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A CLASSICAL LIMIT THEOREM WITHOUT INVARIANCE OR REFLECTION[1]

BY HENRY TEICHER

Rutgers University

A sequence of stopping or first passage times is utilized to derive the limiting distribution of the maximum of partial sums of independent, identically distributed random variables with mean zero and finite variance and concomitantly the limit distribution of the stopping times themselves. The result, due to Erdös and Kac, first appeared in the paper which launched the extremely fruitful invariance principle; reflection enters in the calculations relating to the choice of a specific distribution for the $\{X_n\}$.

Moreover, it is noted when the $\{X_n\}$ are i.i.d. with mean $\mu>0$ and variance $\sigma^2<\infty$ that $\max_{1\le j\le n} S_j/j^\alpha$ has a limiting standard normal distribution for any α in $[0, 1)$.

1. Summary. A sequence of stopping or first passage times is utilized to derive the limiting distribution of the maximum of partial sums of independent, identically distributed (i.i.d.) random variables with mean zero and finite variance and concomitantly the limit distribution of the stopping times themselves. The result, due to Erdös and Kac, first appeared in the paper [2] which launched the extremely fruitful invariance principle; reflection enters in the calculations relating to the choice of a specific distribution for the $\{X_n\}$ (see [1] page 202).

Moreover, it is noted when the $\{X_n\}$ are i.i.d. with mean $\mu > 0$ and variance $\sigma^2 < \infty$ that

$$\frac{\max_{1\le j\le n} S_j/j^\alpha - \mu n^{1-\alpha}}{\sigma n^{\frac{1}{2}-\alpha}}$$

has a limiting standard normal distribution for any α in $[0, 1)$, a related fact which is virtually implicit in Siegmund [5], Heyde [3].

2. Mainstream. Let $\{X_n, n \geqq 1\}$ be i.i.d. random variables with partial sums $\{S_n, n \geqq 1\}$ and define

$$(1) \qquad T_c^* = \inf\{j \geqq 1 : S_j > c\}, \qquad c > 0 .$$

THEOREM 1. *If* $\{X_n, n \geqq 1\}$ *are* i.i.d. *random variables with mean zero, finite variance* σ^2 *and* T_c^* *is as in* (1), *then*

$$(2) \qquad \lim_{c\to\infty} P\left\{T_{cx}^* > \frac{c^2 y}{\sigma^2}\right\} = \lim_{n\to\infty}' P\{\max_{1\le j\le ny} S_j \leqq \sigma x n^{\frac{1}{2}}\}$$
$$= 2\Phi(x/y^{\frac{1}{2}}) - 1 , \qquad x > 0, y > 0$$

where Φ *is the standard normal distribution function.*

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Note. For $y = 1$, the right side of (2) is the positive normal distribution while for $x = 1$ it is one minus the positive stable distribution function of characteristic exponent $\frac{1}{2}$.

PROOF. The first equality of (2) is transparent upon setting $c = \sigma n^{\frac{1}{2}}$ and in proving the second it may clearly be supposed that $y = 1 = \sigma$. Now $T_n = T_n(x) = \inf\{j \geqq 1 : S_j > xn^{\frac{1}{2}}\}$ is a finite stopping time for any positive integer n and

$$\begin{aligned} P\{T_n(x) \leqq n\} &= P\{\max_{1\leqq j\leqq n} S_j > xn^{\frac{1}{2}}\} \\ &= P\{S_n > xn^{\frac{1}{2}}\} + P\{S_n \leqq xn^{\frac{1}{2}}, \max_{1\leqq j\leqq n} S_j > xn^{\frac{1}{2}}\} \\ &\leqq P\{S_n > xn^{\frac{1}{2}}\} + \textstyle\sum_{j=1}^{n} P\{T_n = j, S_n - S_j \leqq 0\}\,. \end{aligned}$$

Hence via independence and identical distributions

$$\tag{3} \begin{aligned} \tfrac{1}{2}P\{\max_{1\leqq j\leqq n} S_j > xn^{\frac{1}{2}}\} &\leqq P\{S_n > xn^{\frac{1}{2}}\} \\ &\quad + \textstyle\sum_{j=1}^{n} P\{T_n = j\}[P\{S_{n-j} \leqq 0\} - \tfrac{1}{2}]\,. \end{aligned}$$

In order to evaluate the limit on the right side of (3) and deduce therefrom

$$\tag{4} \limsup_n P\{\max_{1\leqq j\leqq n} S_j > xn^{\frac{1}{2}}\} \leqq 2[1 - \Phi(x)]$$

it will be shown that $a_{n,j} = P\{T_n = j\}$ satisfies

$$\tag{5} a_{n,n-j} = o(1) \qquad \text{as } n \to \infty \text{ for } j = 0, 1, \cdots.$$

Conceding (5) for the moment, the fact that $b_n = P\{S_n \leqq 0\} - \frac{1}{2} = o(1)$ via the central limit theorem, implies

$$\textstyle\sum_{j=1}^{n} a_{n,j}\, b_{n-j} = \sum_{j=0}^{n-1} a_{n,n-j}\, b_j = o(1)$$

in view of the well-known fact [4] that a Toeplitz matrix (clearly, $\sum_{j=0}^{n-1} a_{n,n-j} \leqq 1$) transforms null sequences into null sequences. Thus, under the proviso (5), formula (3) guarantees (4).

