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THE CENTRAL LIMIT THEOREM FOR RANDOM MOTIONS OF d-DIMENSIONAL EUCLIDEAN SPACE[1]

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Let $g_1, g_2, \cdots$ be random elements of the Euclidean group of motions of d-dimensional Euclidean space R^d ($d \geqq 1$), that are independent and identically distributed. The product $g_1 \cdots g_n$ is represented in the form $t(n)r(n)$, where $t(n)$ is a translation and $r(n)$ is a rotation. In this paper it is shown that under natural conditions $r(n)$ and $n^{-\frac{1}{2}}t(n)$ jointly converge weakly as $n \to \infty$ to the product distribution of the Haar measure on a certain closed subgroup of the rotations group, and a normal distribution on R^d, with mean zero and covariance matrix $\sigma^2 I$ (I is the identity matrix), and the value of σ^2 is identified.

1. Introduction and results. The purpose of this work is to obtain a limiting distribution for the product of n independent and identically distributed random rigid motions of Euclidean space, as $n \to \infty$ (a rigid motion of Euclidean space is a linear transformation of the space which preserves the distance between points, and the group of all these transformations is called the Euclidean group). Such a result is given by V. N. Tutubalin in his paper [6], for Euclidean spaces of dimensions two and three, but without fully identifying the limiting distribution. Here we present a central limit theorem which holds for Euclidean spaces of all dimensions, and completely determine the limiting distribution; moreover, our conditions are weaker than those of [6].

Let $\mathscr{G}$ be the Euclidean group of motions of d-dimensional Euclidean space R^d ($d \geqq 1$), $\mathscr{T}$ the subgroup of $\mathscr{G}$ of all parallel translations (isomorphic to R^d), and $\mathscr{R}$ the subgroup of $\mathscr{G}$ of all rotations about the origin (both proper and improper). We use leftward application of group elements on R^d, and denote $x\bar{g}$ the image of $x \in R^d$ under $g \in \mathscr{G}$. It is well known that each element $g \in \mathscr{G}$ may be written as $g = t(g)r(g)$, where $t(g) \in \mathscr{T}$ and $r(g) \in \mathscr{R}$; and with this representation, if $g_i = t_i r_i$, $i = 1, \cdots, n$, are elements of $\mathscr{G}$, the product $g(n) = g_1 \cdots g_n$ is expressed as $g(n) = t(n)r(n)$, where

$$r(n) = r_1 \cdots r_n ,$$

and

$$t(n) = \sum_{i=1}^{n} t_i \overline{(r_0 \cdots r_{i-1})^{-1}} ,$$

with $r_0 =$ the identity element (see [6]). We represent the points $x \in R^d$ and

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$t \in \mathscr{T}$ by column-vectors denoted X and T respectively, and the elements $r \in \mathscr{R}$ by orthogonal matrices of order d denoted R, by the rule that to the point $x\bar{r}$ corresponds the vector $R^{-1}X$. Consequently the group elements $r(n)$ and $t(n)$ are respectively represented by the matrix

$$R(n) = R_n^{-1} \cdots R_1^{-1}$$

and the vector

$$T(n) = \sum_{i=1}^{n} R_0 \cdots R_{i-1} T_i ,$$

with $R_0 = I$, the identity matrix.

Let $(\Omega, \mathscr{F}, P)$ be a probability space on which is defined a sequence $g_i = t_i r_i$, $i = 1, 2, \cdots$, of random elements of $\mathscr{G}$, that are independent and identically distributed. We seek to obtain a limiting distribution of the product $g(n) = g_1 \cdots g_n$, or equivalently (see [5]), an asymptotic expression of the n-fold convolution of the distribution of g_1 with itself, as $n \to \infty$. By virtue of the above representations we need only to obtain a limiting joint distribution of $R(n)$ and $T(n)$ as $n \to \infty$.

For $R(n)$ the following is clear by a known theorem (see [5]). Let S_1 denote the support of the distribution of R_1 in the orthogonal group $O(d)$, viz. the set $\{x \in O(d) : P[R_1 \in U] > 0$ for all open sets U containing $x\}$, and $G(S_1)$ the closed subgroup of $O(d)$ generated by S_1. *The matrix $R(n)$ converges weakly as $n \to \infty$ if and only if S_1 is not contained in any (nontrivial) coset of any proper closed normal subgroup of $G(S_1)$, and when the weak limit exists it is the Haar distribution on $G(S_1)$.* Indeed, $R(n)$ converges weakly to M if and only if $R(n)^{-1}$ converges weakly to M^{-1}, and M has the Haar distribution on $G(S_1)$ if and only if so does M^{-1}.

In order to obtain a limit theorem for $T(n)$ we introduce the following definition. We say that a random matrix A of order d is *irreducible* if $P[AV \subseteqq V] < 1$ for all nontrivial subspaces V of R^d, for $d \geqq 2$, and if A is not almost surely nonnegative, for $d = 1$. By requiring the R_i to be irreducible we prevent the $T(n)$ from being all restricted to a proper subspace of R^d, for $d \geqq 2$, and from having all the same direction, for $d = 1$, thus avoiding degenerate limits for $T(n)$ (at least for these causes). This definition has the property that if A is an irreducible random contractive matrix then the matrix $I - EA$ is invertible (see Lemma 1).

In the following, a prime denotes matrix transposition, tr stands for matrix trace, and we use the vector norm $||x|| = (\sum_{i=1}^{d} x_i^2)^{\frac{1}{2}}$, with $x = (x_1, \cdots, x_d)'$.

