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A POISSON-TYPE LIMIT THEOREM FOR MIXING SEQUENCES OF DEPENDENT 'RARE' EVENTS[1]

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Certain mixing sequences of dependent 'rare' events are considered and a Poisson limit is established for the probability that k events occur. An asymptotic distribution for the number of upcrossings of a high level by certain stochastic processes is considered as an application.

1. Introduction. The well-known Poisson limit for sequences of Bernoulli (binomial) 'rare' events asserts that if, for each n, the events A_i^n $(i = 1, 2, \cdots, n)$ are mutually independent and $P(A_i^n) = a/n$ $(a > 0;\ i = 1, 2, \cdots, n)$, then for any fixed nonnegative integer k, $P_k^n = \Pr\{\text{exactly } k \text{ among } A_i^n\ (i = 1, 2, \cdots, n) \text{ occur}\} \to e^{-a}a^k/k!$ as $n \to \infty$. This result has been generalized to mutually independent events not necessarily of equal probability and Koopman [1] gives necessary and sufficient conditions. For the dependent case Walsh [4] provides sufficient conditions for convergence of P_k^n to a Poisson limit when each event among A_i^n $(i = 1, 2, \cdots, n)$ is independent only of at least $n - m - 1$ of the others.

Here we consider certain mixing sequences of events in which each A_i^n conceivably depends upon every other event. A Poisson limit for P_k^n is established by generalizing a result of Loynes [2]. As an application we consider the asymptotic distribution of the number of upcrossings of a high level by certain strongly mixing, strictly stationary stochastic processes.

2. Main results. In order to provide a proper setting for treating Poisson limit theorems a probability space $(\Omega_n, \mathscr{A}_n, P_n)$ and sequence of events $\{A_i^n\ (i = 1, 2, \cdots)\}$ herein is introduced for each positive integer n. For notational convenience the index n will be omitted from P_n and, for example, $P(A_i^n)$ will be written instead of $P_n(A_i^n)$.

DEFINITION. A sequence of sequences of events $\{A_i^n\ (i = 1, 2, \cdots)\}$ $(n = 1, 2, \cdots)$ is termed uniformly (or strongly) mixing with mixing function sequence $\{g_n\ (n = 1, 2, \cdots)\}$ if $|P(E^nF^n) - P(E^n)P(F^n)| \leqq g_n(k)$, where $E^n(F^n)$ is an event defined in terms of $\{A_1^n, \cdots, A_m^n\}$ $(\{A_{m+k+1}^n, \cdots\})$, and $g_n(z_n) \to 0$ if $z_n \to \infty$.

DEFINITION. A sequence of sequences of events $\{A_i^n\ (i = 1, 2, \cdots)\}$ $(n = 1, 2, \cdots)$ is termed stationary if for any event E^n defined in terms of $\{A_i^n(i =$

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$1, 2, \cdots)\}$, $P(E^n) = P(T(E^n))$, where $T(E^n)$ denotes the translated event obtained from E^n by replacing each A_i^n in its definition by A_{i+1}^n.

The following theorem is closely related to Loynes' [2] original result. The proof contains certain differences.

THEOREM. *Let* $\{A_i^n\ (i = 1, 2, \cdots)\}\ (n = 1, 2, \cdots)$ *be stationary and uniformly mixing with mixing function sequence* $\{g_n\ (n = 1, 2, \cdots)\}$ *and with* $P(A_i^n) \sim a/n$ $(a > 0)$. *Suppose that there exist two sequences of integers* $\{p_m\ (m = 1, 2, \cdots)\}$ *and* $\{q_m\ (m = 1, 2, \cdots)\}$ (*with* $t_m = m(p_m + q_m)$) *such that as* $m \to \infty$, $m^r g_{t_m}(q_m) \to 0$ (*for any fixed* $r > 0$), $q_m/p_m \to 0$, *and* $p_{m+1}/p_m \to 1$. *Finally assume* $I_{p_m} = \sum_{i=1}^{p_m-1} (p_m - i) P(A_{i+1}^{t_m} A_1^{t_m}) = o(1/m)$ *as* $m \to \infty$. *Then* $P_k^n \to e^{-a}a^k/k!$ *as* $n \to \infty$.

