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A CONVERGENCE THEOREM FOR EXTREME VALUES FROM GAUSSIAN SEQUENCES[1]

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Let $\{X_n, n = 0, \pm 1, \pm 2, \cdots\}$ be a stationary Gaussian stochastic process with means zero, variances one, and covariance sequence $\{r_n\}$. Let $M_n = \max\{X_1, \cdots, X_n\}$ and $S_n =$ second largest $\{X_1, \cdots, X_n\}$. Limit properties are obtained for the joint law of M_n and S_n as n approaches infinity. A joint limit law which is a function of a double exponential law is known to hold if the random variables X_i are mutually independent. When M_n alone is considered Berman has shown that a double exponential law holds in the case of dependence provided either $r_n \log n \to 0$ or $\sum_{n=1}^{\infty} r_n^2 < \infty$. In the present work it is shown that the above conditions are also sufficient for the convergence of the joint law of M_n and S_n. Weak convergence properties of the stochastic processes $M_{[nt]}$ and $S_{[nt]}$ with $0 < a \leqq t < \infty$ are also discussed.

1. Introduction. This paper extends and simplifies a theorem obtained by the author in Section 4 of [5]. The reader is assumed to have some acquaintance with those results.

Let $\{X_n, n = 0, \pm 1, \pm 2, \cdots\}$ be a discrete parameter stationary Gaussian stochastic process, characterized by expectation, and covariance function, respectively:

$$(1.1)\qquad E(X_n) =_\Delta 0\,, \qquad E(X_i X_{i+n}) =_\Delta r_n\,, \qquad r_0 =_\Delta 1\,.$$

This paper treats some of the limit properties of the random variables

$$M_n = \max\{X_1, \cdots, X_n\}\,,$$
$$S_n = \text{second largest}\quad \{X_1, \cdots, X_n\}\,.$$

A double exponential limit law is known to hold for M_n if the random variables X_i are mutually independent, that is $r_n =_\Delta 0$, $n \neq 0$. Berman [1] has shown that the same law holds in the case of dependence provided either

$$(1.2)\qquad r_n \log n \to 0\,, \qquad \text{or}$$

$$(1.3)\qquad \textstyle\sum_{n=1}^{\infty} r_n^2 < \infty\,.$$

The author [5] has shown that the processes $\{M_{[nt]}, S_{[nt]}\}$, properly normalized, and with $0 < a \leqq t < \infty$, converge weakly in the Skorohod space $D^2[a, \infty)$ when

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the Gaussian sequence is strong-mixing and

$$r_n \log n = O(1) . \tag{1.4}$$

The limit law is the same as that which occurs in the independent case.

Condition (1.4) is weaker than (1.2) but we imposed the strong-mixing condition. In many cases strong-mixing is difficult to verify and it is natural in view of Berman's work to see if the weak convergence results mentioned above hold when the strong-mixing assumption is dropped and just (1.2) or (1.3) is assumed. The purpose of this paper is to show that this is, in fact, true. The reader is referred to [6] for some examples of why it is of interest to consider the joint distribution of M_n and S_n. A more extensive discussion of the maxima of stationary Gaussian processes is contained in [3].

2. Some properties of Gaussian distributions. Let (r_{ij}) be a $k \times k$ symmetric positive definite matrix with 1's along the diagonal, and let $\phi_k(x_1, \cdots, x_k; r_{ij}, 1 \leqq i < j \leqq k)$ be the k-dimensional Gaussian density function with mean vector 0 and covariance matrix (r_{ij}); ϕ_k is a function of the x's and the $k(k-1)/2$ parameters r_{ij}. Define:

$$\tilde{Q}_k(c, d, \alpha, \{r_{ij}\}) = \int_{-\infty}^{c} dx_1 \cdots \int_{-\infty}^{c} dx_{\alpha-1} \int_{d}^{\infty} dx_\alpha \int_{-\infty}^{c} dx_{\alpha+1} \cdots \int_{-\infty}^{c} dx_k \times \phi_k(x_1, \cdots, x_k; \{r_{ij}\}) . \tag{2.1}$$

The integral from d to ∞ will always be on the αth dummy variable and we assume that $0 < c \leqq d$.

