The Annals of Probability
1973, Vol. 1, No. 4, 674–689

EPSILON ENTROPY OF STOCHASTIC PROCESSES WITH CONTINUOUS PATHS[1]

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This paper shows that the epsilon entropy in the sup norm of a wide variety of processes with continuous paths on the unit interval is finite. In fact, the class coincides with the class of processes for which proofs of continuity have been given from a covariance condition. This suggests the conjecture that the epsilon entropy of any process continuous on the unit interval is finite in the sup norm of continuous functions. The epsilon entropy considered in this paper is defined as the minimum Shannon entropy of any partition by sets of diameter at most epsilon of the space of continuous functions on the unit interval, where the probability is the one inherited from the given process. The proof proceeds by constructing partitions and estimating their entropy using probability bounds.

1. Introduction. The concept of *epsilon entropy* introduced in [4] was applied to stochastic processes on [0, 1] considered as subspaces of $L_2[0, 1]$. This was further developed for Gaussian Processes in [5]. Here we consider processes with continuous paths and their embedding in $C[0, 1]$. This leads to a different value of epsilon entropy, for this entropy depends on the metric as well as the probabilistic structure of the process. [3] relates the concept of epsilon entropy to information theory.

Epsilon entropy was defined as follows in [4]: Let X be a complete separable metric space with metric d and probability measure μ such that open sets are measurable. For $\varepsilon > 0$, an *epsilon partition* $U = \{U_i\}$ is a finite or denumerable collection of disjoint measurable sets in X, each of diameter no greater than ε, which together cover a set of measure 1. The entropy of this partition is

$$H(U) = \sum \mu(U_i) \log [1/\mu(U_i)] \leqq \infty . \quad (1)$$

The ε-entropy of X is

$$H_\varepsilon(X) = \inf H(U) \quad (2)$$

over all ε-partitions U. This may be infinite.

Since the uniform norm on [0, 1] is at least as large as the L_2 norm, the epsilon entropy of a process in the L_2 norm gives a lower bound for its entropy in $C[0, 1]$.

Received June 5, 1972.

[1] This paper presents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract No. NAS7-100, sponsored by the National Aeronautics and Space Administration.

AMS 1970 *subject classifications.* Primary Probability-stochastic processes; general theory (6040); Secondary Information and Communication, Circuits, Automata, Information Theory (9420).

Key words and phrases. Epsilon entropy, continuous processes, modulus of continuity, partitions, covariance conditions, Wiener process.

Here we prove some upper bounds, showing that $H_\varepsilon(X)$ is finite for the processes considered. The point of departure is an assumption about the covariance function of the process, which implies that the paths are continuous. To estimate the entropy, quantitative statements about the probability with which a certain modulus of continuity is assumed are needed. Hence proofs of continuity which can be found elsewhere are repeated here to obtain this extra information. See [2] for a recent summary on continuity of Gaussian processes.

The results (Theorems 1 and 2) suggest that any simple condition on the covariance which guarantees continuous paths makes the epsilon entropy finite. This raises the question of whether any process with continuous paths has finite entropy. A construction was given in [4] for processes with infinite entropy in $L_2[0, 1]$, which can have continuous paths. However, for Gaussian processes, the L_2-epsilon entropy must be finite [5], so the question is open for the Gaussian case. In fact, it may, as far as we know, now be true that the epsilon entropy of a Gaussian process on the unit interval in any norm whatever may be finite, if the sample functions have finite norm with probability 1 and a set of sample functions of probability 1 is continued in a linear subspace of $L_2[0, 1]$ separable in the given norm.

2. Preliminary lemmas. Here some statements about the modulus of continuity of sample paths are proved which will be used later to get bounds on $H_\varepsilon(X)$. In each lemma, the hypothesis is actually an assumption about the covariance function

$$R(s, t) = E[x(s)x(t)]\,, \tag{3}$$

since

$$E[x(s) - x(t)]^2 = R(s, s) + R(t, t) - 2R(s, t)\,.$$

The first lemma concerns general mean-continuous stochastic processes. The other two lemmas deal with Gaussian processes only.

We assume that the processes we deal with are separable and have finite second moments. The property of separability implies that any inequality such as (5) need only be proved on some dense set of points of [0, 1] ([1] Theorem 2.2, page 54).

LEMMA 1. *Let a separable stochastic process $x(t)$ on* [0, 1] *have finite second moments for all $t \in [0, 1]$, and suppose there are positive constants A, α such that*

$$E[x(s) - x(t)]^2 \leqq A|s - t|^{1+2\alpha} \tag{4}$$

for all $s, t \in [0, 1]$. Then for $M > 0$,

$$|x(s) - x(t)| \leqq \frac{2M}{(1 - 2^{-\alpha})^2}|s - t|^\alpha \left(\log_2 \frac{2}{|s - t|}\right)^{\frac{1}{2}} \log_2 \log_2 \frac{8}{|s - t|} \tag{5}$$

for all s, t, except on a set of x of probability at most BA/M^2, where B is a universal constant.

PROOF. It follows from (4) that for $\delta > 0$

$$\Pr\{|x(s) - x(t)| > \delta\} \leqq A|s - t|^{1+2\alpha}/\delta^2 .$$

Let $s = j/2^n$, $t = (j-1)/2^n$, $\delta = 2^{-n\alpha}Mn^{\frac{1}{2}}\log_2(n+1)$:

$$\Pr\left\{\left|x\left(\frac{j}{2^n}\right) - x\left(\frac{j-1}{2^n}\right)\right| > 2^{-n\alpha}Mn^{\frac{1}{2}}\log_2(n+1)\right\} \leqq \frac{2^{-n}A}{M^2n[\log_2(n+1)]^2} .$$

Summing over n and j,

$$\sum_{n=1}^{\infty}\sum_{j=1}^{2^n}\Pr\left\{\left|x\left(\frac{j}{2^n}\right) - x\left(\frac{j-1}{2^n}\right)\right| > 2^{-n\alpha}Mn^{\frac{1}{2}}\log(n+1)\right\} < \frac{AB}{M^2}$$

where

$$B = \sum_{n=1}^{\infty}\frac{1}{n[\log_2(n+1)]^2} .$$

Hence, except for a set of paths of measure at most BA/M^2, we have

$$\left|x\left(\frac{j}{2^n}\right) - x\left(\frac{j-1}{2^n}\right)\right| \leqq 2^{-n\alpha}Mn^{\frac{1}{2}}\log_2(n+1) \tag{6}$$

for all $n \geqq 1$ and $1 \leqq j \leqq 2^n$.