The proviso (5) can be established by noting that

$$\tag{6} \begin{aligned} a_{n,n-j} &= P\{T_n = n - j\} = P\{\max_{i<n-j} S_i \leqq xn^{\frac{1}{2}}, S_{n-j} > xn^{\frac{1}{2}}\} \\ &\leqq P\{\max_{i<n-j} (S_i - S_{n-j}) < 0\} \\ &= P\{\max(-X_{n-j}, -X_{n-j} - X_{n-j-1}, \cdots, -X_{n-j} - \cdots - X_2) < 0\} \\ &= P\{\max_{i<n-j} (-S_i) < 0\} = P\{\min_{i<n-j} S_i > 0\} = o(1) \end{aligned}$$

since the hypothesis ensures $P\{\liminf S_n' = -\infty\} = 1$.

To obtain the reverse inequality of (4) for the lim inf, observe that for every $\varepsilon > 0$

$$\tag{7} \begin{aligned} &P\{\max_{1\leqq j\leqq n} S_j > xn^{\frac{1}{2}}\} \\ &\quad = P\{S_n > (x + \varepsilon)n^{\frac{1}{2}}\} + P\{S_n \leqq (x + \varepsilon)n^{\frac{1}{2}}, \max_{1\leqq j\leqq n} S_j > xn^{\frac{1}{2}}\} \\ &\quad \geqq P\{S_n > (x + \varepsilon)n^{\frac{1}{2}}\} + \textstyle\sum_{j=1}^{n} P\{T_n = j, S_n \leqq S_j, X_j \leqq \varepsilon n^{\frac{1}{2}}\} \\ &\quad = P\{S_n > (x + \varepsilon)n^{\frac{1}{2}}\} + \textstyle\sum_{j=1}^{n} P\{T_n = j, S_n \leqq S_j\} \\ &\qquad - \textstyle\sum_{j=1}^{n} P\{T_n = j, S_n \leqq S_j, X_j > \varepsilon n^{\frac{1}{2}}\} \end{aligned}$$

whence, denoting the term on the extreme right by q_n,

$$q_n \leqq \sum_{j=1}^n P\{X_j > \varepsilon n^{\frac{1}{2}}\} = nP\{X_1^2 > \varepsilon^2 n\} = o(1)\,, \qquad \varepsilon > 0$$

since $EX_1^2 < \infty$. Thus, it follows from (7) as earlier that

$$\liminf_n P\{\max_{1\leqq j\leqq n} S_j > xn^{\frac{1}{2}}\} \geqq 2[1 - \Phi(x+\varepsilon)]\,, \qquad \varepsilon > 0$$

and letting $\varepsilon \to 0$

$$\liminf_n P\{\max_{1\leqq j\leqq n} S_j > xn^{\frac{1}{2}}\} \geqq 2[1 - \Phi(x)]$$

which, in conjunction with (4) establishes the latter portion of (2).

THEOREM 2. *If* $\{X_n, n \geqq 1\}$ *are* i.i.d. *random variables with mean* $\mu > 0$ *and variance* $\sigma^2 < \infty$, *then for any* α *in* $[0, 1)$

$$\lim_{n\to\infty} P\left\{\max_{1\leqq j\leqq n} \frac{S_j}{j^\alpha} - \mu n^{1-\alpha} \leqq x\sigma n^{\frac{1}{2}-\alpha}\right\} = \Phi(x)\,. \tag{8}$$

PROOF. For any $x \neq 0$ define the positive quantity $c = c_n$ by

$$\frac{cn^{\alpha-\frac{1}{2}} - \mu n^{\frac{1}{2}}}{\sigma} = \frac{cn^\alpha - n\mu}{\sigma n^{\frac{1}{2}}} = -x$$

whence, inverting

$$n = \left(\frac{c}{\mu}\right)^{(1-\alpha)^{-1}} + \frac{x\sigma(c/\mu)^{(2(1-\alpha))^{-1}}}{\mu(1-\alpha)} + O(1)\,. \tag{9}$$

Then, if $T_c = \inf\{j \geqq 1 : S_j > cj^\alpha\}$, $c > 0$

$$\begin{aligned}
&P\{\max_{1\leqq j\leqq n} S_j/j^\alpha - \mu n^{1-\alpha} > -x\sigma n^{\frac{1}{2}-\alpha}\} \\
&\quad = P\{\max_{1\leqq j\leqq n} S_j/j^\alpha > c\} = P\{T_c \leqq n\} \\
&\quad = P\left\{\frac{T_c - (c/\mu)^{(1-\alpha)^{-1}}}{\sigma\mu^{-1}(1-\alpha)^{-1}(c/\mu)^{(2(1-\alpha))^{-1}}} \leqq \frac{n - (c/\mu)^{(1-\alpha)^{-1}}}{\sigma\mu^{-1}(1-\alpha)^{-1}(c/\mu)^{(2(1-\alpha))^{-1}}}\right\} \\
&\quad \to \Phi(x)
\end{aligned}$$

in view of (9) and Theorem 1 of [5] and this is tantamount to (8).

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RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY