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A NOTE ON THE CONVERGENCE OF STABLE AND CLASS L PROBABILITY MEASURES ON BANACH SPACES

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It is proved that the class of stable probability measures and the class of self-decomposible probability measures on Banach spaces are closed under weak convergence.

1. Introduction. Some of the basic properties of stable probability measures on Banach spaces can be found in [4]. It has been shown in [3] that the class L of probability measures on a Banach space is the class of self-decomposible probability measures and every self-decomposible probability measure is infinitely divisible.

Wolfe [6] has shown that if F_n is a sequence of stable distribution functions such that $F_n \to_c F$, then F is stable. He made use of the deep result that the stable distribution functions are absolutely continuous unless they are degenerate, and therefore his proof is not valid for probability measures on Banach spaces. In this note we shall give a proof showing that the class of stable probability measures on a Banach space is closed under weak convergence, using a generalization of Khintchine's convergence of types theorem.

Kubik has shown ([2] page 248) that if F_n is a sequence of distribution functions of class L and $F_n \to_c F$, then F is a distribution function of class L. We shall give a proof of the above for the class L of probability on a Banach space.

Let E be a real separable Banach space, and μ a probability measure on $\mathscr{B}$. For $c > 0$, $T_c\mu$ is the probability measure given by

$$T_c\mu(B) = \mu(B/c)\,,$$

where B is a Borel set of E and $B/c = \{x/c : x \in B\}$. For $c = 0$, $T_c\mu$ is defined to be the degenerate probability measure at zero. A probability measure degenerate at x will be denoted by x. We state the following theorem which is taken from [4].

THEOREM 1. (*Convergence of types theorem*). *Let μ_n and μ be probability measures on $\mathscr{B}$ such that $\mu_n \Rightarrow \mu$, and there exists positive constants a_n's and a sequence $\{x_n\}$ in E such that $T_{a_n}\mu_n * x_n \Rightarrow \mu'$ where μ and μ' are nondegenerate probability measures on $\mathscr{B}(E)$. Then there exists an $a \in R$ and an $x \in E$ such that $\mu' = T_a\mu * x$, $a_n \to a$, $||x_n - x|| \to 0$ as $n \to \infty$.*

The above theorem generalizes Khintchine's convergence of types theorem to Banach spaces.

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THEOREM 2. *Let μ_n be a sequence of stable probability measures on $\mathscr{B}$ converging weakly to μ. Then μ is a stable probability measure.*

PROOF. If μ is degenerate, the proof is trivial. So assume μ is nondegenerate. Let a and b be positive real numbers. Then, by the definition of stable probability measure, there exists a positive real number c_n and an element x_n of E such that

$$T_a \mu_n * T_b \mu_n = T_{c_n} \mu_n * x_n$$

for each n. Since $\mu_n \Rightarrow \mu$, it follows that

$$T_a \mu_n * T_b \mu_n \Rightarrow T_a \mu * T_b \mu .$$

By Theorem 1, there exists a real number $c > 0$ and an element x in E such that $c_n \to c$, $||x_n - x|| \to 0$ and

$$T_a \mu * T_b \mu = T_c \mu * x .$$

Thus μ is stable.

THEOREM 3. *Let μ_n be a sequence in class L of probability measures on $\mathscr{B}$ converging weakly to μ. Then μ is in L.*

PROOF. By the definition of self-decomposible probability measure [3], we have for each c, $0 < c < 1$, a probability measure $\mu_{c,n}$ such that

$$\mu_n = T_c \mu_n * \mu_{c,n} .$$

Since $\mu_n \Rightarrow \mu$, $T_c \mu_n \Rightarrow T_c \mu$. Hence, by ([5] page 58), $\mu_{c,n}$ is compact. Since the class of infinitely divisible probability measures on $\mathscr{B}$ is closed under weak convergence, μ is infinitely divisible. Hence, by ([4] page 78) and the fact that a Banach space contains no nontrivial compact subgroups, the characteristic function of μ does not vanish. Consequently, the characteristic function $\hat{\mu}_{c,n}(f)$, $f \in E^*$, converges pointwise to $\hat{\mu}(f)/\hat{\mu}(cf)$. Therefore, by ([1] page 37) $\mu_{c,n} \Rightarrow \mu_c$. Thus

$$\mu = T_c \mu * \mu_c ,$$

which proves that μ is in L.

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