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# THE PROBABILITY LIMIT IDENTIFICATION FUNCTION EXISTS UNDER THE CONTINUUM HYPOTHESIS

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The paper deals with the existence under the continuum hypothesis of the probability limit identification function, the definition of which has been presented by G. Simons in 1971.

**1. Introduction.** Consider a probability space $(\Omega, \mathscr{A}, P)$ and denote by $\mathscr{E}$ the space of random sequences $\mathscr{X} = (X_1, X_2, \cdots)$, defined on $(\Omega, \mathscr{A}, P)$, whose coordinates converge in probability. Let the probability limit associated with $\mathscr{X} \in \mathscr{E}$ be denoted by $p(\mathscr{X})$, a random variable. A function $f: R^\infty \to R^1$ is called a probability limit identification function (PLIF) on $\mathscr{E}' \subset \mathscr{E}$ if for all $\mathscr{X} \in \mathscr{E}'$,

$$\{\omega \in \Omega : f(\mathscr{X}(\omega)) \neq p(\mathscr{X})(\omega)\} \tag{1}$$

is contained in a $P$-null set of $\mathscr{A}$.

G. Simons (1971) introduced this concept and proved that there exists a PLIF on $\mathscr{E}$ if and only if there exists a PLIF on $\mathscr{E}^*$, the set of $\mathscr{X} \in \mathscr{E}$ whose coordinates are 0–1 variables and whose probability limit is a constant almost surely.

Let $\mathscr{P}' = \{P\mathscr{X}^{-1} : \mathscr{X} \in \mathscr{E}'\}$ (where $P\mathscr{X}^{-1}$ denotes the probability measures induced on $(R^\infty, \mathscr{B}^\infty)$ by $\mathscr{X}$). The purpose of this paper is to prove that there exists a PLIF on $\mathscr{E}' \subset \mathscr{E}$ whenever the cardinality of $\mathscr{P}' \leqq \aleph_1$, where $\aleph_1$ denotes the cardinality of the set of countable ordinal numbers. From this, it easily follows that there exists a PLIF on $\mathscr{E}$ under the continuum hypothesis. The main mathematical tool used in our proof is transfinite induction.

**2. Notation and Definitions.** Let $R^\infty$ be the space of sequences $x = (x_1, x_2, \cdots)$ of real numbers. Consider the usual product topology and denote by $\mathscr{B}^\infty$ its Borel $\sigma$-algebra. Let $\mathscr{P}$ be the set of all probability measures $\mu$ on $\mathscr{B}^\infty$ such that the sequence of coordinates $\{x_1, x_2, \cdots\}$ is convergent in probability $\mu$, i.e.

$$\mathscr{P} = \{\mu : \mu\{x \in R^\infty : |x_n - x_m| > \varepsilon\} \to 0 \quad n, m \to \infty \quad \text{for all} \quad \varepsilon > 0\}. \tag{2}$$

A standard result of measure theory says that for each $\mu \in \mathscr{P}$ there exists a measurable function $G_\mu : (R^\infty, \mathscr{B}^\infty) \to (R^1, \mathscr{B}^1)$ such that $x_n \to G_\mu$ in probability $\mu$, i.e.

$$\mu\{x \in R^\infty : |x_n - G_\mu(x)| > \varepsilon\} \to 0 \qquad n \to \infty \quad \text{for all} \quad \varepsilon > 0. \tag{3}$$

This gives us an opportunity to modify the definition of the PLIF.

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DEFINITION. A map $f: R^\infty \to R^1$ will be called a PLIF on $\mathscr{P}' \subset \mathscr{P}$ if for each $\mu \in \mathscr{P}'$ there exists a $A[\mu] \in \mathscr{B}^\infty$, $\mu(A[\mu]) = 1$ such that

$$A[\mu] \subset \{x \in R^\infty : f(x) = G_\mu(x)\} .$$

Obviously, this definition does not depend on the choice of the $G_\mu$'s.