It will be shown that if s and t are two dyadic points (rational numbers whose denominators are powers of 2 with $|s - t| < 1$, then (5) is valid for any path satisfying (6). To do this, we decompose the interval (s, t) into a finite number of disjoint subintervals by a sequence of points $t_0 = s, t_1, t_2, \cdots, t_m = t$ such that each interval (t_{k-1}, t_k) is an interval of the form $((j-1)2^{-n}, j2^{-n})$, and apply (6) to the right side of the inequality

$$|x(s) - x(t)| \leqq \sum_{k=1}^{m}|x(t_k) - x(t_{k-1})| . \tag{7}$$

If we take the sequence for which m has its minimum value (for given s/t), it can be shown that each possible distance 2^{-n} occurs at most twice among the numbers $|t_k - t_{k-1}|$. Furthermore, the longest subinterval has length $\leqq 2^{-n_0}$, where

$$2^{-n_0} \leqq |s - t| < 2^{-n_0+1} .$$

The application of (6) to (7) implies

$$|x(s) - x(t)| \leqq 2\sum_{n=n_0}^{\infty}2^{-n\alpha}Mn^{\frac{1}{2}}\log_2(n+1) .$$

Using the inequality

$$n^{\frac{1}{2}}\log_2(n+1) \leqq n_0^{\frac{1}{2}}\log_2(n_0+2)(n-n_0+1) , \qquad n \geqq n_0 \geqq 1 ,$$

which is easy to show, we get

$$\begin{aligned}|x(s) - x(t)| &\leqq 2Mn_0^{\frac{1}{2}}\log_2(n_0+2)\sum_{n=n_0}^{\infty}2^{-n\alpha}(n-n_0+1) \\ &= \frac{2Mn_0^{\frac{1}{2}}\log_2(n_0+2)\cdot 2^{-\alpha n_0}}{(1-2^{-\alpha})^2} \\ &\leqq \frac{2M}{(1-2^{-\alpha})^2}|s-t|^{\alpha}\left(\log_2\frac{2}{|s-t|}\right)^{\frac{1}{2}}\log_2\log_2\frac{8}{|s-t|} .\end{aligned}$$

This completes the proof of Lemma 1.

LEMMA 2. *Let a separable Gaussian process on* [0, 1] *with mean zero have the property that for some positive constants* A, β,

$$E[x(s) - x(\varepsilon)]^2 \leqq A|s - t|^{2\beta}, \tag{8}$$

for all s, $t \in [0, 1]$. *Then for* $M > (4A \log 2)^{\frac{1}{2}}$,

$$|x(s) - x(t)| \leqq \frac{2M}{(1 - 2^{-\beta})^2} |s - t|^\beta \left(\log_2 \frac{2}{|s - t|}\right)^{\frac{1}{2}}$$

for all s, t, *except on a set of paths of probability less than*

$$\frac{8}{M}\left(\frac{A}{2\pi}\right) \exp\left(-\frac{M^2}{2A}\right).$$

PROOF. $x(s) - x(t)$ is normal, with mean zero and variance bounded by (8). Hence for $\delta > 0$

$$\Pr\{|x(s) - x(t)| > \delta\} \leqq \Phi\left(\frac{\delta}{A^{\frac{1}{2}}|s - t|^\beta}\right),$$

where

$$\Phi(u) = (2/\pi)^{\frac{1}{2}} \textstyle\int_u^\infty e^{-\frac{1}{2}y^2}\, dy < (2/\pi)^{\frac{1}{2}} u^{-1} e^{-\frac{1}{2}u^2}$$

for $u > 0$. It follows that

$$\Pr\left\{\left|x\left(\frac{j}{2^n}\right) - x\left(\frac{j-1}{2^n}\right)\right| > \delta\right\} \leqq \Phi\left(\frac{2^{n\beta}\delta}{A^{\frac{1}{2}}}\right) < \left(\frac{2A}{\pi}\right)^{\frac{1}{2}} \frac{2^{-n\beta}}{\delta} \exp - \left(\frac{2^{2n\beta}\delta^2}{2A}\right).$$

Put $\delta = 2^{-n\beta} M n^{\frac{1}{2}}$ and sum over n and j:

$$\textstyle\sum_{n=1}^\infty \sum_{j=1}^{2^n} \Pr\left\{\left|x\left(\frac{j}{2^n}\right) - x\left(\frac{j-1}{2^n}\right)\right| > 2^{-n\beta} M n^{\frac{1}{2}}\right\} < q(M),$$

where

$$q(M) = \left(\frac{2A}{\pi}\right)^{\frac{1}{2}} \sum_{n=1}^\infty \frac{2^n}{Mn^{\frac{1}{2}}} \exp\left(-\frac{nM^2}{2A}\right)$$
$$< \left(\frac{2A}{\pi}\right)^{\frac{1}{2}} \frac{2}{M} \exp\left(-\frac{M^2}{2A}\right)\left[1 - 2\exp\left(-\frac{M^2}{2A}\right)\right]^{-1}$$

if $\exp(-M^2/2A) < \frac{1}{2}$. In particular, if $M > (4A \log 2)^{\frac{1}{2}}$,

$$\exp(-M^2/2A) < \tfrac{1}{4},$$

and

$$q(M) < \frac{8}{M}\left(\frac{A}{2}\right)^{\frac{1}{2}} \exp\left(-\frac{M^2}{2A}\right).$$

Except on a set of paths of probability less than $q(M)$, we have

$$\left|x\left(\frac{j}{2^n}\right) - x\left(\frac{j-1}{2^n}\right)\right| \leqq 2^{-n\beta} M n^{\frac{1}{2}}, \qquad n \geqq 1,\ 1 \leqq j \leqq 2^n. \tag{9}$$

Now we proceed to estimate $|x(s) - x(t)|$ for a path satisfying (9), when s, t are dyadic points with $|s - t| < 1$. If

$$2^{-n_0} \leqq |s - t| < 2^{-n_0+1},$$

then $n_0 \geqq 1$, and we get

$$|x(s) - x(t)| \leqq 2 \sum_{n=n_0}^{\infty} 2^{-n\beta} M n^{\frac{1}{2}}$$
$$\leqq 2\, n_0 \sum_{n=n_0}^{\infty} 2^{-n\beta} M(n - n_0 + 1)$$
$$= \frac{2M}{(1-2^{-\beta})^2} n_0^{\frac{1}{2}} 2^{-n_0\beta} \leqq \frac{2M}{(1-2^{-\beta})^2} |s-t|^{\beta} \left(\log_2 \frac{2}{|s-t|}\right)^{\frac{1}{2}}.$$

This completes the proof of Lemma 2.

