The Annals of Probability
1973, Vol. 1, No. 4, 690–698

ON EQUIVALENCE OF PROBABILITY MEASURES[1]

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Let H be a real and separable Hilbert space, Γ the Borel σ-field of H sets, and μ_1 and μ_2 two probability measures on (H, Γ). Several sufficient conditions for equivalence (mutual absolute continuity) of μ_1 and μ_2 are obtained in this paper. Some of these results do not require that μ_1 and μ_2 be Gaussian. The conditions obtained are applied to show equivalence for some specific measures when H is $L_2[T]$.

0. Introduction. Conditions for equivalence of probability measures on the Borel σ-field of a Hilbert space have been the subject of much research during recent years [4]–[8], [13]–[17]. For two Gaussian measures μ_1, μ_2, either μ_1 and μ_2 are equivalent (mutually absolutely continuous, denoted by $\mu_1 \sim \mu_2$) or else μ_1 and μ_2 are orthogonal ($\mu_1 \perp \mu_2$), and general necessary and sufficient conditions for equivalence have been obtained (e.g., [13]).

When one or both of the measures is not Gaussian, few conditions for equivalence are known. Moreover, even when both measures *are* Gaussian the general conditions for equivalence are often difficult to verify, requiring one to prove existence of a Hilbert–Schmidt operator with prescribed spectral properties.

In this paper, several conditions for equivalence are given. Most of these conditions are stated in terms of sample function properties. Several results do not require that both measures be Gaussian; when the two measures *are* Gaussian, the sufficient conditions given here may often be easier to verify than those previously obtained.

1. Definitions and problem statement. Let H be a real and separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and Borel σ-field Γ. Let (Ω, β, P) be a probability space, and suppose that S and N are β/Γ measurable mappings (e.g., $S^{-1}(A) \in \beta$ for all $A \in \Gamma$) of Ω into H. Following Mourier [10], a β/Γ measurable mapping of Ω into H will be called a "random element" in H. Let $(H \times H, \Gamma \times \Gamma)$ be the usual product measurable space; $\Gamma \times \Gamma$ is the smallest σ-field containing all measurable rectangles $A \times B$, $A, B \in \Gamma$.

Define the map $(S, N) : \Omega \to H \times H$ by $(S, N)(\omega) = (S(\omega), N(\omega))$; this map is $\beta/\Gamma \times \Gamma$ measurable, since for A, B, in Γ

$$\{\omega : (S(\omega), N(\omega)) \in A \times B\} = \{\omega : S(\omega) \in A\} \cap \{\omega : N(\omega) \in B\} .$$

Received November 15, 1970; revised November 15, 1972.

[1] This research was partially supported by the National Science Foundation under Grant GU-2059 and by the U.S. Air Force Office of Scientific Research under Contract No. AFOSR-68-1415.

AMS 1970 *subject classifications.* Primary 60G30.

Key words and phrases. Measures on Hilbert space, absolute continuity, Gaussian measures.

Hence (S, N) induces from P a measure $\mu_{S,N}$ on $(H \times H, \Gamma \times \Gamma)$ defined by $\mu_{S,N}[C] = P\{\omega: (S(\omega), N(\omega)) \in C\}$, C in $\Gamma \times \Gamma$. Further, S and N induce measures μ_S and μ_N, respectively, on Γ; e.g., for A in Γ, $\mu_S(A) = P[S^{-1}(A)]$. S and N are independent if and only if $\mu_{S,N} = \mu_S \otimes \mu_N$, where $\mu_S \otimes \mu_N[A \times B] = \mu_S(A)\mu_N(B)$, A, B, in Γ.

Consider the map $f\colon H \times H \to H$, $f(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y}$. f is clearly $\Gamma \times \Gamma/\Gamma$ measurable. Hence $S + N$ is β/Γ measurable, and induces a measure μ_{S+N} from P,

$$\mu_{S+N}[A] = \mu_{S,N}[f^{-1}(A)] = P\{\omega: (S(\omega), N(\omega)) \in f^{-1}(A)\}\,.$$

Let $\mathbf{v}$ denote any fixed element of H. Define $f_{\mathbf{v}}\colon H \to H$ by $f_{\mathbf{v}}(\mathbf{y}) = \mathbf{v} + \mathbf{y}$; $f_{\mathbf{v}}$ is the section of f at $\mathbf{v}$, therefore is Γ/Γ measurable and for A in Γ we define the measure $\mu_{\mathbf{v}+N}$ by $\mu_{\mathbf{v}+N}(A) = \mu_N[f_{\mathbf{v}}^{-1}(A)]$.

The problem considered in this paper is that of obtaining sufficient conditions for equivalence of μ_N and μ_{S+N}. Several of the conditions obtained are stated in terms of the measures $\mu_{\mathbf{v}+N}$, $\mathbf{v} \in H$.