LEMMA 1. *If $r_{ij} \equiv r_{ji}$ then*

$$\frac{\partial \tilde{Q}_k}{\partial r_{hl}} = \int_{-\infty}^{c} \cdots \int_{d}^{\infty} \cdots \int_{-\infty}^{c} \phi_k(x_1, \cdots, x_{h-1}, c, x_{h+1}, \cdots, x_{l-1}, c, x_{l+1}, \cdots, x_k) \times \prod_{j \neq h, j \neq l} dx_j \tag{2.2}$$

when $h \neq \alpha$, $l \neq \alpha$, $h \neq l$, and

$$\frac{\partial \tilde{Q}_k}{\partial r_{\alpha l}} = -\int_{-\infty}^{c} \cdots \int_{-\infty}^{c} \phi_k(x_1, \cdots, x_{\alpha-1}, d, x_{\alpha+1}, \cdots, x_{l-1}, c, x_{l+1}, \cdots, x_k) \times \prod_{j \neq \alpha, j \neq l} dx_j \tag{2.3}$$

with a corresponding expression when $h \neq \alpha$ and $l = \alpha$.

PROOF. A complete proof is contained on page 481 of the paper by Slepian [4]. Only the essential ideas will be given here. The k-variate Gaussian density is given in terms of its characteristic function by

$$\phi_k(x_1, \cdots, x_k; \{r_{ij}\}) = \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_k \exp\left[i\Sigma x_j t_j - \tfrac{1}{2}\Sigma r_{ij} t_i t_j\right] .$$

From this expression it is easy to see that

$$\frac{\partial \phi_k}{\partial r_{ij}} = \frac{\partial^2 \phi_k}{\partial x_i \partial x_j} \qquad j > i$$

which allows integration over x_i and x_j, giving (2.2) and (2.3) and completing the proof of Lemma 1.

If the upper limits of integration in (2.2) are replaced by $(\infty, \cdots, \infty)$ then the value of the integral is increased. Now integrate $k-3$ variables from $-\infty$ to $+\infty$ to obtain

$$\frac{\partial \tilde{Q}_k}{\partial r_{hl}}(c, d, \alpha, \{r_{ij}\}) \leqq \int_d^\infty \phi_3(c, c, x_\alpha; \Sigma(h, l, \alpha))\, dx_\alpha \tag{2.4}$$

where

$$\Sigma(h, l, \alpha) =_\Delta \begin{pmatrix} 1 & r_{hl} & r_{h\alpha} \\ r_{lh} & 1 & r_{l\alpha} \\ r_{\alpha h} & r_{\alpha l} & 1 \end{pmatrix}.$$

We note that if the limits of integration in (2.3) are replaced by $(\infty, \cdots, \infty)$ then

$$\left|\frac{\partial \tilde{Q}_k}{\partial r_{\alpha l}}\right| \leqq \phi_2(c, c; r_{\alpha l}) = (2\pi)^{-1}(1 - r_{\alpha l}^2)^{-\frac{1}{2}} \exp[-c^2/(1 + r_{\alpha l})]. \tag{2.5}$$

Since $\{X_n\}$ is a stationary process, r_{ij} is a function of the difference $j - i$, $i < j$; we write $r_{j-i} = r_{ij}$. The function $\tilde{P}_k$ is defined as

$$\tilde{P}_k(c, d, \alpha, r_1, \cdots, r_{k-1}) = \tilde{Q}_k(c, d, \alpha, \{r_{ij}\})$$

and the partial derivatives are given by the chain rule as

$$\partial \tilde{P}_k / \partial r_j = \sum_{l-h=j} \partial \tilde{Q}_k / \partial r_{hl}.$$

Let the sequences $\{a_n\}$ and $\{b_n\}$ be defined as

$$\begin{aligned} a_n &= (2 \log n)^{-\frac{1}{2}} \\ b_n &= (2 \log n)^{\frac{1}{2}} - \tfrac{1}{2}(2 \log n)^{-\frac{1}{2}}(\log \log n + \log 4\pi). \end{aligned} \tag{2.6}$$

It is known (Cramér, page 374) that when $r_n =_\Delta 0$, $n \neq 0$

$$\lim_{n\to\infty} P\{M_n \leqq a_n x + b_n\} = \exp(-e^{-x}) =_\Delta G(x)$$

for all x.

Both (1.2) and (1.3) imply that $r_n \to 0$; therefore, there exists a positive number δ such that

$$\sup_n |r_n| = \delta < 1.$$

Define: $\delta(n) = \sup_{k \geqq n} |r_k|$, $q_n = [n^\beta]$, $\delta_n = \delta([q_n/2])$ where $0 < \beta < (1-\delta)^2/2(1 + 2\delta)^2$. Clearly (1.2) implies that

$$\lim_{n\to\infty} \delta(n) \log n = 0, \quad \text{and} \tag{2.7}$$

$$\lim_{n\to\infty} \delta_n \log n = 0. \tag{2.8}$$

3. Convergence theorems. In this section we extend Berman's results to the joint laws of M_n and S_n.

THEOREM 1. *Let* $\{X_n, n = 0, \pm 1, \pm 2, \cdots\}$ *be a stationary Gaussian sequence satisfying* (1.1). *If*

$$\lim_{n\to\infty} r_n \log n = 0$$

then

$$\begin{aligned}\lim_{n\to\infty} P\{M_n \leqq a_n x + b_n, S_n \leqq a_n y + b_n\} \\ &= G(y)\{1 + \log[G(x)/G(y)]\} \qquad y < x \\ &= G(x) \qquad y \geqq x .\end{aligned}$$

The following two lemmas will be needed in the proof. For convenience let $c_n = a_n y + b_n$ and $d_n = a_n x + b_n$, and to avoid technical details we will assume that n is so large that $c_n > 0$.

LEMMA 2. *Assume that the conditions of Theorem* 1 *are satisfied and*

$$\gamma_n = (1 - 4\delta_n)/(1 + 2\delta_n)$$
$$\hat{\gamma}_n = (1 - 3\delta_n - \delta)/(1 + 2\delta)$$

where n is so large that $(1 - 2\delta_n - \delta) > 0$. *Then*

(3.1) $$\lim_{n\to\infty} n^2[1 - \Phi(b_n\gamma_n)]\phi_2(c_n, c_n, \delta_n) \sum_{j=q_n+1}^{n-1} |r_j| = 0$$

and

(3.2) $$\lim_{n\to\infty} n^{1+\beta}[1 - \Phi(b_n\hat{\gamma}_n)]\phi_2(c_n, c_n, \delta_n) \sum_{j=q_n+1}^{n-1} |r_j| = 0$$

where $\Phi(\cdot)$ *is the standardized Gaussian distribution function.*

PROOF. Berman [1] as part of the proof of his Theorem 3.1 has shown that

(3.3) $$\lim_{n\to\infty} n\phi_2(c_n, c_n, \delta_n) \sum_{j=q_n+1}^{n-1} |r_j| = 0 .$$

To prove (3.1) and (3.2) we note that

(3.4) $$1 - \Phi(x) \leqq (2\pi)^{-\frac{1}{2}}x^{-1}\exp(-x^2/2) \qquad x > 0 ,$$

(3.5) $$b_n^2 = 2\log n - \log\log n + O(1) , \qquad \text{and}$$

(3.6) $$b_n = (2\log n)^{\frac{1}{2}} + o(1) .$$

Therefore

$$n[1 - \Phi(b_n\gamma_n)] = \frac{O(1)\cdot\exp((1 - \gamma_n^2)\log n)}{\exp[((1 - \gamma_n^2)/2)\log\log n] + o(1)} .$$

Since $1 - \gamma_n^2 = \delta_n \cdot O(1)$ and $\delta_n \log n \to 0$, $n[1 - \Phi(b_n\gamma_n))$ is bounded and (3.1) follows from (3.3). Similarly

$$n^\beta[1 - \Phi(b_n\hat{\gamma}_n)] = \frac{O(1)\cdot\exp[(\beta - \hat{\gamma}_n^2)\log n]}{\exp[((1 - \hat{\gamma}_n^2)/2)\log\log n] + o(1)}$$

and (3.2) follows because $\beta < \hat{\gamma}_n^2$ and $1 - \hat{\gamma}_n^2 > 0$ for sufficiently large n.