LEMMA 3. *Let $x(t)$ be a separable Gaussian process on* $[0, 1]$ *with mean* 0 *and*

$$E[x(s) - x(t)]^2 \leqq A\left(\log \frac{2}{|s-t|}\right)^{-a}$$

for all s, $t \in [0, 1]$, where A and a are constants with $a > 1$. Let β and M be positive numbers such that

$$M^2 > 4A2^{a\beta},$$

and define

$$\gamma = \tfrac{1}{2}\beta(a-1).$$

Then $x(t)$ has a modulus of continuity $\omega_x(\delta)$ satisfying the inequality

$$\omega_x(\delta) \leqq \frac{2M}{1-2^{-\gamma}} (\log \delta^{-1})^{-\frac{1}{2}(a-1)} \tag{10}$$

for $\delta \leqq \exp(-2^{n\beta})$, except on a set of probability at most

$$\frac{(2A/\pi)^{\frac{1}{2}}}{M} \sum_{j=n+1}^{\infty} 2^{-\frac{1}{2}j\beta + \frac{1}{2}a\beta} \exp\left[2^{j\beta}\left(2 - \frac{M^2}{2A} 2^{-a\beta}\right)\right].$$

PROOF. By hypothesis, if $|s - t| \leqq 2e^{-\mu}$,

$$E[x(s) - x(t)]^2 \leqq A\mu^{-a}.$$

$x(s) - x(t)$ is normal with mean zero. Hence

$$\Pr\{|x(s) - x(t)| > q\} < \left(\frac{2A}{\pi}\right)^{\frac{1}{2}} \frac{1}{q\mu^{a/2}} \exp\left(-\frac{q^2\mu^a}{2A}\right), \tag{11}$$

if $q > 0$.

A sequence of partitions of the unit interval, such that each is a refinement of the preceding, will be constructed. We define a sequence of positive integers k_n and associated quantities d_n, μ_n such that

$$d_n = 1/k_n = 2e^{-\mu_n}, \qquad n \geqq 1,$$

by induction, taking k_n to be the first positive integer such that

(i) $\mu_n \geqq 2^{\beta n}$,
(ii) k_n is a multiple of k_{n-1} if $n \geqq 2$.

Then for all n,

$$e^{-2^{\beta n}} < d_n \leqq 2e^{-2^{\beta n}}.$$

Also, put $d_0 = 2/e$, $\mu_0 = 1$. The nth partition is to be a partition into k_n intervals of length d_n.

Let Q_n be the set of pairs of vertices (s, t) of the nth partition such that

$$|s - t| \leqq d_{n-1} .$$

Then by (11), if $(s, t) \in Q_n$,

$$\Pr\{|x(s) - x(t)| > q\} < \left(\frac{2A}{\pi}\right)^{\frac{1}{2}} \frac{1}{q\mu_{n-1}^{a/2}} \exp\left(\frac{q^2\mu_{n-1}^a}{2A}\right).$$

There are less than $k_n{}^2$ elements in Q_n. Hence, if we take $q = 2^{-\gamma n}M$,

$$\begin{aligned} p_n &= \Pr\{\max_{(s,t)\in Q_n} |x(s) - x(t)| > 2^{-\gamma n}M\} \\ &< \left(\frac{2A}{\pi}\right)^{\frac{1}{2}} \frac{2^{\gamma n}k_n{}^2}{M\mu_{n-1}^{a/2}} \exp\left(-\frac{2^{-2\gamma n}M^2\mu_{n-1}^a}{2A}\right), \end{aligned}$$

which implies

$$\begin{aligned} p_n &< \left(\frac{2A}{\pi}\right)^{\frac{1}{2}} \frac{1}{M}\, 2^{\gamma n - \frac{1}{2}a\beta(n-1)} \exp\left[2^{\beta n+1} - \frac{M^2}{2A}\, 2^{-2\gamma n + a\beta(n-1)}\right] \\ &= \left(\frac{2A}{\pi}\right)^{\frac{1}{2}} \frac{1}{M}\, 2^{-\frac{1}{2}\beta n + \frac{1}{2}a\beta} \exp\left[2^{\beta n}\left(2 - \frac{M^2}{2A}\, 2^{-n\beta}\right)\right]. \end{aligned}$$

Suppose that s and t are two points of $[0, 1]$, such that each is a vertex of some partition. The set of such points is dense on $[0, 1]$, so it is sufficient to consider the modulus of continuity of $x(t)$ on this set. If

$$d_{n_0} \leqq |s - t| \leqq d_{n_0 - 1} ,$$

there is a finite set of points $\{t_0 = s, t_1, t_2, \cdots, t_r = t\}$ such that each pair (t_{j-1}, t_j) is in some Q_n with $n \geqq n_0$, and there are at most two pairs in each Q_n. If $x(t)$ lies in the set of x for which

$$\max_{(s',t')\in Q_n} |x(s') - x(t')| \leqq 2^{-\gamma n}M , \qquad n \geqq n_0 ,$$

then

$$|x(s) - x(t)| \leqq 2 \textstyle\sum_{n=n_0}^{\infty} 2^{-\gamma n}M = \dfrac{2M}{1 - 2^{-\gamma}}\, 2^{-\gamma n_0} .$$

We have

$$2^{\beta n_0} > \log d_{n_0}{}^{-1} > \log |s - t|^{-1} .$$

Hence

$$|x(s) - x(t)| \leqq \frac{2M}{1 - 2^{-\gamma}} \left(\log \frac{1}{|s - t|}\right)^{-\gamma/\beta} . \tag{12}$$

If N is any positive integer, (12) holds wherever $|s - t| \leqq d_{N-1}$, except for a set of x of probability less than $\sum_{n=N}^{\infty} p_n$. Thus, except with this probability, inequality (10) is valid for

$$\delta \leqq \exp[-2^{\beta(N-1)}] .$$

This completes the proof of Lemma 3.

3. Upper Bounds. Theorem 1 below gives bounds for $H_\varepsilon(X)$ for stochastic processes whose covariance function satisfies the condition

$$R(s, s) + R(t, t) - 2R(s, t) \leqq A|s - t|^{2\beta} .$$

Two lemmas which will be proved next cover the Gaussian and general cases separately, and give preliminary bounds which are quite different. These are used in the proof of Theorem 1 to get a bound which is the same for both cases, when valid. This is reminiscent of Theorem 9, [4], which gives an asymptotic bound for the epsilon entropy of a general mean continuous process which is sharp for Gaussian processes which satisfy the hypotheses there.

LEMMA 4. *If a separable Gaussian process $x(t)$ on $[0, 1]$ has $x(0) = 0$, mean zero, and*

$$E[x(s) - x(t)]^2 \leqq A|s - t|^{2\beta}$$

for all $s, t \in [0, 1]$, where A, β are positive constants, then the paths are continuous with probability 1, and in the uniform norm, the ε-entropy of the process is at most

$$C(\beta)\frac{A^{\frac{1}{2}\beta}}{\varepsilon^{1/\beta}}\left[1 + \left(\log\frac{A^{\frac{1}{2}}}{\varepsilon}\right)^{1/2\beta}\right]$$

for $\varepsilon \leqq A^{\frac{1}{2}}$, where $C(\beta)$ depends only on β.

PROOF. The ε-entropy of $x(t)$ is the same as the $\varepsilon/A^{\frac{1}{2}}$ entropy of $x(t)/A^{\frac{1}{2}}$, which satisfies the condition

$$E\left[\frac{x(s)}{A^{\frac{1}{2}}} - \frac{x(t)}{A^{\frac{1}{2}}}\right]^2 \leqq |s - t|^{2\beta} .$$

Thus it is sufficient to prove the lemma when $A = 1$.

Let S_n be the set of $x(t)$ whose modulus of continuity ω_x satisfies the inequality

$$\omega_x(\delta) \leqq \frac{2^{n+2}}{(1 - 2^{-\beta})^2}\,\delta^\beta\left(\log_2\frac{2}{\delta}\right)^{\frac{1}{2}} , \qquad 0 < \delta \leqq 1 ,$$

for $n = 1, 2, \cdots$. Applying Lemma 2 with $M = 2^{n+1} (> (4 \log 2)^{\frac{1}{2}})$, we have

$$\Pr(X - S_n) \leqq 2^{2-n}/(2\pi)^{\frac{1}{2}} e^{-2^{2n}} . \tag{13}$$

Let $T_1 = S_1$, and

$$T_n = S_n - S_{n-1} , \qquad n \geqq 2 .$$

By considering ε-partitions of X which are refinements of the partition into T_1, T_2, $\cdots$, we can show

$$H_\varepsilon(X) \leqq \textstyle\sum_{n=1}^\infty P(T_n)[\log P(T_n)^{-1} + H_\varepsilon(T_n)] .$$

By (13), for $n \geqq 2$

$$\Pr(T_n) \leqq \Pr(X - S_{n-1}) \leqq 2^{1-n}/(2\pi)^{\frac{1}{2}} e^{-2^{2n-2}} .$$

Hence

$$\begin{aligned} H_\varepsilon(X) \leqq \varepsilon^{-1} + H_\varepsilon(T_1) + (2\pi)^{-\frac{1}{2}} &\textstyle\sum_{n=2}^\infty 2^{1-n} e^{-2^{2n-2}} \\ &\times [2^{2n-2} + (n-1)\log 2 + \log(2n)^{\frac{1}{2}} + H_\varepsilon(T_n)] . \end{aligned} \tag{14}$$

Define $\delta_n < 2e^{-2/\beta}$ by

$$\frac{2^{n+2}}{(1-2^{-\beta})^2}\,\delta_n{}^\beta\left(\log_2\frac{2}{\delta_n}\right)^{\frac{1}{2}} = \frac{\varepsilon}{4}\,.$$

Then in S_n, $\omega_x(\delta_n) \leqq \varepsilon/4$. Let $C_1, C_2, \cdots$ denote constants depending on β. Then we have

$$\log_2 2/\delta_n < C_1\delta_n{}^{-\beta}\,,$$

so $\delta_n{}^{\beta/2} > C_2 2^{-n}\varepsilon$,

$$\log_2\frac{2}{\delta_n} < 1 + \frac{2}{\beta}\left(n + \log_2\frac{1}{C_2\varepsilon}\right)$$

and

$$\frac{1}{\delta_n} = \left[\frac{2^{n+4}}{(1-2^{-\beta})^2}\frac{1}{\varepsilon}\left(\log_2\frac{2}{\delta_n}\right)^{\frac{1}{2}}\right]^{1/\beta}$$

$$(15)\qquad < C_3\varepsilon^{-1/\beta}2^{n/\beta}\left[1 + \frac{2}{\beta}\left(n + \log_2\frac{1}{C_2\varepsilon}\right)\right]^{1/2\beta}$$

$$< \varepsilon^{-1/\beta}2^{n/\beta}[C_4 n^{1/2\beta} + C_5(\log\varepsilon^{-1})^{1/2\beta}]\,.$$

S_n can be partitioned into ε-sets as follows: Map each curve $x(t)$ into a polygonal curve $\bar{x}(t)$ with vertices at $t = t_j = j\delta_n$, $j = 0, 1, \cdots, [1/\delta_n]$ such that

$$\bar{x}(0) = 0\,,$$
$$\bar{x}(t_j) - \bar{x}(t_{j-1}) = 0 \quad \text{or} \quad \pm\varepsilon/4\,,$$
$$\bar{x}(t) = \text{const.} \quad \text{for} \quad t \geqq t[1/\delta_n]\,,$$
$$|\bar{x}(t_j) - x(t_j)| \leqq \varepsilon/8\,.$$

This is possible because $x(t)$ varies by at most $\varepsilon/4$ in each interval (t_j, t_{j+1}). We have $|x(t) - \bar{x}(t)| \leqq \varepsilon/2$. Hence the set of $x(t)$ which map into a given $\bar{x}(t)$ form an ε-set. Use these sets to partition S_n.

Restricting this partition to T_n gives an ε-partition of T_n. It contains at most $3^{1/\delta_n}$ sets. Hence

$$H_\varepsilon(T_n) \leqq \delta_n{}^{-1}\log 3\,.$$

Combining this with (14) and (15),

$$H_\varepsilon(X) \leqq \varepsilon^{-1} + 2^{1/\beta}\varepsilon^{-1/\beta}\log 3[C_4 + C_5(\log\varepsilon^{-1})^{1/2\beta}]$$
$$+ (2\pi)^{-\frac{1}{2}}\sum_{n=2}^{\infty}2^{1-n}e^{-2^{2n-2}}\Big\{2^{2n-2} + (n-1)\log 2$$
$$+ \log(2\pi)^{\frac{1}{2}} + \frac{2^{n/\beta}\log 3}{\varepsilon^{1/\beta}}[C_4 n^{1/2\beta} + C_5(\log\varepsilon^{-1})^{1/2\beta}]\Big\}$$
$$= C_6 + C_7 e^{-1/\beta} + C_8\varepsilon^{-1/\beta}(\log\varepsilon^{-1})^{1/2\beta}\,,$$
$$\leqq C_9\,\varepsilon^{-1/\beta}[1 + (\log\varepsilon^{-1})^{1/2\beta}]$$

for $\varepsilon \leqq 1$. Lemma 4 is proved.