2. Covariance operators; Gaussian measures. Suppose $E\|S(\omega)\|^2 < \infty$; then there exists [10] an element $\mathbf{m}_S$ of H and an operator R_S in H such that $\langle \mathbf{m}_S, \mathbf{u}\rangle = E\langle S(\omega), \mathbf{u}\rangle$, $\langle R_S\mathbf{u}, \mathbf{v}\rangle = E\langle S(\omega) - \mathbf{m}_S, \mathbf{u}\rangle \cdot \langle S(\omega) - \mathbf{m}_S, \mathbf{v}\rangle$, for all $\mathbf{u}$, $\mathbf{v}$ in H. The operator R_S is a "covariance" operator; i.e., it is linear, bounded, nonnegative, self-adjoint, and trace-class.

If $E\|N(\omega)\|^2 < \infty$, then S and N have a "cross-covariance" operator $R_{SN}\colon H \to H$, defined by $\langle R_{SN}u, v\rangle = E\langle S(\omega) - \mathbf{m}_S, \mathbf{v}\rangle\langle N(\omega) - \mathbf{m}_N, \mathbf{u}\rangle$ for all $\mathbf{u}$, $\mathbf{v}$ in H; moreover, $R_{SN} = R_S^{\frac{1}{2}} V R_N^{\frac{1}{2}}$ for a bounded linear operator V, with $\|V\| \leqq 1$ [2]. R_{SN} is thus trace-class, and $R_{NS} = R_{SN}^*$, where * denotes adjoint.

S is said to be a Gaussian element, and μ_S a Gaussian measure, if $\langle S, \mathbf{u}\rangle$ is a Gaussian random variable for all $\mathbf{u}$ in H. μ_S then has a covariance operator R_S and a mean element m_S, and $E\|S(\omega)\|^2 < \infty$ [10].

When μ_N and μ_{S+N} are Gaussian measures, a number of conditions for equivalence have been given by various authors; the conditions most useful for our purposes are the following ([13], [16] Chapters 19 and 20).

LEMMA 1. *$\mu_N \sim \mu_{S+N}$ if and only if*

(a) *$\mathbf{m}_S$ is in the range of $R_N^{\frac{1}{2}}$;*

(b) *$R_{S+N} = R_N^{\frac{1}{2}} W R_N^{\frac{1}{2}} + R_N$, where W is a Hilbert–Schmidt operator that does not have -1 as an eigenvalue.*

In dealing with the equivalence of Gaussian measures, one often needs the following results on the range of square roots of covariance operators [1]:

LEMMA 2. *Suppose R_1 and R_2 are covariance operators. Then*

(1) range $(R_1^{\frac{1}{2}}) \subset$ range $(R_2^{\frac{1}{2}})$ *if and only if the following (equivalent) conditions are satisfied:*

(a) *$R_1^{\frac{1}{2}} = R_2^{\frac{1}{2}} G$ for G linear and bounded;*

(b) *$R_1 = R_2^{\frac{1}{2}} Q R_2^{\frac{1}{2}}$ for Q linear and bounded;*

(c) *$\langle R_1\mathbf{u}, \mathbf{u}\rangle \leqq k\langle R_2\mathbf{u}, \mathbf{u}\rangle$ for all $\mathbf{u}$ in H and some finite scalar k.*

(2) range $(R_1^{\frac{1}{2}})$ = range $(R_2^{\frac{1}{2}})$ *if and only if the following (equivalent) conditions are satisfied*:

(a) $R_1^{\frac{1}{2}} = R_2^{\frac{1}{2}}G$, *for G linear and bounded with bounded inverse*
(b) $R_1 = R_2^{\frac{1}{2}}QR_2^{\frac{1}{2}}$ *for Q linear and bounded with bounded inverse.*

Note that one can use part 2b of Lemma 2 to restate part b of Lemma 1 as follows:

(i) Range $(R_N^{\frac{1}{2}})$ = range $(R_{S+N}^{\frac{1}{2}})$, and
(ii) $R_{S+N} = R_N^{\frac{1}{2}}(I + W)R_N^{\frac{1}{2}}$, with W Hilbert–Schmidt.

3. Applications. In most applications H is $L_2[T]$ (Lebesgue measure) for some compact interval T of the real line. In such cases, S and N are random functions corresponding to measurable stochastic processes (S_t), (N_t), whose sample functions belong almost surely to $L_2[T]$. It is easy to verify that a measurable stochastic process (S_t) with sample functions a.s. in $L_2[T]$ is a β/Γ measurable function; one uses the facts that $\langle S, \mathbf{u}\rangle$ is $\beta/B[R]$ measurable ($B[R] \equiv$ Borel sets of the real line) for all $\mathbf{u}$ in $L_2[T]$, and that Γ is the smallest σ-field such that all bounded linear functionals on $L_2[T]$ are $\Gamma/B[R]$ measurable.

4. Equivalence conditions for independent S, N. In this section it is assumed that $\mu_{S,N} = \mu_S \otimes \mu_N$.

THEOREM 1. *If* $\mu_{\mathbf{v}+N} \sim \mu_N$ a.e. $d\mu_S(\mathbf{v})$, *then* $\mu_{S+N} \sim \mu_N$.