LEMMA 3. *If the conditions of Theorem* 1 *hold and* $y < x$, *then*

(3.7) $$\sum_{\alpha=1}^{n} |\tilde{P}_n(c_n, d_n, \alpha, r_1, \cdots, r_{n-1}) - \tilde{P}_n(c_n, d_n, \alpha, r_1, \cdots, r_{q_n}, 0, \cdots, 0)| \to 0 .$$

PROOF. By the law of the mean, there exist numbers r_i' between 0 and r_i, $i = q_n + 1, \cdots, n-1$, such that

$$\tilde{P}_n(c_n, d_n, \alpha, r_1, \cdots, r_{n-1}) - \tilde{P}_n(c_n, d_n, \alpha, r_1, \cdots, r_{q_n}, 0, \cdots, 0) \\ = \sum_{j=q_n+1}^{n-1} r_j(\partial \tilde{P}_n/\partial r_j)(c_n, d_n, \alpha, r_1, \cdots, r_{q_n}, r'_{q_n+1}, \cdots, r'_{n-1})$$

and therefore the sum in (3.7) is less than

$$\sum_{\alpha=1}^{n} \sum_{j=q_n+1}^{n-1} |r_j| \sum_{l-h=j} |(\partial \tilde{Q}_n/\partial r_{hl})(c_n, d_n, \alpha, r_1, \cdots, r_{q_n}, r'_{q_n+1}, \cdots, r'_{n-1})| .$$

We now consider three cases:

(i) $l = \alpha$ or $h = \alpha$ (both cannot occur),
(ii) $|l - \alpha| > q_n/2$ and $|h - \alpha| > q_n/2$,
(iii) $|l - \alpha| \leqq q_n/2$ or $|h - \alpha| \leqq q_n/2$ (both cannot occur), $h \neq \alpha$ and $l \neq \alpha$.

In the first case (2.5) applies and

$$\sum_{\alpha=1}^{n} \sum_{j=q_n+1}^{n-1} |r_j| \sum_{l-h=j, l=\alpha \text{ or } h=\alpha} |\partial \tilde{Q}_n/\partial r_{hl}| \leqq 2n\phi_2(c_n, c_n, \delta_n) \sum_{j=q_n+1}^{n-1} |r_j|$$

which goes to 0 with n by (3.3).

For the second case we use (2.4) so that

$$\left|\frac{\partial \tilde{Q}_k}{\partial r_{hl}}\right| \leqq \int_{d_n}^{\infty} \phi_3(c_n, c_n, x_\alpha; \Sigma'(h, l, \alpha))\, dx_\alpha$$

where $\Sigma'(h, l, \alpha)$ contains some primed elements. Now we compute the conditional distribution of x_α given the first two variates, represented here by c_n (cf. Cramér, page 314). Thus

$$\int_{d_n}^{\infty} \phi_3(c_n, c_n, x_\alpha; \Sigma'(h, l, \alpha))\, dx_\alpha = \phi_2(c_n, c_n, r_{hl})(1 - \Phi((d_n - \mu_n)/\sigma_n))$$

with (suppressing the primes on the elements of Σ')

$$\mu_n = c_n(r_{h\alpha} + r_{l\alpha})/(1 + r_{hl}) \\ \sigma_n^{\ 2} = (1 - r_{hl}^2 - r_{h\alpha}^2 - r_{l\alpha}^2 + 2r_{hl}r_{h\alpha}r_{l\alpha})/(1 - r_{hl}^2) .$$

By assumption

(3.8) $$\max(|r_{h\alpha}|, |r_{l\alpha}|, |r_{hl}|) \leqq \delta_n$$

and using this fact we obtain

$$(d_n - \mu_n)/\sigma_n \geqq b_n(1 - 3\delta_n)/(1 + 2\delta_n)$$

provided n is taken so large that $1 - 3\delta_n > 0$. Summarizing we have

$$\sum_{\alpha=1}^{n} \sum_{j=q_n+1}^{n-1} |r_j| \sum_{l-h=j, |l-\alpha|>q_n/2, |h-\alpha|>q_n/2} |\partial \tilde{Q}_n/\partial r_{hl}| \\ \leqq n^2[1 - \Phi(b_n\gamma_n)]\phi_2(c_n, c_n, \delta_n) \sum_{j=q_n+1}^{n-1} |r_j|$$

where $\gamma_n = (1 - 3\delta_n)/(1 + 2\delta_n)$. Applying (3.1) completes the proof.