Let us make some simple observations about PLIF's. Considering $\mathscr{E}' \subset \mathscr{E}$, one can easily see that all PLIF's on $\mathscr{P}' = \{P\mathscr{X}^{-1} : \mathscr{X} \in \mathscr{E}'\}$ are PLIF's on $\mathscr{E}'$ in the sense of Section 1. and that they are $\mu$-measurable with respect to each $\mu \in \mathscr{P}$. Suppose that $f$ is a PLIF on $\mathscr{P}$ and $x = (x_1, x_2, \cdots) \in R^\infty$ is such that $\lim_{n\to\infty} x_n \in R^1$. Then $f(x) = \lim_{n\to\infty} x_n$ since a unit mass at the point $x$, $\varepsilon_x$, belongs to $\mathscr{P}$. On the other hand it follows from the fact that a sequence convergent in probability has a subsequence convergent almost surely, that the elements of $\mathscr{P}$ are supported by the set

$$A = \{x \in R^\infty : x = (x_1, x_2, \cdots) \quad \text{has a finite cluster point}\} .$$

Hence, an arbitrary value is admissible for $f$ at an $x \notin A$.

Now, let us introduce some notation and a definition. Denote the set of natural numbers by $N$ and the set of strictly increasing sequences of its elements by $\mathscr{N}$. Take $p, q \in \mathscr{N}$. We shall say that $p$ is a subsequence of $q$ and write $p \preceq q$ if there is a $k \in N$ such that $p_i = q_{n_i}$ for $i \geqq k$, where $i \leqq n_i \in N$ for all $i$. The relation $\preceq$ is reflexive and transitive.

Take $p = (p_1, p_2, \cdots) \in \mathscr{N}$, $x = (x_1, x_2, \cdots) \in R^\infty$ and put $x(p) = (x_{p_1}, x_{p_2}, \cdots)$, Denote by $C(p)$ the set of those $x \in R^\infty$ for which $\lim_{n\to\infty} x_{p_n}$ exists and is an element of $R^1$. Put $\lim x(p) = \lim_{n\to\infty} x_{p_n}$ for $x \in C(p)$. The following is a reformulation of the known Riesz theorem.

REMARK 1. $\mu \in \mathscr{P}$ if and only if there is a measurable function

$$G : (R^\infty, \mathscr{B}^\infty) \to (R^1, \mathscr{B}^1)$$

such that for each $p' \in \mathscr{N}$ there exists $p \preceq p'$ that

$$\mu\{x \in C(p) : \lim x(p) = G(x)\} = 1$$

holds.

(The measurability of the set $\{x \in C(p) : \lim x(p) = G(x)\}$ follows from the fact that both $G$ and $C(p)$ are $\mathscr{B}^\infty$-measurable.)

**3. The existence of probability limit identification functions.** First, let us prove a lemma that will be utilized in the course of the transfinite construction.

LEMMA 1. *Let $(p^1, p^2, \cdots)$ be a non-increasing finite or countable sequence of elements of $\mathscr{N}$ and consider $\mu \in \mathscr{P}$. Then there exists $p \in \mathscr{N}$ such that*

$$\mu\{x \in C(p) : \lim x(p) = G_\mu(x)\} = 1$$

*and*

$$p \preceq p_n \qquad \textit{for all } n \in N .$$

(The function $G_\mu$ is defined by (3).)

PROOF. The finite case does not give us any troubles. Suppose that $(p^1, p^2, \cdots)$ is a countable sequence. Using a diagonalization argument we are able to find $p' \in \mathscr{N}$ such that $p' \preceq p^n$ for all $n \in N$. The assertion now follows readily from Remark 1.

The following simple lemma will be useful also in the sequel.

LEMMA 2.

$$\text{card}\, \mathscr{P} = \mathfrak{c} \quad (= \text{card}\, R^1)\,.$$

PROOF. Denote by $\varepsilon_x$ a unit mass at a point $x \in R^\infty$. It is very simple to see that

$$\{\varepsilon_{(a,a,\cdots)} : a \in R^1\} \subset \mathscr{P}.$$

hence we have card $\mathscr{P} \geqq \mathfrak{c}$. On the other hand the set of all probabilities on $\mathscr{B}^\infty$ has no more than a continuum of points since it can be topologized as a separable metric space (see [1] page 239).

Now, here is our transfinite construction.

THEOREM. *Consider* $\mathscr{P}' \subset \mathscr{P}$ *such that* card $\mathscr{P}' \leqq \aleph_1$. *Then there is a* PLIF *on* $\mathscr{P}'$.