PROOF. For A in Γ,

$$(*)\qquad \begin{aligned} \mu_{S+N}(A) &= \mu_{S,N}[f^{-1}(A)] = \mu_S \otimes \mu_N[f^{-1}(A)] \\ &= \int_H \mu_N[f_{\mathbf{v}}^{-1}(A)]\, d\mu_S(\mathbf{v}) \\ &= \int_H \mu_{\mathbf{v}+N}(A)\, d\mu_S(\mathbf{v})\,. \end{aligned}$$

Suppose $\mu_{\mathbf{v}+N} \sim \mu_N$ a.e. $d\mu_S(\mathbf{v})$. Then, $\mu_N(A) = 0 \Rightarrow \mu_{\mathbf{v}+N}(A) = 0$ a.e. $d\mu_S(\mathbf{v}) \Rightarrow \mu_{S+N}(A) = 0$, from (*). Also, $\mu_{S+N}(A) = 0 \Rightarrow \mu_{\mathbf{v}+N}(A) = 0$ a.e. $d\mu_S(\mathbf{v}) \Rightarrow \mu_N(A) = 0$. Hence $\mu_{S+N} \sim \mu_N$. (Note that $\mu_{\mathbf{v}+N} \perp \mu_N$ a.e. $d\mu_S(\mathbf{v})$ does not imply $\mu_{S+N} \perp \mu_N$, since the set $A_{\mathbf{v}}$ satisfying $\mu_{\mathbf{v}+N}(A_{\mathbf{v}}) = 1 - \mu_N(A_{\mathbf{v}}) = 0$ can vary with $\mathbf{v}$.)

Although the above result assumes that $\mu_{S,N} = \mu_S \otimes \mu_N$, it can be applied to yield conditions for equivalence when S and N are not independent. For example, suppose that $N = N_1 + N_2$, where N_1 and N_2 are β/Γ measurable transformations inducing measures μ_{N_1} and μ_{N_2}. If $\mu_{N_1,N_2} = \mu_{N_1} \otimes \mu_{N_2}$ and $\mu_{S,N_2} = \mu_S \otimes \mu_{N_2}$, one can use the above result to determine if $\mu_N \sim \mu_{N_2}$ and $\mu_{S+N} \sim \mu_{N_2}$. If both these equivalences hold, then $\mu_{S+N} \sim \mu_N$. This simple modification should be useful in many applications, especially in practical signal detection problems, where the noise usually contains an additive Gaussian component that is independent of the signal and of the remainder of the noise.

In order to apply Theorem 1, one must first determine sufficient conditions for $\mu_{\mathbf{v}+N} \sim \mu_N$, $\mathbf{v} \in H$. Several such conditions are known when μ_N is Gaussian, and are utilized to obtain the following corollary.

COROLLARY. *Suppose that μ_N is Gaussian with $\|\mathbf{m}_N\| = 0$, and $\mu_{S,N} = \mu_S \otimes \mu_N$. Then $\mu_{S+N} \sim \mu_N$ if any of the following conditions is satisfied:*

(1) $\mathbf{v} \in$ range $(R_N^{\frac{1}{2}})$ a.e. $d\mu_S(\mathbf{v})$;

(2) $E\|S(\omega)\|^2 < \infty$, $\mathbf{m}_S \in$ range $(R_N^{\frac{1}{2}})$, *and* $R_S = R_N^{\frac{1}{2}} W R_N^{\frac{1}{2}}$ *with W trace-class;*

(3) *H is $L_2[T]$ for a compact interval T, μ_N is the measure induced by a measurable mean-square-continuous stationary stochastic process with rational spectral density $\hat{R}_N$, $E\|S(\omega)\|^2 < \infty$, and*

(a) *there exists a rational spectral density function $\hat{R}_0$ such that $\langle R_S\mathbf{u}, \mathbf{u}\rangle \leqq k \int_{-\infty}^{\infty} \hat{R}_0(\lambda)|\hat{\mathbf{u}}(\lambda)|^2 \, d\lambda$ for all $\mathbf{u}$ in H ($\hat{\mathbf{u}}$ is the Fourier transform of $\mathbf{u}$, $\mathbf{u}(t) \equiv 0$ for $t \notin T$) and some finite scalar k, with*

$$\int_{-\infty}^{\infty} \frac{\hat{R}_0(\lambda)}{\hat{R}_N(\lambda)} \, d\lambda < \infty \, ,$$

(b) $\mathbf{m}_S \in$ range $(R_N^{\frac{1}{2}})$.

(4) *H is as in (3), μ_N is induced by a measurable mean-square continuous stationary stochastic process with a spectral density function, $\hat{R}_N$, and*

$$\int_{-\infty}^{\infty} \frac{|\hat{\mathbf{v}}(\lambda)|^2}{\hat{R}_N(\lambda)} \, d\lambda < \infty \qquad \text{a.e.} \quad d\mu_S(\mathbf{v}) \, .$$

PROOF. (1) From Lemma 1, $\mathbf{v}$ in range $(R_N^{\frac{1}{2}}) \Leftrightarrow \mu_{\mathbf{v}+N} \sim \mu_N$.