The third case is similar to the second one except (3.8) is no longer satisfied. But either $|r_{\alpha l}| < \delta_n$ or $|r_{\alpha h}| < \delta_n$ and, of course, $|r_{hl}| < \delta_n$. Conditioning as before and recalling that $q_n = [n^\beta]$ we obtain for large n

$$\sum_{\alpha=1}^{n} \sum_{j=q_n+1}^{n-1} |r_j| \sum_{l-h=j, |l-\alpha| \leqq q_n/2, |h-\alpha| \leqq q_n/2} |\partial \tilde{Q}_n / \partial r_{hl}| \leqq n^{1+\beta}[1 - \Phi(b_n \hat{\gamma}_n)] \phi_2(c_n, c_n, \delta_n) \sum_{j=q_n+1}^{n-1} |r_j|$$

where $\hat{\gamma}_n = (1 - 3\delta_n - \delta)/(1 + 2\delta)$. This converges to 0 because of (3.2).

Proof of Theorem 1. When $y \geqq x$, Berman's result applies so we consider $y < x$. Then

$$\begin{aligned} P\{M_n \leqq a_n x + b_n, S_n \leqq a_n y + b_n\} &= P\{M_n \leqq c_n\} + \textstyle\sum_{i=1}^{n} (P\{X_\alpha > c_n; X_i \leqq c_n, 1 \leqq i \leqq n, i \neq \alpha\} \\ &\quad - P\{X_\alpha > d_n; X_i \leqq c_n, 1 \leqq i \leqq n, i \neq \alpha\}) . \end{aligned} \tag{3.9}$$

The first term in (3.9) converges to $G(y)$ by Berman's result. Each term in the sum of (3.9) is of the form treated in Lemma 3. Hence we need only find the limit of

$$\sum_{\alpha=1}^{n} \tilde{P}_n(c_n, c_n, \alpha, r_1, \cdots, r_{q_n}, 0, \cdots, 0) - \tilde{P}_n(c_n, d_n, \alpha, r_1, \cdots, r_{q_n}, 0, \cdots, 0) . \tag{3.10}$$

This can be accomplished by using the proof developed in [5] for a strong-mixing sequence. For each n we are essentially considering a Gaussian sequence which is q_n-dependent. If

$$p_n = \frac{n - n^{1-\beta}}{n^{1-2\beta}} , \qquad k_n = n^{1-2\beta}$$

then

(a) $k_n \to \infty$, $p_n \to \infty$
(b) $n/k_n p_n \to 1$, $n = k_n(p_n + q_n)$

and we split the sequence of n random variables into k_n blocks of p_n random variables separated by k_n blocks of q_n random variables. Since $q_n/n = n^{\beta-1} \to 0$ it is easy to show that only the blocks of size p_n need to be considered. These blocks may now be treated as independent of each other because $r_{q_n+1}, r_{q_n+2}, \cdots$ are all equal to zero in (3.10). In order to complete the proof as outlined in [5] we must verify that

$$\lim_{n\to\infty} k_n \sum_{j=1}^{p_n-1} (p_n - j) P\{X_1 > c_n, X_{j+1} > c_n\} = 0 .$$

This can be accomplished by using the mean-value theorem on $P\{X_1 > c_n, X_{j+1} > c_n\}$ as a function of r_j. The details are contained in the proof of Theorem 3 of [5].

Theorem 1 may also be proved when $\sum_n r_n^2 < \infty$. Only minor modifications of the proof given above are required.

4. Concluding remarks. The weak convergence results of Theorem 2 of [5] are also valid. The convergence of the finite dimensional distributions of $(M_{[nt]} - b_n)/a_n$ and $(S_{[nt]} - b_n)/a_n$ can be proved in manner similar to that given above. Even if just the one-dimensional process $M_{[nt]}$ is considered, it is necessary to verify the convergence of the second maximum since this is an essential part of the tightness proof given in Theorem 2 of [5]. We are able to use that tightness proof in this case because it depends on the form of the limit law for S_n and not on the strong-mixing property.

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