LEMMA 5. *If a separable stochastic process $x(t)$ on $[0, 1]$ has finite second moments, $x(0) = 0$, and*

$$E[x(s) - x(t)]^2 \leqq A|s-t|^{1+2\alpha}$$

for all $s, t \in [0, 1]$, *where* A *and* α *are positive constants, then the paths are continuous with probability* 1, *and the* ε*-entropy of the process in the uniform norm satisfies the inequality*

$$H_\varepsilon \leqq C(\alpha)(A^{\frac{1}{2}}/\varepsilon)^{1/\alpha}[1 + (\log (A^{\frac{1}{2}}/\varepsilon))^{1/2\alpha}(\log\log (A^{\frac{1}{2}}/\varepsilon))^{1/\alpha}]$$

for $\varepsilon < A^{\frac{1}{2}}$, *where* $C(\alpha)$ *depends only on* α.

PROOF. We can assume $A = 1$, as in the proof of Lemma 4. Let S_n be the set of $x(t)$ for which

$$\omega_x(\delta) < \frac{2^{n+2}}{(1-2^{-\alpha})^2}\, \delta^\alpha \left(\log_2 \frac{2}{\delta}\right)^{\frac{1}{2}} \log_2 \log_2 \frac{8}{\delta}\,.$$

By Lemma 1,

$$\Pr (X - S_n) \leqq 2^{-2n-2}BA\,.$$

Proceeding from here as in the proof of Lemma 4 leads to the given inequality.

Next we prove two lemmas which will be applied to the decomposition (21) used in the proof of Theorem 1.

LEMMA 6. *Let* $x(t)$ *be a stochastic process with continuous paths.*

(i) *Suppose* $x(t) = \sum_{j=1}^n x_j(t)$, *where each* $x_j(t)$ *is a stochastic process with continuous paths, with the probabilistic metric space* X_j. *Then if* $\varepsilon = \sum_{j=1}^n \varepsilon_j$, $\varepsilon_j > 0$,

$$H_\varepsilon(X) \leqq \textstyle\sum_{j=1}^n H_{\varepsilon_j}(X_j)\,.$$

(ii) *Let the range of* t *be partitioned into* n *intervals* $I_1, \cdots, I_n$. *If* $x_j(t)$ *is the restriction of* $x(t)$ *to* I_j, *then*

$$H_\varepsilon(X) \leqq \textstyle\sum_{j=1}^n H_\varepsilon(X_j)\,.$$

PROOF. The proof depends on Lemma 3 of [4], which states the following: If Y is a probability space which is a product space,

$$Y = \textstyle\prod_{j=1}^n Y_j$$

and we consider the Y_j's as measure spaces under the marginal distributions of Y, then for any partition U of Y which is a product of partitions U_j of Y_j, we have

$$H(U) \leqq \textstyle\sum H(U_j)\,.$$

To prove (i), take $Y = \prod X_j$. The measure on Y is the joint distribution of $x_1(t), \cdots, x_n(t)$. Let U_j be an ε_j-partition of X_j, with $H(U_j) = H_{\varepsilon_j}(X_j)$ (this exists by Theorem 2 of [4]). Then for the product partition of Y we have

$$H(U) \leqq \textstyle\sum H_{\varepsilon_j}(X_j)\,.$$

Define a metric d_Y on Y by

$$d_Y((x_1, \cdots, x_n), (x_1', \cdots, x_n')) = d(x_1, x_1') + \cdots + d(x_n, x_n')\,.$$

Then the sets of U have diameters $\leqq \varepsilon$, so $H_\varepsilon(Y) \leqq H(U)$, and

$$H_\varepsilon(Y) \leqq \textstyle\sum H_{\varepsilon_j}(X_j)\,.$$

The natural mapping of Y into X,

$$\phi = (x_1, \cdots, x_n) \to x_1 + \cdots + x_n ,$$

does not increase distances: if ξ, $\xi' \in Y$,

$$d(\phi(\xi), \phi(\xi')) \leqq d_Y(\xi, \xi') .$$

The measure on X is that which is induced from the measure on Y by ϕ. From these facts it is easy to show that

$$H_\varepsilon(X) \leqq H_\varepsilon(Y) ,$$

which proves (i).

To prove (ii), we proceed similarly. Here we need to consider the linear subspace Y' of Y in which the components $x_1, \cdots, x_n$ are the restrictions to $I_1, \cdots, I_n$ of some continuous function.

For (ii) we start with ε-partitions U_j of X_j such that $H(U_j) = H_\varepsilon(X_j)$, and define distance in Y by

$$d_Y((x_1, \cdots, x_n), (x_1', \cdots, x_n')) = \max \{d(x_1, x_1'), \cdots, d(x_n, x_n')\} .$$

Then the product partition U is an ε-partition of Y. The measure in Y is concentrated on Y'. Restricting the sets of U to Y' we get an ε-partition U' of Y', with

$$H_\varepsilon(Y') \leqq H(U') = H(U) \leqq \sum H_\varepsilon(X_j) .$$

Now the mapping

$$\phi : (x_1, \cdots, x_n) \to x ,$$

where $x(t)$ is the function which equals $x_j(t)$ on I_j, $j = 1, \cdots, n$, is a mapping of Y' into X which does not increase distance and is consistent with the measures. Hence we conclude

$$H_\varepsilon(X) \leqq H_\varepsilon(Y') ,$$

which proves (ii).

LEMMA 7. *Let $\{x_1, \cdots, x_n\}$ have a joint distribution with finite second moments, such that $E(x_1^2) \leqq C$, and*

$$E[(x_{j+1} - x_j)^2] \leqq C , \qquad j = 1, 2, \cdots, n-1 ,$$

where C is a fixed constant. Then there is a bound $h_n(C)$ for the 1-entropy of this process in the norm

$$d(x, x') = \max |x_j' - x_j| , \qquad j = 1, \cdots, n$$

which depends only on C and n, such that $h_n(C) = O(n)$ as $n \to \infty$.