(2) Let $\{\lambda_n\}$, $\{\mathbf{e}_n\}$ be the nonzero eigenvalues and an associated set of orthonormal eigenvectors of R_N. Then

$$E \sum_1^N \frac{\langle S(\omega), \mathbf{e}_n\rangle^2}{\lambda_n} = \sum_1^N \left\{\frac{\langle R_S \mathbf{e}_n, \mathbf{e}_n\rangle}{\lambda_n} + \frac{\langle \mathbf{m}_S, \mathbf{e}_n\rangle^2}{\lambda_n}\right\}$$

and (2) follows, since $S(\omega) \in \overline{\text{range}\,(R_N^{\frac{1}{2}})}$ almost surely.

(3) According to Hájek [7],

$$\int_{-\infty}^{\infty} \frac{\hat{R}_0(\lambda)}{\hat{R}_N(\lambda)} \, d\lambda < \infty$$

implies that $R_0 = R_N^{\frac{1}{2}} W R_N^{\frac{1}{2}}$, W trace-class. By Lemma 2, the condition $\langle R_S\mathbf{u}, \mathbf{u}\rangle \leqq k \int_{-\infty}^{\infty} \hat{R}_0(\lambda)|\hat{\mathbf{u}}(\lambda)|^2 \, d\lambda$ implies that range $(R_S^{\frac{1}{2}}) \subset$ range $(R_0^{\frac{1}{2}})$, and this last condition implies (Lemma 2) that $R_S^{\frac{1}{2}} = R_0^{\frac{1}{2}} G$ for G bounded; while the representation $R_0 = R_N^{\frac{1}{2}} W R_N^{\frac{1}{2}}$ with W trace-class implies $R_0^{\frac{1}{2}} = R_N^{\frac{1}{2}} Q$, Q Hilbert–Schmidt. Hence $R_S = R_N^{\frac{1}{2}} Q G G^* Q^* R_N^{\frac{1}{2}}$, and QGG^*Q^* is trace-class, so that (3) follows from (2).

(4) The integrability condition implies [9] that $\mathbf{v} \in$ range $(R_N^{\frac{1}{2}})$ a.e. $d\mu_S(\mathbf{v})$. To see this, note that by Lemma 2, $\mathbf{v}$ is in range $(R_N^{\frac{1}{2}})$ if and only if there exists a finite scalar k such that $\langle \mathbf{v}, \mathbf{u}\rangle^2 \leqq k\langle R_N\mathbf{u}, \mathbf{u}\rangle$, all $\mathbf{u}$ in H. The condition given

in (4) implies that

$$\langle \mathbf{v}, \mathbf{u}\rangle^2 \leqq \int_{-\infty}^{\infty} \frac{|\hat{\mathbf{v}}(\lambda)|^2}{\hat{R}_N(\lambda)} d\lambda \langle R_N \mathbf{u}, \mathbf{u}\rangle \qquad \text{a.e.} \quad d\mu_S(\mathbf{v}),$$

so that $\mathbf{v} \in$ range $(R_N^{\frac{1}{2}})$ a.e. $d\mu_S(\mathbf{v})$.

Kadota and Shepp [8] have proved part (1) of the above corollary for $H = L_2[0, b]$, $b < \infty$, under the additional assumptions that μ_N has continuous covariance function and R_N is strictly positive definite. An inspection of their proof shows that it requires the additional hypothesis that $(S_t + N_t)$ is continuous in probability. Part (1) of the corollary was previously stated for μ_{S+N} absolutely continuous with respect to μ_N ($\mu_{S+N} \ll \mu_N$) by Pitcher [12].

The proof of Theorem 1 also yields the following result: If $\mu_{\mathbf{x}+N} \ll \mu_N$ a.e. $d\mu_S(\mathbf{x})$, then $\mu_{S+N} \ll \mu_N$; if $\mu_N \ll \mu_{\mathbf{x}+N}$ a.e. $d\mu_S(\mathbf{x})$, then $\mu_N \ll \mu_{S+N}$.

In most practical signal detection problems, one can assume the presence of an additive component in the noise process (N) that is Gaussian, stationary, zero-mean, and independent of the remainder of the noise and of the signal process (S). Moreover, this component has a spectral density function that is constant out to a "very high frequency." This process, due to thermal noise in electronic equipment, is commonly called "white noise," and one can show formally that orthogonality is impossible when the "very high frequency" is assumed to be infinite. Parts (3) and (4) of the above corollary give mathematical meaning to this result. The result given in (4) has previously been used by Root ([16] Chapter 20) to conclude that orthogonality is impossible when white noise is present and the signal process consists of a single (known) function.

