots\}$ so that $(\tilde{S}_{T(k,n)} - \tilde{S}_{T(k-1,n)})/2^n$ is approximately $(S_{T(k,n)} - S_{T(k-1,n)})/2^n$, $\tilde{S}_j = \tilde{S}_{T(k-1,n)}$ if $T(k-1, n) \leqq j < T(k, n)$, and $\{\tilde{S}_{T(k,n)} \mid k = 0, \cdots, 4^m\}$ is a martingale.

With this in mind, we define two triangular arrays of conditional probabilities (except for a factor of 2) $x(k, n, +)$ and $x(k, n, -)$, where $n = 0, 1, \cdots$ and $k = 0, 1, \cdots, 4^m$. (Remember $n(m) \leqq n < n(m+1)$.) Let $x(0, 0, +) = x(0, 0, -) = 0$, $x(1, 0, +) = 2P[Y^0_{\sigma(1,0)} \leqq -1]$, and $x(1, 0, -) = 2P[Y^0_{\sigma(1,0)} \geqq 1]$. When $n \geqq 1$, let

$$x(k, n, +) = x(k, n, -) = 0 \qquad \text{if} \quad 0 \leqq k \leqq M(n) \wedge 4^m .$$

Let

$$x(k, n, +) = 2P[Y^n_{T(k,n)} - Y^n_{T(n-1)} \leqq 2^m - (Y^n_{T(n-1)} - Y^n_{\sigma(M(n),m)}) \mid \mathscr{F}_{T(n-1)}]$$

and

$$x(k, n, -) = 2P[Y^n_{T(k,n)} - Y^n_{T(n-1)} \geqq 2^{-m} - (Y^n_{T(n-1)} - Y^n_{\sigma(M(n),m)}) \mid \mathscr{F}_{T(n-1)}]$$

if $M(n) + 1 = k \leqq 4^m$. Let

$$x(k, n, +) = 2P[Y^n_{T(k,m)} - Y^n_{T(k-1,m)} \leqq -2^{-m} \mid \mathscr{F}_{T(k-1,m)}]$$

and

$$x(k, n, -) = 2P[Y^n_{T(k,m)} - Y^n_{T(k-1,m)} \geqq 2^{-m} \mid \mathscr{F}_{T(k-1,m)}]$$

if $M(n) + 2 \leqq k \leqq 4^m$. These will be used to define a triangular array of random variables $W(k, n)$ as follows. Set

$$W(0, 0) = 0 , \qquad W(1, 0) = x(1, 0, +)1_{[Y^0_{\sigma(1,0)} \geqq +1]} - x(1, 0, -)1_{[Y^0_{\sigma(1,0)} \leqq -1]} .$$

For $n = 1, 2, \cdots$ set $W(k, n) = 0$ if $0 \leqq k \leqq M(n)$, and set

$$W(k, n) = 2^{-m}x(k, n, +)1_{[Y^n_{T(k,n)} - Y^n_{T(n-1)} > 0]} - 2^{-m}x(k, n, -)1_{[Y^n_{T(k,n)} - Y^n_{T(n-1)} < 0]}, \quad \text{if } M(n) + 1 = k \leqq 4^m .$$

Let

$$W(k, n) = 2^{-m}x(k, n, +)1_{[Y^n_{T(k,n)} - Y^n_{T(k-1,n)} \geqq 2^{-m}]} - 2^{-m}x(k, n, -)1_{[Y^n_{T(k,n)} - Y^n_{T(k-1,n)} \leqq -2^{-m}]}, \quad \text{if } M(n) + 2 \leqq k \leqq 4^m .$$

Set $\tilde{S}_0 = 0$, and $\tilde{S}_{T(0)} = \tilde{S}_{T(1,0)} = W(1, 0)$. For $n = 1, 2, \cdots$ let $\tilde{S}_{T(k,n)} = \tilde{S}_{T(n-1)} + 2^n \sum_{j=0}^{k} W(j, n)$, if $0 \leqq k \leqq 4^m$. By (8), for each $\omega \in \Omega$, this defines a function $\tilde{S}.(\omega)$ on some subset of $\{0, 1, 2, \cdots\}$, say G. (G depends on ω.) For $j \notin G$, let $\tilde{S}_j = \tilde{S}_{\bar{j}}$, where $\bar{j} = \sup\{k : k \in G, k \leqq j\}$. Some useful properties of $\{\tilde{S}_j \mid j = 0, 1, \cdots\}$ are contained in

(9) LEMMA.

(a) $E[\tilde{S}_{T(k+1,n)} \mid \mathscr{F}_{T(k,n)}] = \tilde{S}_{T(k,n)}$ a.s., *for* $k = 0, 1, \cdots, 4^m - 1$.

(b) *If* $k < M(n) \leqq 4^m$, *then* $\tilde{S}_{T(k+1,n)} - \tilde{S}_{T(k,n)} = 0$, *and if* $M(n) < k < 4^m$, *then the regular conditional P-distribution of* $\tilde{S}_{T(k+1,n)} - \tilde{S}_{T(k,n)}$, *given* $\mathscr{F}_{T(k,n)}$, *is concentrated on* 2 *points* a.s.

(c) $|\tilde{S}_{T(k+1,n)} - \tilde{S}_{T(k,n)}| \leqq 2^{-m+1}2^n$, *for* $k = 0, 1, \cdots, 4^m - 1$.

(d) $\sup_{0 \leqq j \leqq T(4^m,n)} |S_j - \tilde{S}_j|/2^n \to_P 0$ *as* $n \to \infty$.

We now proceed to construct the process $\{S_j' \mid j = 0, 1, \cdots\}$ on $(\Omega', \mathscr{A}', P')$. Assume that $\{\xi_{k,n} \mid k = 0, 1, \cdots, 4^m; n = 0, 1, \cdots\}$ is a triangular array of independent, identically distributed random variables on $(\Omega', \mathscr{A}', P')$ which are uniform on $[0, 1]$. These will be needed for auxiliary randomization. The construction will be done in stages.

First stage. Here we construct a process with the same distribution as $S_0, S_1, \cdots, S_{T(1,0)}$. Let $\tau(0, 0) = 0$, $\tau(1, 0) = \inf\{t \mid B_t = 2P[S_{\sigma(1,0)} \leqq -1]$ or $B_t = -2P[S_{\sigma(1,0)} \geqq 1]\}$, and set $W'(1, 0) = B_{\tau(1,0)}$. Let $F'(1, 0, +)$ and $F'(1, 0, -)$ denote the regular conditional cumulative distribution functions of $S_{\sigma(1,0)}$, given $[S_{\sigma(1,0)} \geqq 1]$, and of $S_{\sigma(1,0)}$, given $[S_{\sigma(1,0)} \leqq -1]$, respectively. These are (nonrandom) functions on $(-\infty, \infty)$. Let $F'(1, 0, +)^{-1}$ and $F'(1, 0, -)^{-1}$ denote their left continuous inverses, respectively. Define

$$S'_{T'(1,0)} = F'(1, 0, +)^{-1}(\xi_{1,0})1_{[W'(1,0) \geqq 0]} + F'(1, 0, -)^{-1}(\xi_{1,0})1_{[W'(1,0) < 0]} .$$

Clearly, the P'-distribution of $S'_{T'(1,0)}$ equals the P-distribution of $S_{T(1,0)}$. Now, generate $S_0', S_1', \cdots, S'_{T'(1,0)-1}$ in such a way as to have the P'-distribution of $(S_0', \cdots, S'_{T'(1,0)})$ equal the P-distribution of $(S_0, \cdots, S_{T(1,0)})$. This may be done by using the uniform random variables (defined on $(\Omega', \mathscr{A}', P')$) and the appropriate regular conditional cumulative distribution functions.

*n*th *stage.* For $n \geqq 1$, assume $S_0', \cdots, S'_{T'(n-1)}$ and $\tau(4^m, n-1) = \tau(n-1)$ are given, with the P'-distribution of $(S_0', \cdots, S'_{T'(n-1)})$ equal to the P-distribution of $(S_0, \cdots, S_{T(n-1)})$. We will now construct $S'_{T'(0,n)}, \cdots, S'_{T'(4^m,n)}$. Like $M(n)$, define $M'(n)$ to be the number of changes in size of at least 2^{-m} completed by $S_0'/2^n, S_1'/2^n, \cdots, S'_{T'(n-1)}/2^n$.

If $M'(n) \geqq 4^m$, then set $T'(n) = T(n-1)$, $\tau(n-1) = \tau(0, n) = \cdots = \tau(4^m, n) = \tau(n)$, and go to the $(n+1)$st stage.