PROOF. For fixed m, partition the positive integers into consecutive blocks of size p_m and q_m alternately, beginning with the initial block $\{1, 2, \cdots, p_m\}$ of size p_m. Let $P_m(Q_m)$ denote those positive integers falling into size $p_m(q_m)$ blocks. (The subscript m will be supressed where convenient.) The proof consists of three parts: (i) showing that asymptotically if an event A_i^t occurs then $i \in P_m$, (ii) showing that if k among $A_i^t\ (i = 1, 2, \cdots, t)$ occur then asymptotically all k indices lie in separate p_m blocks, and (iii) using Bonferroni's inequalities to obtain the asymptotic probability of the event D_k^t defined as "exactly k among A_i^t $(i = 1, 2, \cdots, t)$ occur and each such A_i^t has subscript i in a separate block of P_m."

First, write $P_k^t = P(B_k^t) + P(C_k^t)$ say, where B_k^t and C_k^t are defined respectively as "exactly k among $A_i^t\ (i = 1, 2, \cdots, t)$ occur and all such i's are in P_m" and "exactly k among $A_i^t\ (i = 1, 2, \cdots, t)$ occur and some such i's are in Q_m." From the hypothesis, $P(C_k^t) \leqq \sum_{i \in Q_m, i \leqq t} P(A_i^t) \sim mqa/t \to 0$ as $m \to \infty$. Thus, if either P_k^t or $P(B_k^t)$ has a limit as $m \to \infty$, both have and the two are equal. Next write $P(B_k^t) = P(D_k^t) + P(F_k^t)$ say where D_k^t is defined above and F_k^t is defined as "B_k^t occurs and some such i's lie in the same block in P_m." From the hypothesis $P(F_k^t) \leqq mI_p \to 0$ as $m \to \infty$. Thus if either $P(B_k^t)$ or $P(D_k^t)$ has a limit as $m \to \infty$, both have and the two are equal.

To evaluate $P(D_k^t)$ write D_k^t as "exactly k among $G_i^t\ (i = 1, 2, \cdots, m)$ occur" where G_i^t is defined as "exactly one $A_j^t\ (j = (i-1)(p+q) + 1, \cdots, (i-1)(p+ + q) + p)$ occurs." Using Bonferroni's inequalities it follows that for any even integer v, $v + k \leqq m$, $L_{v,k}^t \leqq P(D_k^t) \leqq U_{v,k}^t$, where $L_{v,k}^t = S_k^t - \binom{k+1}{k} S_{k+1}^t + \cdots - \binom{k+v-1}{k} S_{k+v-1}^t$, and $U_{v,k}^t = L_{v,k}^t + \binom{k+v}{k} S_{k+v}^t$ and $S_r^t = \sum_C P(G_{i_1}^t \cdots G_{i_r}^t)$ with $C = \{(i_1, \cdots, i_r) \mid 1 \leqq i_1 < \cdots < i_r \leqq m\}, 0 < r \leqq m$. Via the uniform mixing and stationarity assumptions $|P(G_{i_1}^t \cdots G_{i_r}^t) - P^r(G_1^t)| \leqq r g_t(q)$. Now again using Bonferroni's inequalities, $T_1^t - T_2^t \leqq P(G_1^t) \leqq T_1^t$, where $T_1^t = \sum_{j=1}^p P(A_j^t)$ and $T_2^t = I_p$. From the hypotheses, $mT_1^t \sim mpa/t \to a$ and $mT_2^t \to 0$ as $m \to \infty$. Therefore, $P(G_1^t) \sim a/m$ as $m \to \infty$, so that $|P(G_{i_1}^t \cdots G_{i_r}^t) - [a/m + o(1/m)]^r| \leqq r g_t(q)$ and hence $|S_r^t - \binom{m}{r}[a/m + o(1/m)]^r| \leqq r\binom{m}{r} g_t(q)$. This implies that $S_r^t \to a^r/r!$ as $m \to \infty$, and so $L_{v,k}^t \to a^k/k!\,[\sum_{j=0}^{v-1}(-a)^j/j!]$ and $U_{v,k}^t \to a^k/k!\,[\sum_{j=0}^{v}(-a)^j/j!]$ as $t \to \infty$. Since v is arbitrary it can be concluded that $P(D_k^t) \to e^{-a}a^k/k!$ as

$t \to \infty$, and so $P_k^n \to e^{-a}a^k/k!$ as $n \to \infty$ along the sequence $\{t_m\}$. However, since an arbitrary positive integer n lies between two consecutive values of t, the special properties of the sequence guarantee that $P_k^n \to e^{-a}a^k/k!$ as $n \to \infty$ in any manner.