Theorem 1 is stated for translates of measures on a real separable Hilbert space, H. This is the most useful form, since for a Gaussian measure μ on H necessary and sufficient conditions for equivalence of μ to a translate of μ are wellknown. However, inspection of the definitions and the proof shows that Theorem 1 holds for any pair of probability spaces (X, β, μ_S), $(Y, \mathscr{F}, \mu_N)$, and any $\beta \times \mathscr{F}/\mathscr{F}$ measurable function f; i.e., $(\mu_S \otimes \mu_N) \circ f^{-1} \sim \mu_N$ if $\mu_N \circ f_{\mathbf{x}}^{-1} \sim \mu_N$ a.e. $d\mu_S(\mathbf{x})$, where $f_{\mathbf{x}}(\mathbf{y}) = f(\mathbf{x}, \mathbf{y})$.

5. Dependent S, N. When the assumption that $\mu_{S,N} = \mu_S \otimes \mu_N$ is not valid, the equivalence of μ_{S+N} and μ_N is not implied by $\mu_{\mathbf{v}+N} \sim \mu_N$ a.e. $d\mu_S(\mathbf{v})$. As a counter-example, suppose that μ_N is Gaussian and that $S(\omega) = kR_N^{1/p}N(\omega)$, for p a positive integer and some scalar k. S is then Gaussian, and for $p = 1$ or 2, $S(\omega) \in$ range $(R_N^{1/p})$ a.e. $dP(\omega)$, implying $\mu_{\mathbf{v}+N} \sim \mu_N$ a.e. $d\mu_S(\mathbf{v})$. One has $R_{S+N} = R_N^{\frac{1}{2}}[k^2R_N^{2/p} + 2kR_N^{1/p} + I]R_N^{\frac{1}{2}}$. From Lemma 1 and Lemma 2, $\mu_{S+N} \perp \mu_N$ if and only if $k^2R_N^{2/p} + 2kR_N^{1/p} + I$ has zero as an eigenvalue; this will occur if and only if $k = -\lambda_i^{-1/p}$ for some nonzero eigenvalue λ_i of R_N. As a specific example, $\mu_{S+N} \perp \mu_N$ if H is $L_2[0, 1]$, μ_N is Wiener measure, and

$$S_t(\omega) = -((2n+1)^2/4)\pi^2 \int_0^t \int_u^1 N_v(\omega)\, dv\, du = -((2n+1)^2/4)\pi^2[R_N N]_t(\omega),$$

for any integer n.

6. Dependent S, N; Gaussian measures. In this section, it is assumed that μ_N, μ_S, and μ_{S+N} are Gaussian. As previously noted, this implies the existence of covariance operators R_N, R_S, and R_{S+N} and mean elements $\mathbf{m}_N$, $\mathbf{m}_S$, and $\mathbf{m}_{S+N}$; we assume $\|\mathbf{m}_N\| = 0$. Let $\mu_{(S+N)}$ denote the Gaussian measure defined by

$$\mu_{(S+N)}(A) = \int_{f^{-1}(A)} d\mu_S \otimes \mu_N \qquad \text{for } A \in \Gamma$$

$(f(\mathbf{u}, \mathbf{v}) = \mathbf{u} + \mathbf{v})$. $\mu_{(S+N)}$ has covariance operator $R_S + R_N$ and mean element $\mathbf{m}_S$. Finally, we assume that the range of R_N is dense in H; in cases where this is not satisfied, one can obtain the results given in this section by defining H to be the closure of range (R_N). We proceed to obtain some new sufficient conditions for equivalence of μ_{S+N} and μ_N.

LEMMA 3. *Suppose* $\mu_S\{\text{range}\,(R_N^{\frac{1}{2}})\} = 1$. *Then* $R_S = R_N^{\frac{1}{2}} W R_N^{\frac{1}{2}}$ *for a covariance operator* W, *and* $\mathbf{m}_S \in \text{range}\,(R_N^{\frac{1}{2}})$.

PROOF. By the Corollary to Theorem 1, $\mu_S\{\text{range}\,(R_N^{\frac{1}{2}})\} = 1 \Rightarrow \mu_{(S+N)} \sim \mu_N$; from Lemma 1 this implies $R_S = R_N^{\frac{1}{2}} W R_N^{\frac{1}{2}}$, W Hilbert–Schmidt, and $\mathbf{m}_S \in \text{range}\,(R_N^{\frac{1}{2}})$. Let $g \equiv R_N^{-\frac{1}{2}}\mathbf{m}_S$.

Since μ_S is Gaussian, $\langle S, \mathbf{u}\rangle$ is a Gaussian random variable for all $\mathbf{u} \in H$. Define $Y\colon \Omega \to H$ by

$$\begin{aligned} Y(\omega) &= R_N^{-\frac{1}{2}} S(\omega) && \text{for} \quad S(\omega) \in \text{range}\,(R_N^{\frac{1}{2}}) \\ &= \mathbf{0} && \text{for} \quad S(\omega) \notin \text{range}\,(R_N^{\frac{1}{2}})\,. \end{aligned}$$

