*The Annals of Probability*
1973, Vol. 1, No. 3, 509–512

# ON KHINTCHINE'S ESTIMATE FOR LARGE DEVIATIONS

BY DAVID G. KOSTKA

*Texas A & M University*

The large deviation estimate, used in classical proofs of the law of the iterated logarithm for i.i.d. random variables, implies the random variables satisfy a condition more stringent than a finite variance. Thus it is impossible to prove the law of the iterated logarithm in its full strength (i.e. assuming only a finite second moment) by using such a deviation estimate in a "straightforward" manner.

**1. Introduction.** Let $\{X_k\}_{k \geqq 1}$ be a sequence of independent, identically distributed random variables with mean zero and variance one; let $S_n = X_1 + \cdots + X_n$. It is well known that the law of the iterated logarithm (L.I.L.) for $S_n$ holds if and only if the random variables $X_k$ have finite variance. Hartman and Wintner (1941) showed the sufficiency of a finite variance, and Strassen (1966) showed the necessity.

In classical proofs of the law of the iterated logarithm a key estimate (see Lamperti (1966) pages 41–49) is

$$P(S_n/n^{\frac{1}{2}} \geqq a_n) = \exp[-(a_n^2/2)(1 + o(1))] \tag{1.1}$$

as $n \uparrow \infty$ where $a_n = (1 \pm \varepsilon)(2 \lg \lg n)^{\frac{1}{2}}$, $\varepsilon > 0$. Petrov (1966) showed that such a large deviation estimate is readily deduced from Berry–Esséen type theorems when the random variables $X_k$ satisfy a condition more stringent than a finite variance.

The purpose of this note is to point out that estimate (1.1) implies the $\{X_k\}$ satisfy a condition which is stronger than a finite variance. Thus, it is impossible to prove the law of the iterated logarithm in its full strength (i.e. assuming only a finite second moment) by using Gaussian tail estimates of the form (1.1) in a "straightforward" manner.

**2. Large deviations of the form** (1.1). Pinsky (1969) derived estimates of the form (1.1) by examining the convergence rate in Trotter's method of operators and thus avoided the use of the Berry–Esséen theorem. However, estimates of this form can be derived in more generality by using an extended Berry–Esséen theorem (see Katz (1963) or Petrov (1965)), of the following type.

Assume $E[X_1^2 h(|X_1|)] < \infty$ where $h(x) \uparrow \infty$, $x/h(x) \uparrow$ as $x \uparrow \infty$ then

$$P(S_n/n^{\frac{1}{2}} \geqq a) = \int_a^\infty \frac{\exp(-t^2/2)}{(2\pi)^{\frac{1}{2}}}\, dt + O\{(h(n^{\frac{1}{2}}))^{-1}\} \tag{2.1}$$

Received May 30, 1972; revised September 28, 1972.

*AMS* 1970 *subject classifications.* Primary 60F10; Secondary 60F05, 60G50.

*Key words and phrases.* Large deviations, law of the iterated logarithm, Gaussian tail estimates.

where the error is uniform in $a \in (-\infty, \infty)$. For a suitable choice of $a = a_n$ the error term in (2.1) can be absorbed into the Gaussian term. As in Petrov (1966), this yields the following result.

(2.2) PROPOSITION. *Assume $E[X_1^2 h(|X_1|)] < \infty$ where $h(x) \uparrow \infty$, $x/h(x) \uparrow$ as $x \uparrow \infty$. If $\{a_n\}$ is a sequence increasing to $+\infty$ so that*

$$\exp[a_n^2/2(1 + o(1))]/h(n^{\frac{1}{2}}) \to 0$$

*then for each $\varepsilon > 0$*

(2.3) $$\exp[-(a_n^2/2)(1 + \varepsilon)] \leqq P(S_n/n^{\frac{1}{2}} \geqq a_n) \leqq \exp[-(a_n^2/2)(1 - \varepsilon)]$$

*for $n \geqq N(\varepsilon)$*

Immediate consequences of this proposition are estimates of the form (1.1) that are used in classical proofs of the L.I.L.