PROOF. It is sufficient to show this result for $n = 2^k$, $k = 0, 1, 2, \cdots$, since if $2^k < n < 2^{k+1}$, we can take

$$h_n = h_{2^{k+1}} .$$

Let X be 2^k-dimensional space, with a probability distribution satisfying the given conditions. Express X as a product space,

$$X = X_1 \times X_2 ,$$

where $X_1 = \{(x_1, \cdots, x_{2^{k-1}})\}$, $X_2 = \{(x_{2^{k-1}+1}, \cdots, x_{2^k})\}$. If we define

$$y_j = x_{j+2^{k-1}} - x_{2^{k-1}} , \qquad j = 1, \cdots, 2^{k-1} ,$$

then

$$X_2 = (x_{2^{k-1}}, \cdots, x_{2^{k-1}}) + Y_2 ,$$

where $Y_2 = \{(y_1, \cdots, y_{2^{k-1}})\}$. X_1 and Y_2 are spaces of dimension 2^{k-1} satisfying the given conditions of the lemma. It follows from this way of decomposing X that

$$H_{\varepsilon_1+\varepsilon_2}(X) \leqq H_{\varepsilon_1}(X_1) + H_{\varepsilon_1}(Y_2) + H_{\varepsilon_2}(\{x_{2^{k-1}}\}) . \tag{16}$$

The hypotheses imply

$$E(x_{2^{k-1}}^2) \leqq 2^{2k-2}C .$$

It follows from [4], Lemma 5, that

$$H_{\varepsilon_2}(\{x_{2^{k-1}}\}) \leqq B\left(1 + \log^+ \frac{2^{k-1}C^{\frac{1}{2}}}{\varepsilon_2}\right),$$

where B is a universal constant. Let K_k be the supremum of possible $(1 - 2^{-k-1})$-entropies of 2^k-dimensional spaces satisfying the hypotheses. Then taking $\varepsilon_1 = 1 - 2^{-k}$, $\varepsilon_2 = 2^{-k-1}$ in (16) implies

$$K_k \leqq 2K_{k-1} + B[1 + \log^+ (2^{2k}C^{\frac{1}{2}})] , \qquad k \geqq 1 ,$$

so that

$$K_k \leqq 2^k\{K_0 + B \textstyle\sum_{j=1}^{k} 2^{-j}[1 + \log^+ (2^{2j}C^{\frac{1}{2}})]\} .$$

Since

$$H_{\frac{1}{2}}(\{x_1\}) \leqq B(1 + \log^+ 2C^{\frac{1}{2}}) ,$$

we have

$$K_k \leqq 2^kB \textstyle\sum_{j=0}^{\infty} 2^{-j}[1 + \log^+ (2^{2j+1}C^{\frac{1}{2}})] .$$

The lemma follows from $h_{2^k} \leqq K_k$.

THEOREM 1. *Let a separable stochastic process on* [0, 1] *have finite second moments, and suppose there are positive constants A and $a \leqq 2$ such that*

$$Ex(0)^2 \leqq A ,$$

$$E(x(s) - x(t))^2 \leqq A|s - t|^a , \qquad s, t \in [0, 1] . \tag{17}$$

Under either of the following conditions:

(i) *the process is Gaussian,*

(ii) $a > 1$,

the ε-entropy satisfies the inequality

$$H_\varepsilon(X) \leqq C(a)A^{1/a}\varepsilon^{-2/a} \tag{18}$$

for $\varepsilon < A^{\frac{1}{2}}$, where $C(a)$ is a number depending only on a.

PROOF. As in the proof of Lemma 4, we can assume $A = 1$.

For a given positive integer k, define the process $z(t)$ as follows:

(i) $z(j/k) = x(j/k) - x(0)$, $0 \leqq j \leqq k$

(ii) $z(t)$ linear, $(j-1)/k \leqq t \leqq j/k$; $1 \leqq j \leqq k$.

Put $z_j = z(j/k)$. Then for $1 \leqq j \leqq k$

$$E(z_j - z_{j-1})^2 = E\left[x\left(\frac{j}{k}\right) - x\left(\frac{j-1}{k}\right)\right]^2 \leqq k^{-\alpha} \tag{19}$$

by (17), and if s and t both belong to the same interval $[(j-1)/k, j/k]$,

$$E[z(s) - z(t)]^2 = [k(s-t)]^2 E(z_j - z_{j-1})^2 \leqq |s-t|^\alpha . \tag{20}$$

Define the stochastic process $y(t)$ by

$$x(t) = x(0) + y(t) + z(t) . \tag{21}$$

Then by Lemma 6,

$$H_\varepsilon(X) = H_{\varepsilon/3}(X_0) + H_{\varepsilon/3}(Y) + H_{\varepsilon/3}(Z) , \tag{22}$$

where X_0 is the 1-dimensional space of values of $x(0)$, and Y and Z are $C[0, 1]$ with the measures induced by the processes $y(t)$, $z(t)$.

In Z, the measure is concentrated on the $k + 1$-dimensional subspace of polygonal curves with vertices at the points j/k. In this subspace, two curves $z(t)$, $z'(t)$ have distance

$$d(z, z') = \max_{0 \leqq j \leqq k} |z_j - z_j'| .$$

If we map this subspace into the $(k + 1)$-dimensional space Z^* by

$$z(t) \to \zeta = \left(\frac{3}{\varepsilon} z_0, \cdots, \frac{3}{\varepsilon} z_k\right) = (\zeta_0, \cdots, \zeta_k) ,$$

and take the measure on this space which is induced by the mapping Lemma 7 can be applied. We have $E(\zeta_0^2) = 0$, and

$$E(\zeta_j - \zeta_{j-1})^2 = 9\varepsilon^{-2}k^{-\alpha} , \qquad 1 \leqq j \leqq k ,$$

by (19). The mapping multiplies all distances by $3/\varepsilon$, hence

$$H_{\varepsilon/3}(Z) = H_1(Z^*) \leqq h_{k+1}(9\varepsilon^{-2}k^{-\alpha}) . \tag{23}$$

X_0 is the 1-dimensional space of $x(0)$, which has second moment $\leqq 1$. Hence by [4], Lemma 5,

$$H_{\varepsilon/3}(X_0) \leqq B[1 + \log^+ (3/\varepsilon)] , \tag{24}$$

where B is a universal constant. It remains to consider the space Y.

Note first that we can easily reduce the proof to the case where $x(t)$ has mean zero. For, if $Ex(t) = m(t)$, and $x'(t) = x(t) - m(t)$, the identity

$$E[x'(s) - x'(t)]^2 + [m(s) - m(t)]^2 = E[x(s) - x(t)]^2$$

shows that the process $x'(t)$ satisfies the hypotheses of the Theorem, and this new process has the same ε-entropy, since $x(t) \to x'(t)$ is an isometry.