Otherwise, $M'(n) \leqq 4^m - 1$. For $k = 1, \cdots, 4^m$, suppose $S_0', \cdots, S'_{T'(n-1)}, \cdots, S'_{T'(k-1,n)}$, and $\tau(k-1, n)$ are given ($\tau(0, n) = \tau(n-1)$). Define $x'(k, n, +)$ and $x'(k, n, -)$ as functions of $S_0', \cdots, S'_{T'(k-1,n)}$ just like $x(k, n, +)$ and $x(k, n, -)$ were computed from $S_0, \cdots, S_{T(k-1,n)}$ Let

$$\tau(k, n) = \inf\{t \mid t \geqq \tau(k-1, n), (B_t - B_{\tau(k-1,n)})/2^n = 2^{-m}x'(k, n, +) \text{ or } = -2^{-m}x'(k, n, -)\},$$

and set

$$W'(k, n) = (B_{\tau(k,n)} - B_{\tau(k-1,n)})/2^n .$$

Now, let $F(k, n, +)$ and $F(k, n, -)$ denote the regular conditional cumulative distribution functions of $S_{T(k,n)} - S_{T(k-1,n)}$, given $S_0, \cdots, S_{T(k-1,n)}$ and $S_{T(k,n)} - S_{T(k-1,n)} \geqq 0$, and of $S_{T(k,n)} - S_{T(k-1,n)}$, given $S_0, \cdots, S_{T(k-1,n)}$ and $S_{T(k,n)} - S_{T(k-1,n)} \leqq 0$, respectively. These are functions of $S_0, \cdots, S_{T(k-1,n)}$, and $\operatorname{sgn}(S_{T(k,n)} - S_{T(k-1,n)})$. In the same way, define $F'(k, n, +)$ and $F'(k, n, -)$ from $S_0', \cdots, S'_{T'(k-1,n)}$, and $\operatorname{sgn} W'(k, n)$. Let $F'(k, n, +)^{-1}$ and $F'(k, n, -)^{-1}$ denote their left continuous inverses, respectively. Finally, set

$$S'_{T'(k,n)} - S'_{T'(k-1,n)} = F'(k, n, +)^{-1}(\xi_{k,n})1_{[W'(k,n)>0]} + F'(k, n, -)^{-1}(\xi_{k,n})1_{[W'(k,n)<0]} .$$

Generate $S'_{T'(k-1,n)+1}, \cdots, S'_{T'(k,n)-1}$ in such a way that the P'-distribution of $S_0', \cdots, S'_{T'(k,n)}$ equals the P-distribution of $S_0, \cdots, S_{T(k,n)}$. When this procedure is carried out for $k = 1, \cdots, 4^m$, go to the $(n+1)$st stage.

This method constructs a process $\{S_j' \mid j = 0, 1, \cdots\}$ satisfying (4a). Define $\mathscr{A}_t'$ to be the smallest σ-field with respect to which $\{B_s \mid 0 \leqq s \leqq t\}$ and all the uniform random variables are measurable. Then compute $\{T'(k, n) \mid k, n\}$ from $\{S_j' \mid j = 0, 1, \cdots\}$ just like $\{T(k, n) \mid k, n\}$ is computed from $\{S_j \mid j = 0, 1, \cdots\}$. To define $\{T_t \mid 0 \leqq t < \infty\}$ satisfying (4d) let $T_{\tau(k,n)} = T'(k, n)$, for $n = 0, 1, \cdots$ and $k = 0, 1, \cdots, 4^m$. Complete $\{T_t \mid 0 \leqq t < \infty\}$ so that it has jumps of size 1, is right continuous, and is non-decreasing. Since $\{\{\tau(k, n) \mid k = 0, 1, \cdots, 4^m\} \mid n = 0, 1, \cdots\}$ forms a non-decreasing sequence of stopping times with respect to $\{\mathscr{A}_t' \mid 0 \leqq t \leqq \infty\}$, we may relabel them as $\tau_1, \tau_2, \cdots$, which obviously satisfy (4c). By construction, (4e) is fulfilled. In view of (9), in order to verify (4f) and (4g), it suffices to show

(10) LEMMA. *$\tau(4^m, n)/4^n \to_P 1$ as $n \to \infty$.*

4. The converse. Suppose (4) holds, we prove (5). Define $Z_k'^n = S_k'/n^{\frac{1}{2}}$, and let $\mathscr{F}_{\sigma(M,0,Z'^n)}$ be the σ-field generated by $Z_0'^n, Z_1'^n, \cdots, Z'^n_{\sigma(M,0)}$. Fix an integer $M \geqq 0$, and $\varepsilon > 0$. For each n, let $r_n = \inf_k \{\tau_k \mid \sigma(M, 0, Z'^n) \leqq T_{\tau_k}\}$, and denote by U_n the first change in size of at least 1 completed by $B_{\cdot}/n^{\frac{1}{2}}$ after r_n. As a consequence of (4g),

$$\max_{0\leq t\leq\sigma(M+1,0,Z^n)} |S'_{T_{nt}} - B_{nt}|/n^{\frac{1}{2}} \to_P 0 \qquad \text{as } n\to\infty .$$

Hence, by repeating the argument in (4) of [1] and using (4f), it can be shown that

$$P[Z'^n_{\sigma(M+1,0)} - Z'^n_{\sigma(M,0)} \in [1, 1+\varepsilon] \,|\, \mathscr{F}'_{\sigma(M,0,Z'^n)}] - P[U_n \in [1, 1+\varepsilon] \,|\, \mathscr{F}'_{\sigma(M,0,Z'^n)}] \to_P 0 \qquad \text{as } n\to\infty .$$

By (4e), $\mathscr{F}'_{\sigma(M,0,Z'^n)} \subset \mathscr{A}'_{r_n}$. (5) follows, since $P[U_n = 1 \,|\, \mathscr{A}'_{r_n}] = P[U_n = 1] = \frac{1}{2}$.

5. Proofs of lemmas.

PROOF OF (9). (a), (b), and (c) follow from the definitions. (d) will be proved once it is shown that

(11) $$\max_{0\leq j\leq T(n-N)} |\tilde{S}_j|/2^n \to_P 0 \qquad \text{as } n, N\to\infty$$

with $n - N\to\infty$

(12) $$\max_{0\leq j\leq T(n-N)} |S_j|/2^n \to_P 0 \qquad \text{as } n-N\to\infty$$

with $n - N\to\infty$ and

(13) $$\max_{T(n-N)\leq j\leq T(n)} |(S_j - S_{T(n-N)}) - (\tilde{S}_j - \tilde{S}_{T(n-N)})|/2^n \to_P 0$$

as $n\to\infty$, for all N.

(11) follows from the inequality

$$E[\sup_{0\leq j\leq T(n)} |\tilde{S}_j|/2^n] \leq E\sum_{k=1}^n (1/2^{n-k})[\sup_{T(k-1)\leq j\leq T(k)} |\tilde{S}_j - \tilde{S}_{T(k-1)}|/2^k] .$$

The r.h.s. is bounded independently of n, by (9a), (9b), (9c), and [4].

For (12), check from the definitions that $Y_{\cdot}^n$ satisfies (3) and $T(n-N) \leq \sigma(4^m, m, Y^{n-N})$ for any $N < n$. By the proof of (4) in [1] it follows that

$$P[\sup_{0\leq j\leq\sigma(4^m,m,Y^{n-N})} |Y_j^{n-N}| > x] \to P'[\sup_{0\leq t\leq 1} |B_t| > x] \qquad \text{as } n\to\infty ,$$

for any N and x. This fact combined with the inequality

$$\max_{0\leq j\leq T(n-N)} |S_j|/2^n \leq (1/2^N) \max_{0\leq j\leq\sigma(4^m,m,y^{n-N})} |Y_j^{n-N}|$$

implies (12).

For (13), let m_r satisfy $n(m_r) \leq r < n(m_r + 1)$. Then

$$\begin{aligned}
&\sup_{T(n-N)\leq j\leq T(n)} |(S_j - S_{T(n-N)}) - (\tilde{S}_j - \tilde{S}_{T(n-N)})|/2^n \\
&\quad\leq 2^{-m_n} + \sum_{r=n-N}^n \sum_{k=1}^{4^{m_r}} |(Y^r_{T(k,r)} - Y^r_{T(k-1,r)}) - W(k, r)| \\
&\quad\leq 2^{-m_n} + N4^{m_n} \sup_{n-N\leq r\leq n} \sup_{0\leq k\leq 4^{m_r}} |(Y^r_{T(k,r)} - Y^r_{T(k-1,r)}) - W(k, r)| .
\end{aligned}$$

By the definition of $W(k, r)$ and (7), it follows that the r.h.s. is bounded by $2^{-m_n} + N4^{m_n}\varepsilon_{m_n}$, with probability at least $1 - N\varepsilon_{m_n}$, for large n. Let $n\to\infty$, proving the lemma.

PROOF OF (10). Since (5) implies (3), $M(n)/4^m \to_P \frac{3}{4}$ as $n\to\infty$ is a consequence of (4), (21), and (25) of [1]. Hence, by (7) and the definition of $\tau(k, n)$,

$$\sum_{k=1}^{4^m} [(B_{\tau(k,n)} - B_{\tau(k-1,n)})/2^n]^2 \to \tfrac{3}{4} \qquad \text{in } L_1\text{-norm, as } n\to\infty .$$

Because

$$\{\sum_{k=1}^{j}[(\tau(k, n) - \tau(k-1, n)) - (B_{\tau(k,n)} - B_{\tau(k-1,n)})^2]/4^n \mid j = 1, \cdots, 4^m\}$$

is a martingale, $E[\tau(4^m, n)/4^n] \to 1$ as $n \to \infty$. Now, the hypotheses of (7) in [1] are easily checked, and (10) follows.

6. Applications.

(14) COROLLARY. *Suppose* $\{S_j \mid j = 0, 1, \cdots\}$ *has independent increments. Then* (2) *and* (4) *are equivalent.*

PROOF. Since the increments are independent, (3) and (5) are equivalent.

For (15), suppose $\{S_j \mid j = 0, 1, \cdots\}$ is a process on $(\Omega, \mathscr{A}, P)$ with $X_j = S_j - S_{j-1}$. Assume $X_0 = 0$. Let $\{\mathscr{A}_j \mid j = 0, 1, \cdots\}$ be an increasing family of sub-σ-fields of $\mathscr{A}$ such that X_j is $\mathscr{A}_j$ measurable. Set $v_m = \sum_{j=1}^{m} E(X_j^2 \mid \mathscr{A}_{j-1})$. Assume $E(X_{j+1} \mid \mathscr{A}_j) = 0$, $EX_j^2 < \infty$, and $v_m \to \infty$ as $m \to \infty$. Define $\hat{T}_n = \inf\{m \mid v_m \geqq n\}$. The phrase "without loss of generality" in (15) below is used in a specific sense (following Strassen [1]), namely: there is a new probability space on which processes $\{S_j' \mid j = 0, 1, \cdots\}$ and $\{T_t' \mid 0 \leqq t < \infty\}$ are defined such that these processes have the same distribution as $\{S_j \mid j = 0, 1, \cdots\}$ and $\{\hat{T}_t \mid 0 \leqq t < \infty\}$, and such that $\{B_t \mid 0 \leqq t < \infty\}$ is defined on the new space.

(15) COROLLARY.

(a) $E[(1/n) \sum_{i=1}^{\hat{T}_n} X_i^2 1_{[X_i^2 > n\varepsilon]}] \to 0$ *as* $n \to \infty$, *for all* $\varepsilon > 0$, *if and only if, without loss of generality,*

(b) *there is a Brownian motion* $\{B_t \mid 0 \leqq t < \infty\}$ *such that* $E[\sup_{0 \leqq t \leqq 1} |B_{nt} - S_{\hat{T}_{nt}}|/n^{\frac{1}{2}}]^2 \to 0$ *as* $n \to \infty$.

PROOF. Assume (15a). It suffices to verify (5), which implies (4), and then check

(16) $$\sup_{0 \leqq t \leqq 1} |Y_{T_{nt}}^n - Y_{\hat{T}_{nt}}^n| \to_P 0 \quad \text{as } n \to \infty$$

(where $\{T_t \mid 0 \leqq t < \infty\}$ is the time change given in (4)) and

(17) $\{\sup_{0 \leqq j \leqq \hat{T}_n} (Y_j^n)^2 \mid n = 0, 1, \cdots\}$ is uniformly integrable.

Assuming (15 b), (15 a) follows from the main Theorem in [1].

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