(2.4) COROLLARY. *If $E(X_1^2(\lg|X_1|)^{1+\delta}) < \infty$, then*

$$P(S_n/n^{\frac{1}{2}} \geqq a_n) = \exp[-a_n^2/2(1 + o(1))]$$

*where*

$$a_n = (1 \pm \varepsilon)(2 \lg \lg n)^{\frac{1}{2}}, \qquad 0 \leqq \varepsilon \leqq \delta/3 \text{ and}$$

$$\lg x = \log_e x \quad \text{if } x \geqq 1,$$
$$= 1 \quad \text{otherwise.}$$

**3. A partial converse of equation** (1.1). We can get a partial converse to Corollary (2.5) which shows that estimates of the form (1.1) imply the random variables $\{X_k\}$ satisfy a condition more stringent than a finite variance. Inspiration for the proof came from a private communication of Howard Stratton via Mark Pinsky. The first lemma we need appears in Baum and Katz (1965) in a more specialized form than the following.

(3.1) LEMMA. *Assume $\{b_n\}$ is an increasing sequence of real numbers such that $nP(X_1 > b_n) \to 0$ as $n \to \infty$ and $\{X_k\}$ are symmetric, then*

$$P(S_n > b_n) \geqq CnP(X_1 > b_n)$$

*for some $C > 0$.*

PROOF.

$$\begin{aligned}
P(S_n > b_n) &\geqq P \textstyle\bigcup_{i=1}^n \{(X_i > b_n) \cap (\sum_{j=1, j\neq i}^n X_j \geqq 0) \cap (\bigcap_{j=1, j\neq i}^n (X_j \leqq b_n))\} \\
&= \textstyle\sum_{i=1}^n P\{(X_i > b_n) \cap (\sum_{j=1, j\neq i}^n X_j > 0) \cap (\bigcap_{j=1, j\neq i}^n X_j \leqq b_n)\} \\
&= \textstyle\sum_{i=1}^n P(X_i > b_n) P\{(\sum_{j=1, j\neq i}^n X_j \geqq 0) \cap (\bigcap_{j=1, j\neq i}^n X_j \leqq b_n)\} \\
&\geqq \textstyle\sum_{i=1}^n P(X_i > b_n)[P(\sum_{j=1, j\neq i}^n X_j \geqq 0) - P(\overline{\bigcap_{j=1, j\neq i}^n X_j \leqq b_n})] \\
&= \textstyle\sum_{i=1}^n P(X_i > b_n)[P(\sum_{j=1, j\neq i}^n X_j \geqq 0) - P(\bigcup_{j=1, j\neq i}^n X_j > b_n)] \\
&\geqq \textstyle\sum_{i=1}^n P(X_i > b_n)[\frac{1}{2} - (n-1)P(X_i > b_n)] \\
&\geqq nP(X_i > b_n)(\frac{1}{2} - \delta) \qquad \text{for } n \geqq N(\delta).
\end{aligned}$$

This gives the desired result.

The next lemma we need appears in Davis (1968).

(3.2) LEMMA. *Let $X$ be a random variable and $\{a_n\}$ a non-decreasing positive sequence such that*

$$na_n^2 = O(\sum_{k=1}^n a_k^2)\,, \qquad a_n/a_{n-1} = O(1)$$

*then*

$$\sum_{n=1}^\infty a_n^2 P(|X| > a_n n^{1/2}) < \infty$$

*implies $E(X^2) < \infty$.*

(3.3) PROPOSITION. *Let $\{X_k\}_{k\geq 1}$ be a sequence of independent, identically distributed, symmetric random variables with $E(X_k) = 0$ and $E(X_k^2) = 1$. If*

$$P(S_n/n^{\frac{1}{2}} \geqq a_n) = \exp[(-a_n^2/2)(1 + o(1))]$$

*for $a_n = (1+\varepsilon)(2 \lg\lg n)^{\frac{1}{2}}$, $\varepsilon > 0$, then $E(X_1^2(\lg |X_1|)^\varepsilon) < \infty$.*

PROOF. If $b_n = n^{\frac{1}{2}}a_n = (1+\varepsilon)(2n \lg\lg n)^{\frac{1}{2}}$ then the previous lemma applies since

$$nP(X_1 > b_n) \leqq \frac{nE(X_1^2)}{b_n^2} \to 0$$

by Chebyshev's inequality.

Now,

$$\begin{aligned}\infty &> \sum_{n=1}^\infty \frac{(\lg n)^{2\varepsilon}}{n(\lg n)^{(1+\varepsilon)^2(1+o(1))}} \\ &\geqq \sum_{n=1}^\infty \frac{(\lg n)^{2\varepsilon}}{n} \exp\left[\frac{-((1+\varepsilon)(2\lg\lg n)^{\frac{1}{2}})^2(1+o(1))}{2}\right] \\ &\geqq C \sum_{n=1}^\infty \frac{(\lg n)^{2\varepsilon}}{n} P(S_n/n^{\frac{1}{2}} \geqq (1+\varepsilon)(2\lg\lg n)^{\frac{1}{2}})\end{aligned}$$

by the assumptions of the proposition

$$\geqq C \sum_{n=1}^\infty \frac{(\lg n)^{2\varepsilon}}{n} nP(X_1 \geqq (1+\varepsilon)(2n\lg\lg n)^{\frac{1}{2}})$$

by the lemma above

$$\begin{aligned}(3.4) \qquad &\geqq C \sum_{n=1}^\infty (\lg n)^{2\varepsilon} P(X_1^2(\lg X_1)^\varepsilon \geqq Cn \lg\lg n (\lg n)^\varepsilon) \\ &\geqq C \sum_{n=1}^\infty (\lg n)^{2\varepsilon} P(X_1^2(\lg X_1)^\varepsilon \geqq Cn(\lg n)^{2\varepsilon})\,.\end{aligned}$$

In the above $C$ stands for various positive constants. The convergence of series (3.4) combined with Lemma (3.2) gives the desired result that

$$E(X_1^2(\lg |X_1|)^\varepsilon) < \infty\,.$$

**Acknowledgment.** The author's original proof was based on a modification of Pinsky's method. The remark that Proposition 2.2 is a consequence of Katz extensions of the Berry–Esséen theorem is due to the referee.

## REFERENCES

[1] BAUM, L. and Katz, M. (1965). Convergence rates in the law of large numbers. *Trans. Amer. Math. Soc.* **120** 108–123.

[2] DAVIS, J. (1968). Convergence rates for probabilities of moderate deviations. *Ann. Math. Statist.* **39** 2016–2028.

[3] HARTMAN, P. and WINTNER, A. (1941). On the law of the iterated logarithm. *Amer. J. Math.* **63** 169–176.

[4] KATZ, M. (1963). Note on the Berry-Esseén theorem. *Ann. Math. Statist.* **34** 1107–1108.

[5] LAMPERTI, J. (1966). *Probability.* Benjamin, New York.

[6] PETROV, V. V. (1965). An estimate of the deviation of the distribution of a sum of independent random variables form the normal law. *Soviet Math. Dokl.* **6** 242–244.

[7] PETROV. V. V. (1966). On a relation between an estimate of the remainder in the central limit theorem and the law of the iterated logarithm. *Theor. Probability Appl.* **11** 454–458.

[8] PINSKY, M. (1969). An elementary derivation of Khintchine's estimate for large deviations. *Proc. Amer. Math. Soc.* **22** 288–290.

[9] STRASSEN, V. (1966). A converse to the law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **4** 265–268.

DEPARTMENT OF MATHEMATICS
TEXAS A & M UNIVERSITY
COLLEGE STATION, TEXAS 77843