Let $y_j(t)$ be the restriction of $y(t)$ to the interval $[(j-1)/k, j/k]$. This process induces a measure on the space Y_j of continuous functions on the jth interval. By Lemma 6,

$$H_{\varepsilon/3}(Y) \leqq \textstyle\sum_{j=1}^{k} H_{\varepsilon/3}(Y_j)\,. \tag{25}$$

For $s, t \in [(j-1)/k, j/k]$ we have from (17) and (20)

$$E[y_j(s) - y_j(t)]^2 \leqq 4|s-t|^a\,. \tag{26}$$

Now we make a change of scale to map the interval of definition onto [0, 1]. Define

$$y_j'(t) = y_j\left(\frac{t+j-1}{k}\right), \qquad 0 \leqq t \leqq 1\,,$$

with the associated function space Y_j'. Clearly Y_j and Y_j' have the same ε-entropy. From (26),

$$E[y_j'(s) - y_j'(t)]^2 \leqq 4k^{-a}|s-t|^a\,, \qquad s, t \in [0, 1]\,.$$

Under condition (i) of the hypotheses, if we assume $Ex(t) = 0$, all the hypotheses of Lemma 4 are satisfied by $y_j'(t)$, with $\beta = a/2$, $A = 4k^{-a}$. The conclusion of that lemma takes the form

$$H_{\varepsilon/3}(Y_j') \leqq F(a, \varepsilon k^{a/2}/6) \tag{27}$$

for $\varepsilon \leqq 6k^{-a/2}$, where F is a function which is monotonic decreasing in the second argument. Under condition (ii). (27) is valid according to Lemma 5, with a different function F.

We estimate the terms on the right in (25) by (27):

$$H_{\varepsilon/3}(Y) \leqq kF(a, \varepsilon k^{a/2}/6)\,.$$

Combining with (22), (23) and (24),

$$H_\varepsilon(X) \leqq B[1 + \log^+(3/\varepsilon)] + h_{k+1}(9\varepsilon^{-2}k^{-a}) + kF(a, \varepsilon k^{a/2}/6)\,, \tag{28}$$

for $\varepsilon \leqq 6k^{-a/2}$.

Assume $\varepsilon \leqq 1$ and put $k = [(\varepsilon/6)^{-2/a}]$. Then $k \leqq (\varepsilon/6)^{-2/a} < 2k$, so $9\varepsilon^{-2}k^{-a} < 2^{a-2}$ and $\varepsilon k^{a/2}/6 \leqq 1$. From (28) we have

$$H_\varepsilon(X) \leqq B[1 + \log^+(3/\varepsilon)] + h_{k+1}(2^{a-2}) + F(a, 1)(\varepsilon/6)^{-2/a}\,. \tag{29}$$

According to Lemma 7,

$$h_{k+1}(2^{a-2}) = O(k) = O(\varepsilon^{-2/a})\,.$$

Thus the right side of (29) is bounded by a constant multiple of $\varepsilon^{-2/a}$. This completes the proof of Theorem 1.

THEOREM 2. *If $x(t)$ is a separable Gaussian process on* [0, 1] *with $Ex(0)^2 \leqq A$ and*

$$E[x(s) - x(t)]^2 \leqq A \log\left(\frac{2}{|s-t|}\right)^{-a}$$

for all $s, t \in [0, 1]$, *where* A *and* a *are constants with* $a > 1$, *then the paths are continuous with probability* 1 *and the* ε*-entropy of the process in the uniform norm is finite for* $\varepsilon > 0$. *In fact,*

$$H_\varepsilon(X) \leqq C \exp[C'\varepsilon^{-2/(a-1)}] \tag{30}$$

for $\varepsilon \leqq 1$, *where the constants* C, C' *depend only on* A *and* a.

PROOF. First, we can assume $x(t)$ has mean zero, as in the proof of Theorem 1. If $x(0)$ is not zero with probability 1, and $x(t) = x(0) + y(t)$, then by Lemma 6

$$H_\varepsilon(X) \leqq H_{\varepsilon/2}(X_0) + H_{\varepsilon/2}(Y),$$

where X_0 is the 1-dimensional space of values of $x(0)$ and Y is $C[0, 1]$ within $y(t)$ measure. The first term on the right has a bound of the form (30). Hence if we know the theorem is true for $y(t)$, it is also true for $x(t)$. Thus we may assume $x(0) \equiv 0$.

Let β be any positive number, and $\gamma = \frac{1}{2}\beta(a-1)$. Define S_k, $k = 1, 2, \cdots$, to be the subset of X on which

$$w_x(\delta) \leqq \frac{2^{k+1}}{1-2^{-\gamma}}\left(\log\frac{1}{\delta}\right)^{-\frac{1}{2}(a-1)} \tag{31}$$

for $\delta \leqq \exp(-2^{\beta n_k})$, where n_k will be specified later. By Lemma 3, if

$$2^{2k-2} > 2^{a\beta}A, \tag{32}$$

$$\Pr(X - S_k) < \left(\frac{2A}{\pi}\right)^{\frac{1}{2}} 2^{-k} \textstyle\sum_{j=n_k+1}^{\infty} 2^{-\frac{1}{2}j\beta+\frac{1}{2}a\beta} \exp\left[2^{j\beta}\left(2 - \frac{1}{2A}2^{2k-a\beta}\right)\right].$$

Let k_0 be the first value of k for which (32) is true. Then for $k \geqq k_0$, in this series the ratio of each term to the preceding is at most $2^{-\beta/2}$. Hence the series is dominated by $(1 - 2^{-\beta k})^{-1}$ times the first term, and

$$\Pr(X - S_k) < C_1 2^{-k-\frac{1}{2}\beta n_k} \exp[2^{\beta n_k+\beta}(2 - (2A)^{-1}2^{2k-a\beta})], \tag{33}$$

where C_1 depends only on A, a, β.

We will choose n_k so that (31) holds for $\delta \leqq \delta_k$, where

$$\frac{2^{k+1}}{1-2^{\gamma}}\left(\log\frac{1}{\delta_k}\right)^{-\frac{1}{2}(a-1)} = \frac{\varepsilon}{4}.$$

Solving for δ_k,

$$\delta_k = \exp\left\{-\left[\frac{2^{k+3}}{(1-2^{-\gamma})\varepsilon}\right]^{2/(a-1)}\right\} = \exp\left[-C_2\left(\frac{2^k}{\varepsilon}\right)^{2/(a-1)}\right]. \tag{34}$$

We need to choose n_k so that $\delta_k \leqq \exp(-2^{\beta n_k})$, or

$$2^{\beta n_k} \leqq C_2\left(\frac{2^k}{\varepsilon}\right)^{2/(a-1)} = C_2\left(\frac{2^k}{\varepsilon}\right)^{\beta/\gamma},$$

$$n_k \leqq \frac{k}{\gamma} + \frac{1}{\gamma}\log_2\frac{1}{\varepsilon} + \frac{1}{\beta}\log_2 C_2.$$

Suppose $\varepsilon \leqq 1$, and let n_k be the integral part of this expression. Then we have from (33), for $k \geqq k_0$,

$$(35) \qquad \Pr(X - S_k) < C_3 2^{-k-\frac{1}{2}k\beta/\gamma} \varepsilon^{\beta/2\gamma} \exp[2^{\beta k/\gamma}\varepsilon^{-\beta/\gamma} C_2(2 - (2A)^{-1}2^{2k-\alpha\beta})] .$$

This approaches zero as $k \to \infty$, so that $\{S_k\}$ is an increasing sequence of sets whose union has probability 1.