PROOF. (Some known facts about the ordinal numbers will be used in the course of the proof without any references; for these see [2].)
It is clear that we can restrict ourselves to the case card $\mathscr{P}' = \aleph_1$. Let $\Omega$ be the first uncountable ordinal number. Since

$$\text{card}\,\{\alpha : \alpha \text{ ordinal}, \alpha < \Omega\} = \aleph_1$$

we are able to enumerate the elements of the set $\mathscr{P}'$ by the ordinal numbers $\alpha < \Omega$, i.e.

$$\mathscr{P}' = \{\mu_\alpha : \alpha < \Omega\}\,.$$

Put $G_\beta = G_{\mu_\beta}$ where $G_{\mu_\beta}$ is the probability limit determined by (3). Using a transfinite construction we shall show that there is a net $\{p_\beta : \beta < \Omega\} \subset \mathscr{N}$ such that

$$(4) \qquad \beta < \beta' \Rightarrow p_{\beta'} \preceq p_\beta$$

and

$$(5) \qquad \mu\{x \in C(p_\beta) : \lim x(p_\beta) = G_\beta(x)\} = 1$$

for all $\beta$, $\beta' < \Omega$.

Let $p_0 \in \mathscr{N}$ be a sequence satisfying (5) for $\beta = 0$ and suppose that we have constructed $p_\beta \in \mathscr{N}$ for all ordinal numbers which are smaller than an ordinal $\alpha < \Omega$ such that (4), (5) hold for all $\beta$, $\beta' < \alpha$.
There are two possibilities. If the $\alpha$ is an isolated ordinal number then there is an $\alpha' < \alpha$ such that $\alpha' + 1 = \alpha$. It follows from Lemma 1 that we can find a $p_\alpha \preceq p_{\alpha'}$ such that (5) is true for $\beta = \alpha$. It is easy to see that the elements of the set $\{p_\beta : \beta \leqq \alpha\}$ satisfy conditions (4) and (5) for $\beta$, $\beta' < \alpha + 1$.

If the $\alpha$ is a limit ordinal number then we have an increasing sequence of ordinal numbers $\beta_1 < \beta_2 < \cdots$ such that the $\alpha$ is the smallest ordinal which is larger than each $\beta_n$. Hence the $p_{\beta_n}$'s form a non-increasing sequence. Employing Lemma 1 once more we arrive at some $p_\alpha \in \mathscr{N}$ such that

$$p_\alpha \preceq p_{\beta_n} \qquad \text{for all } n \in N$$

and (5) holds for $\beta = \alpha$.

Take $\beta < \alpha$. Then there is a $n_0 \in N$ such that $\beta < \beta_{n_0} < \alpha$. Hence $p_\alpha \preceq p_\beta$. It follows from these considerations that we have obtained the set $\{p_\beta : \beta < \alpha + 1\}$, the elements of which satisfy (4) and (5) for all $\beta$, $\beta' < \alpha + 1$. Now, the existence of the net $\{p_\beta : \beta < \Omega\}$ that satisfies (4) and (5) for all $\beta$, $\beta' < \Omega$ follows from the principle of transfinite induction.

Let us finish the proof. Put

$$A_\alpha = \{x \in C(p_\alpha) : \lim x(p_\alpha) = G_\alpha(x)\}$$

and define $f(x) = G_\alpha(x)$ for $x \in A_\alpha$ and $\alpha < \Omega$. $f$ is well defined on the set $A = \cup \{A_\alpha, \alpha < \Omega\}$ since if $x \in A_\alpha \cap A_{\alpha'}$ then

$$G_\alpha(x) = \lim x(p_\alpha) = \lim x(p_{\alpha'}) = G_{\alpha'}(x)$$

which follows from the fact that either $p_\alpha \preceq p_{\alpha'}$ or $p_{\alpha'} \preceq p_\alpha$. It is clear that all extensions of $f$ from $A$ to $R^\infty$ are PLIF's on $\mathscr{P}'$.

REMARK 2. Considering the PLIF's in the sense of Section 1 it follows from Theorem and Lemma 2 that under the continuum hypothesis there is a map $f: R^\infty \to R^1$ which is a PLIF on $\mathscr{E}$ for all probability spaces $(\Omega, \mathscr{A}, P)$.

At least two problems with regard to PLIF's seem to be open.

Is there any $\mathscr{B}^\infty$-measurable PLIF?

Is is possible to prove the existence of PLIF's without using the continuum hypothesis?

Also the author has not managed to describe the maximal subset of $R^\infty$ where the values of the PLIF's are determined. Both the topological and geometrical descriptions of the set $\mathscr{P}$ seem to be necessary for these purposes (e.g. to characterize the subsets of $R^\infty$ which are contained in a $\mu$-null set of $\mathscr{B}^\infty$ for each $\mu \in \mathscr{P}$).

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