For any $A \in \Gamma$, $R_N^{\frac{1}{2}}[A] \equiv \{\mathbf{x} : \mathbf{x} = R_N^{\frac{1}{2}}\mathbf{u} \text{ for } \mathbf{u} \in A\}$ is an element of Γ, since $R_N^{\frac{1}{2}}$ is one-to-one [11]. Moreover, $Y^{-1}(A) = \{\omega : S(\omega) \in R_N^{\frac{1}{2}}[A]\}$ if $\mathbf{0} \in A$; if $\mathbf{0} \in A$, then $Y^{-1}(A) = \{\omega : S(\omega) \in R_N^{\frac{1}{2}}[A]\} \cup \{\omega : S(\omega) \notin \text{range}\,(R_N^{\frac{1}{2}})\}$. In either case, $Y^{-1}(A) \in \beta$, so that Y is β/Γ measurable and thus induces from P a measure μ_Y on (H, Γ). For any $\mathbf{u}$ in H we show that $\langle Y, \mathbf{u}\rangle$ is a Gaussian random variable; first, note that there exists $\{\mathbf{u}_n\}$ such that $R_N^{\frac{1}{2}}\mathbf{u}_n \to \mathbf{u}$. Using $R_S = R_N^{\frac{1}{2}} W R_N^{\frac{1}{2}}$, W bounded, and $\mathbf{m}_S = R_N^{\frac{1}{2}}\mathbf{g}$, one has that $\langle S, \mathbf{u}_n\rangle \to \langle Y, \mathbf{u}\rangle$ almost surely and in $L_2(\Omega, \beta, P)$. Hence $\langle Y, \mathbf{u}\rangle$ in a Gaussian rv for all $\mathbf{u}$ in H, μ_Y is Gaussian with covariance operator R_Y and mean element $\mathbf{m}_Y$, $R_S = R_N^{\frac{1}{2}} R_Y R_N^{\frac{1}{2}}$, and $\mathbf{m}_S = R_N^{\frac{1}{2}}\mathbf{m}_Y$.

THEOREM 2. *If* $\mu_S\{\text{range}\,(R_N^{\frac{1}{2}})\} = 1$, *then* $\mu_{S+N} \sim \mu_N$ *if and only if* range $(R_{S+N}^{\frac{1}{2}}) \supset$ range $(R_N^{\frac{1}{2}})$.

PROOF. $\mu_S[\text{range}\,(R_N^{\frac{1}{2}})] = 1$ implies, by Lemma 3, that $m_S \in \text{range}\,(R_N^{\frac{1}{2}})$ and $R_S = R_N^{\frac{1}{2}} W R_N^{\frac{1}{2}}$, W trace-class. From Lemma 1, $\mu_{S+N} \sim \mu_N$ if and only if $R_{S+N} = R_N^{\frac{1}{2}}(I + Q)R_N^{\frac{1}{2}}$, where Q is Hilbert–Schmidt and does not have -1 as an eigenvalue. $R_S = R_N^{\frac{1}{2}} W R_N^{\frac{1}{2}}$ implies $R_S^{\frac{1}{2}} = R_N^{\frac{1}{2}} G$, G Hilbert–Schmidt; hence

$$R_{S+N} = R_S + R_{SN} + R_{NS} + R_N = R_N^{\frac{1}{2}}[W + GV + V^*G^* + I]R_N^{\frac{1}{2}}\,,$$

where V is an operator of norm $\leqq 1$ satisfying $R_{SN} = R_S^{\frac{1}{2}} V R_N^{\frac{1}{2}}$ [2]. Now $Z \equiv W + GV + V^*G^*$ is Hilbert–Schmidt, so that $\mu_{S+N} \sim \mu_N$ if and only if Z does not have -1 as an eigenvalue. From Lemma 2, this is precisely the condition for range $(R^{\frac{1}{2}}_{S+N}) \supset$ range $(R_N{}^{\frac{1}{2}})$.

The results summarized in Lemma 2 can be used to determine whether range $(R_N{}^{\frac{1}{2}}) \subset$ range $(R^{\frac{1}{2}}_{S+N})$, and thus $\mu_{S+N} \sim \mu_N$, whenever $\mu_{\mathbf{v}+N} \sim \mu_N$ a.e. $d\mu_S(\mathbf{v})$. The following corollary gives two additional conditions for $\mu_{S+N} \sim \mu_N$.

COROLLARY. *$\mu_{S+N} \sim \mu_N$ if $\mu_S\{\text{range}\,(R_N{}^{\frac{1}{2}})\} = 1$ and either of the two following conditions is satisfied:*

(a) *There exists no non-null $\mathbf{u}$ in H satisfying $GG^*\mathbf{u} = V^*V\mathbf{u} = -GV\mathbf{u} = -V^*G^*\mathbf{u} = \mathbf{u}$, where G, V are operators satisfying $R_S{}^{\frac{1}{2}} = R_N{}^{\frac{1}{2}}G$, $R_{SN} = R_S{}^{\frac{1}{2}}VR_N{}^{\frac{1}{2}}$, $||V|| \leqq 1$ (this condition is also necessary for $\mu_{S+N} \sim \mu_N$).*