To partition S_k in to ε-sets, note that any $x(t) \in S_k$ can be mapped into a polygonal curve $x^*(t)$ with vertices at the points $t = j\delta_k$, $j = 0, 1, \cdots, [1/\delta_k] = j_k$ such that

$$\begin{aligned} &x^*(0) = 0 , \\ &x^*((j\delta_k) - x^*(j-1)\delta_k) = 0 \quad \text{or} \quad \pm\varepsilon/4 , \qquad j = 1, \cdots, j_k , \\ &x^*(t) = x(j_k\delta_k) , \qquad t > j_k\delta_k , \end{aligned}$$

and

$$|x^*(j\delta_k) - x(j\delta_k)| \leqq \varepsilon/8 , \qquad j = 0, \cdots, j_k .$$

This is possible because $x(t)$ varies by at most $\varepsilon/4$ on each of the intervals $[j\delta_k, (j+1)\delta_k]$. The curve $x^*(t)$ associated with $x(t)$ in this way has

$$|x^*(t) - x(t)| \leqq \varepsilon/2 , \qquad 0 \leqq t \leqq 1 .$$

Hence we get an ε-partition of S_k by grouping the curves $x(t)$ corresponding to the same $x^*(t)$ in one set. The number of such sets is at most $3^{1/\delta_k}$. It follows that S_k, or any subset of S_k, has ε-entropy at most $(\log 3)/\delta_k$.

Let k_1 be a positive integer. If we define $T_{k_1} = S_{k_1}$ and

$$T_k = S_k - S_{k-1} , \quad k > k_1 ,$$

then $\{T_{k_1}, T_{k_1+1}, T_{k_1+2}, \cdots\}$ is a partition of X of probability 1. Hence we can estimate the ε-entropy of X by

$$H_\varepsilon(X) \leqq \textstyle\sum_{k=k_1}^{\infty} \Pr(T_k)\left[\log \dfrac{1}{\Pr(T_k)} + H_\varepsilon(T_k)\right] .$$

Choose k_1 so that

$$2^{2k_1-\alpha\beta} > 4A + 4A \cdot 2^{\beta/\gamma} .$$

Then $k_1 \geqq k_0$, and by (35), if $k \geqq k_1$

$$\begin{aligned} \Pr(T_{k+1}) &< C_3 2^{-k-\frac{1}{2}k\beta/\gamma}\varepsilon^{\beta/2\gamma} \exp[-2C_2 \cdot 2^{\beta(k+1)/\gamma}\varepsilon^{-\beta/\gamma}] \\ &< C_3 2^{-k} \exp(-2C_2 2^{\beta(k+1)/\gamma}\varepsilon^{-\beta/\gamma}) . \end{aligned}$$

Hence if k_1 is large enough that

$$C_3 2^{-k_1} \exp(-2C_2 2^{\beta k_1/\gamma}) < 1/e ,$$

$$\begin{aligned} H_\varepsilon(X) \leqq \frac{1}{e} + \frac{\log 3}{\delta_{k_1}} &+ 2C_3 \textstyle\sum_{k=k_1+1}^{\infty} 2^{-k} \exp(-2C_2 2^{\beta k/\gamma}\varepsilon^{-\beta/\gamma}) \log \dfrac{2^{k-1}}{C_3} \\ &+ 4C_2 C_3 \varepsilon^{-\beta/\gamma} \textstyle\sum_{k=k_1+1}^{\infty} 2^{-k+\beta k/\gamma} \exp(-2C_2 2^{\beta k/\gamma}\varepsilon^{-\beta/\gamma}) \\ &+ 2C_3 \log 3 \textstyle\sum_{k=k_1+1}^{\infty} 2^{-k} \exp(-C_2 2^{\beta k/\gamma}\varepsilon^{-\beta/\gamma}) , \end{aligned}$$

using the relations

$$H_\varepsilon(T_k) = \log 3/\delta_k ,$$
$$\delta_k = \exp(-C_2 2^{\beta k/\gamma} \varepsilon^{-\beta/\gamma}) .$$

Each of the three series converges uniformly for $\varepsilon \leqq 1$, to a bounded function of ε. Hence

$$\begin{aligned} H_\varepsilon(X) &\leqq \frac{\log 3}{\delta_{k_1}} + C_4 \varepsilon^{-\beta/\gamma} \\ &= (\log 3) \exp(C_2 2^{\beta k_1/\gamma} \varepsilon^{-\beta/\gamma}) + C_4 \varepsilon^{-\beta/\gamma} \\ &\leqq C \exp(C_2 2^{\beta k_1/\gamma} \varepsilon^{-\beta/\gamma}) . \end{aligned}$$

This completes the proof.

EXAMPLE. The Wiener Process on [0, 1] has covariance $E(x(s)x(t)) = \min(s, t)$ and so

$$E(x(s) - x(t))^2 = |s - t| .$$

Then Theorem 1 can be used with $A = a = 1$ to conclude

$$H_\varepsilon(X) \leqq C/\varepsilon^2 .$$

For $L_2[0, 1]$, [2] had

$$\frac{17}{32\varepsilon^2} \lesssim H_\varepsilon(X) \lesssim \frac{1}{\varepsilon^2} ,$$

for $\varepsilon \to 0$ (the notation "$A(\varepsilon) \lesssim B(\varepsilon)$" means "$\limsup A(\varepsilon)/B(\varepsilon) \leqq 1$"). Thus, the entropy is at most a constant times larger in the uniform norm than in the L_2-norm, and the same comment applies for all the examples of Theorem 5 of [5]. In information-theoretic terms, it is possible to transmit the sample functions of the Wiener process on the unit interval with a finite average number of bits in such a way that the sample functions are with probability 1 known everywhere on the unit interval, not merely at some "sampling instants," to within ε. Furthermore, the partitions needed in Theorem 1 are constructively defined, so that one could in principal actually carry this out.

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