REMARKS. Suppose that the conditions of the preceding theorem are satisfied. If s and t ($0 \leqq s < t$) are two real numbers it can be seen that as $n \to \infty$, $P_k^n(s, t) = \Pr\{\text{exactly } k \text{ among } A^n_{[sn]+1}, A^n_{[sn]+2}, \cdots, A^n_{[tn]} \text{ occur}\} \to e^{-a(t-s)}[a(t-s)]^k/k!$ (and similarly the joint probability of the exact numbers of occurrences among several such disjoint finite sequences of events tends to the corresponding product). Thus, $P_k^n(0, t/a) \to e^{-t}t^k/k!$ as $n \to \infty$, so judged in terms of "time units" of length n/a, the waiting time until the k-th occurrence of an event A_i^n has a distribution function $F_k^n(t)$ satisfying the relation as $n \to \infty$,

$$\Pr\left\{\frac{\text{time to } k\text{th event}}{n/a} > t\right\} = 1 - F_k^n(t) = \sum_{j=0}^{k-1} P_j^n(0, t/a) \to \sum_{j=0}^{k-1} e^{-t}t^j/j! \; .$$

Hence, the asymptotic distribution function, say $F_k(t)$, of the waiting time until the kth occurrence of an event A_i^n (using "time units" of length n/a) is that of a random variable with a Gamma (k) distribution.

3. Application. Associate with each realization of a discrete-time stochastic process $\{X_i \; (i = 1, 2, \cdots)\}$ a continuous (sample) function $\{X(t), 0 < t < \infty\}$ obtained by joining the points (i, X_i) ($i = 1, 2, \cdots$) in sequence by line segments. An upcrossing by $\{X(t), 0 < t < \infty\}$ of a level u_n occurs in the (time) interval $(i-1, i]$ if and only if the event $A_i^n = \{Y_i \in S_n\}$ occurs, where $Y_i = (X_{i-1}, X_i)$ and $S_n = \{(x, y) \mid x < u_n, y > u_n\}$.

By a discrete-time stochastic process being strongly (or uniformly) mixing (with mixing function g) it is meant that if A and B are events defined in terms of $(X_1, \cdots, X_m)$ and $(X_{m+k+1}, \cdots)$ respectively for some integers $m, k \geqq 1$, then $|P(AB) - P(A)P(B)| \leqq g(k)$, where $g(k) \to 0$ as $k \to \infty$. The following consequence of the theorem of the preceding section was suggested by Loynes' [2] theorem.

THEOREM. *Let $\{X_i \; (i = 1, 2, \cdots)\}$ be a strongly mixing, strictly stationary stochastic process (with mixing function g) and $\{u_n\}$ a sequence of levels chosen so that $nP(A_i^n) \to a$ ($a > 0$) as $n \to \infty$. Suppose there exist sequences of integers $\{p_m\}$ and $\{q_m\}$ such that as $m \to \infty$,* (a) $q_m/p_m \to 0$, (b) $p_m/p_{m+1} \to 1$, (c) $m^r g(q_m) \to 0$ *for any fixed $r > 0$, and* (d) $I_{p_m} = o(1/m)$. *Let $Z_n(t) = \sharp$ of upcrossings of u_n by $X(\cdot)$ in $[0, nt]$. Then the sequence $Z_n(\cdot)$ of stochastic processes converges weakly to the Poisson process $P_a(\cdot)$. In particular, for any two real numbers s and t ($0 \leqq s < t$),* $\Pr\{$*exactly k upcrossings by $X(t)$ of u_n in $[ns, nt]$*$\} \to e^{-a(t-s)}[a(t-s)]^k/k!$ *as $n \to \infty$.*

The convergence of the finite-dimensional distributions of $Z_n(\cdot)$ to those of $P_a(\cdot)$ has already been indicated; and Straf [3] has shown that this is sufficient for weak convergence, in this setting of count processes converging to a Poisson process.

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REFERENCES

[1] KOOPMAN, B. O. (1950). Necessary and sufficient conditions for Poisson's distributions. *Proc. Amer. Math. Soc.* **1** 737–740.

[2] LOYNES, R. M. (1965). Extreme values in uniformly mixing stationary stochastic processes. *Ann. Math. Statist.* **36** 993–999.

[3] STRAF, M. (1969). A general Skorohod space and its application to the weak convergence of stochastic processes with several parameters. Ph. D. dissertation. Univ. of Chicago.

[4] WALSH, J. E. (1955). The Poisson distribution as a limit for dependent binomial events with unequal probabilities. *Bull. Operations Res. Soc. Amer.* **3** 198–209.

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