(b) *There exists a scalar $k < 1$ such that for all $\mathbf{u}$ in H, either*

$$\langle R_S\mathbf{u}, \mathbf{u}\rangle \leqq k\langle R_N\mathbf{u}, \mathbf{u}\rangle \qquad \text{or} \qquad \langle R_{SN}\mathbf{u}, \mathbf{u}\rangle^2 \leqq k\langle R_S\mathbf{u}, \mathbf{u}\rangle\langle R_N\mathbf{u}, \mathbf{u}\rangle .$$

PROOF. (a) From the theorem, it is sufficient to show that the stated condition is necessary and sufficient for range $(R^{\frac{1}{2}}_{S+N}) \supset$ range $(R_N{}^{\frac{1}{2}})$, or, by Lemma 2, for $(I + GG^* + GV + V^*G^*)\mathbf{u} = \mathbf{0}$ to imply $||\mathbf{u}|| = 0$. If $(I + GG^* + GV + V^*G^*)\mathbf{u} = \mathbf{0}$, then $||\mathbf{u}||^2 + ||G^*\mathbf{u}||^2 + 2\langle G^*\mathbf{u}, V\mathbf{u}\rangle = 0$. The LHS of this last equality is $\geqq ||\mathbf{u}||^2 + ||G^*\mathbf{u}||^2 - 2||G^*\mathbf{u}||\,||V\mathbf{u}|| \geqq ||\mathbf{u}||^2 + ||G^*\mathbf{u}||^2 - 2||G^*\mathbf{u}||\,||\mathbf{u}|| = (||\mathbf{u}|| - ||G^*\mathbf{u}||)^2 \geqq 0$, with equality throughout if and only if $G^*\mathbf{u} = -V\mathbf{u}$ and $||G^*\mathbf{u}|| = ||\mathbf{u}||$. Further, $||\mathbf{u}||^2 + ||G^*\mathbf{u}||^2 + 2\langle G^*\mathbf{u}, V\mathbf{u}\rangle \geqq ||\mathbf{u}||^2 + ||G^*\mathbf{u}||^2 - 2||V^*G^*\mathbf{u}||\,||\mathbf{u}|| \geqq ||\mathbf{u}||^2 + ||G^*\mathbf{u}||^2 - 2||G^*\mathbf{u}||\,||\mathbf{u}|| = (||\mathbf{u}|| - ||G^*\mathbf{u}||)^2 \geqq 0$, with equality throughout if and only if $||G^*\mathbf{u}|| = ||\mathbf{u}||$ and $V^*G^*\mathbf{u} = -\mathbf{u}$. Hence, if $G^*\mathbf{u} = -V\mathbf{u}$ and $V^*G^*\mathbf{u} = -\mathbf{u}$ only for $\mathbf{u} = \mathbf{0}$, range $(R^{\frac{1}{2}}_{S+N}) \supset$ range $(R_N{}^{\frac{1}{2}})$. It is clear that $V^*G^*\mathbf{u} = -\mathbf{u}$ and $G^*\mathbf{u} = -V\mathbf{u}$ together imply $(I + GG^* + GV + V^*G^*)\mathbf{u} = \mathbf{0}$, so that the condition is also necessary for range $(R^{\frac{1}{2}}_{S+N}) \supset$ range $(R_N{}^{\frac{1}{2}})$.

(b) The conditions given in (b) are obvious consequences of (a).

REMARK. The usual conditions that one must verify in order to show $\mu_N \sim \mu_{S+N}$ are (from Lemma 1 and Lemma 2) (a) range $(R^{\frac{1}{2}}_{S+N}) =$ range $(R_N{}^{\frac{1}{2}})$, (b) $\mathbf{m}_S \in$ range $(R_N{}^{\frac{1}{2}})$, and (c) $R_N{}^{-\frac{1}{2}}R_{S+N}R_N{}^{-\frac{1}{2}} - I$ has a bounded extension to H which is Hilbert–Schmidt. The significance of Theorem 2 is that conditions (b) and (c) need not be verified if one knows that $\mu_S[\text{range}\,(R_N{}^{\frac{1}{2}})] = 1$. Moreover, one need not prove that range $(R^{\frac{1}{2}}_{S+N}) \subset$ range $(R_N{}^{\frac{1}{2}})$. The proof that $\mu_S[\text{range}\,(R_N{}^{\frac{1}{2}})] = 1$ can often be made by examining the S sample functions, and this can be a great deal easier than proving that range $(R^{\frac{1}{2}}_{S+N}) \subset$ range $(R_N{}^{\frac{1}{2}})$ and that $R_N{}^{-\frac{1}{2}}R_{S+N}R_N{}^{-\frac{1}{2}} - I$ has a Hilbert–Schmidt extension.

Finally, we note that if $\mu_S[\text{range}\,(R_N{}^{\frac{1}{2}})] = 1$, then one has $\mu_{\alpha S+N} \sim \mu_N$ for all real scalars α except those in a countable set $\mathscr{A}$. $\mathscr{A}$, which can be empty, consists of those scalars α such that $\alpha^2GG^*\mathbf{u} = -\alpha GV\mathbf{u} = -\alpha V^*G^*\mathbf{u} = \mathbf{u}$ for some non-null $\mathbf{u}$ in H, where G and V are as defined in Theorem 2. The possible limit points of $\mathscr{A}$ are $\pm\infty$. This follows from (a) of the Corollary and the fact that G is compact.

7. Examples. Suppose that H is $L_2[0, \mathrm{b}]$, $b < \infty$, and that μ_N is Gaussian with null mean function and covariance function defined as follows:

(1) $R_1(t, s) = \min(t, s)$;
(2) $R_2(t, s) = b - \max(t, s)$;
(3) $R_3(t, s) = b - |t - s|$;
(4) $R_4(t, s) = e^{-\alpha|t-s|}$, $\alpha > 0$;
(5) $R_5(t, s) = \int_{-\infty}^{\infty} e^{i\lambda(t-s)} \hat{R}(\lambda)\, d\lambda$, where $\hat{R}$ is a rational spectral density function with denominator of degree exactly two greater than the degree of the numerator;
(6) $R_6(t, s) = R_i(t, s) + \int_0^t \int_0^s K(u, v)\, du\, dv$, where $K(u, v)$ is a product-measurable function satisfying $\int_0^b \int_0^b K^2(u, v)\, du\, dv < \infty$, $\int_0^b \mathbf{u}_t \int_0^b K(t, s)\mathbf{u}_s\, ds\, dt \geqq 0$ for all $\mathbf{u}$ in $L_2[0, b]$, $K(u, v) = K(v, u)$ for u, v in $[0, b]$, and $R_i(t, s)$ is any one of the functions defined in (1)–(5).

Suppose that μ_S is induced by the stochastic process (S_t) defined by

(a) $S_t(\omega) = \int_0^t Y_s(\omega)\, ds$
(b) $S_t(\omega) = \int_t^b Y_s(\omega)\, ds$, or
(c) $S_t(\omega) = c(\omega) + \int_0^t Y_s(\omega)\, ds$, all t in $[0, b]$,

where in (a), (b) and (c) (Y_t) is a measurable stochastic process with sample functions almost surely in $L_2[0, b]$. c is an a.s. finite random variable.

Assume that one of the following two conditions is satisfied: (A) S and N are independent; (B) μ_S and μ_{S+N} are Gaussian, and the extensions of $R_N^{-\frac{1}{2}} R_S R_N^{-\frac{1}{2}}$, $-R_N^{-\frac{1}{2}} R_{NS} R_N^{-\frac{1}{2}}$ and $-R_N^{-\frac{1}{2}} R_{SN} R_N^{-\frac{1}{2}}$ do not have a common eigenvector corresponding to the eigenvalue 1. One then has the following results:

I. For S defined as in (a), N defined by (1), (3), (4), (5), or (6), with R_6 defined for $i = 1, 3, 4$ or 5, $\mu_{S+N} \sim \mu_N$.

II. For S defined as in (b), N defined by (2)–(6), with R_6 defined for $i = 2, 3, 4$ or 5, $\mu_{S+N} \sim \mu_N$.

III. For S defined as in (c), N defined by (3)–(6), with R_6 defined for $i = 3, 4$ or 5, $\mu_{S+N} \sim \mu_N$.

These results are unchanged if any of the N covariance functions are multiplied by a positive real scalar. (The result given in I for the Wiener process (covariance function R_1) corresponding to assumption a is well known [17].)

To obtain the preceding results, let R_i denote the integral operator with kernel $R_i(t, s)$. The range space of $R_i^{\frac{1}{2}}$ can be described as follows [3]:

(1′) $R_1^{\frac{1}{2}}$ has range containing all absolutely continuous functions on $[0, b]$ that vanish at 0 and have $L_2[0, b]$ derivative.

(2′) Range $(R_2^{\frac{1}{2}})$ contains all absolutely continuous functions on $[0, b]$ that vanish at b and have $L_2[0, b]$ derivative.

(3′–5′) $R_3^{\frac{1}{2}}$, $R_4^{\frac{1}{2}}$ and $R_5^{\frac{1}{2}}$ have the same range; their range space contains all absolutely continuous functions on $[0, b]$ having $L_2[0, b]$ derivative.

(6′) Range $(R_6^{\frac{1}{2}})$ = range $(R_i^{\frac{1}{2}})$ when $i = 1, 3, 4$ or 5; for $i = 2$, range $(R_6^{\frac{1}{2}}) \subset$ range $(R_3^{\frac{1}{2}})$.

The results stated in I–III now follow directly from the corollaries to the two theorems.

Suppose that R_6 is as defined above, except now the integral operator K having $K(t, s)$ as kernel may be negative. If $I + K$ is positive, then R_6 is still a covariance operator for $i = 1$ and 3. If $I + K$ is strictly positive, then the results stated under I, II and III above remain valid for R_6 defined by these two kernels. One obtains this by noting that $I + K$ strictly positive implies range $(R_6^{\frac{1}{2}})$ = range $(R_i^{\frac{1}{2}})$ for $i = 1$ and $i = 3$ [3].

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