The main result is the central limit theorem for $T(n)$.

THEOREM. *If*

(i) $E||T_1||^2 < \infty$, *and*

(ii) R_1 *is irreducible*,

then the vector $n^{-\frac{1}{2}}T(n)$ converges weakly as $n \to \infty$ to the normal distribution with mean zero and covariance matrix $\sigma^2 I$, where

$$\sigma^2 = d^{-1}\{E||T_1||^2 + 2 \operatorname{tr} [(I - ER_1)^{-1} ET_1 E(T_1' R_1)]\} .$$

REMARKS. (1) This theorem is independent of whether $R(n)$ converges or not; for example, in R^2, if r_1 is the rotation by $\pi/2$, then R_1 is irreducible, and hence the theorem holds, but $R(n)$ does not converge.

(2) It is easily seen that if M is a random orthogonal matrix independent of $T(n)$ then $n^{-\frac{1}{2}}MT(n)$ has the same weak limit as $n^{-\frac{1}{2}}T(n)$ (the theorem in [6] is of this form).

(3) Note that $E(T_1'R_1)$ is $EO\bar{g}_1$ and $E||T_1||^2 = E||O\bar{g}_1||^2$, where O is the origin of R^d.

Finally, *under the conditions of the theorem, when $R(n)$ converges weakly, $R(n)$ and $n^{-\frac{1}{2}}T(n)$ jointly converge weakly as $n \to \infty$ to the product distribution of the Haar measure on $G(S_1)$ and the normal distribution N on R^d with mean zero and covariance matrix $\sigma^2 I$, and hence, even if $R(n)$ does not converge, the image $x\overline{g(n)}$ of $x \in R^d$ under $g(n)$ is approximately distributed as $n^{\frac{1}{2}}N$ for large n.*

In [6] the central limit theorem for $T(n)$ is proved for $d = 2$ and $d = 3$, with the assumptions $\mathscr{R} =$ the group of all proper rotations about the origin, and $G(S_1) =$ the group of all proper matrices. This condition on $G(S_1)$ is much stronger than our assumption of irreducibility of R_1; e.g., in the two-dimensional example in [6] the rotation r_i, which is given by the angle $\alpha_i = \pm\alpha$, each with probability $\frac{1}{2}$, the constant α is required to be incommensurable with respect to the length of the unit circle, whereas our condition is that α be different from 0 or π. Also, the parameter σ^2 is not identified. The proof in [6] employs group representation theoretic methods special for R^2 and R^3. Our approach is different, and has also been used to establish a central limit theorem related to a problem of another kind (see [2] and [3]).

REMARK. If we consider the sequence of random functions $\{n^{-\frac{1}{2}}T([nt]),\ t \geqq 0\}$ ([ ] denotes integral part), under the assumption that $E||T_1||^{2+\nu} < \infty$ for some $\nu > 0$ it is possible to show that it is tight (by using the full strength of Lemma 3) and has asymptotically stationary and independent increments, and thence to conclude that it converges weakly as $n \to \infty$ (in the function space $D[0, \infty)^d$) to a d-dimentional Brownian motion process with variance parameter σ^2.

2. The proof. We use the matrix norm $||A|| = \sup\{||Ax|| : ||x|| = 1\}$, and the following basic inequalities for the conditional expectation $E[\ \ |\mathscr{B}]$ with respect to a Borel field $\mathscr{B}$ (which hold almost surely): for a random vector X in R^d and for a random matrix A of order d

$$||E[X|\mathscr{B}]|| \leqq d^{\frac{1}{2}}E[||X|| \,|\mathscr{B}] \qquad \text{and} \qquad ||E[A|\mathscr{B}]|| \leqq d^{\frac{1}{2}}E[||A|| \,|\mathscr{B}] .$$

These are a consequence of the elementary inequalities

$$(\textstyle\sum_{i=1}^d x_i^2)^{\frac{1}{2}} \leqq \sum_{i=1}^d |x_i| \leqq d^{\frac{1}{2}}(\sum_{i=1}^d x_i^2)^{\frac{1}{2}} .$$

We will repeatedly employ the basic inequalities, and elementary properties of matrices, norms and (conditional) expectation, without stating them in detail. [ ] stands for the integral part of a real number, and when a real number appears where an integer should obviously be it is interpreted as its integral part.

First we prove the property of irreducible random contractive matrices mentioned in the Introduction.

LEMMA 1. *If A is an irreducible random contractive matrix then the matrix $I - EA$ is invertible.*

PROOF. This is obvious for $d = 1$. For $d \geqq 2$, if $I - EA$ is singular there is a nonzero vector x such that $EAx = x$, which implies that $Ax = x$ almost surely, because the point EAx is an average of points that lie in the convex set $\{z: ||z|| \leqq ||x||\}$ and hence can be x only if A fixes x with probability one; but then the subspace spanned by x is almost surely invariant under A.

We will first prove the theorem under the assumption that $E||T_1||^{2+\nu} < \infty$ for some $\nu > 0$.

LEMMA 2. *There is a constant K such that*

$$E||\sum_{i=j+1}^{k} R_0 \cdots R_{i-1} T_i||^\gamma \leqq (KE||T_1||^2(k-j))^{\gamma/2}$$

for each $0 \leqq j < k$ and each $0 < \gamma \leqq 2$.

COROLLARY 1. *There is constant K such that*

$$E||n^{-\frac{1}{2}} \sum_{i=\beta n+1}^{\alpha n} R_0 \cdots R_{i-1} T_i||^\gamma \leqq (K(\alpha - \beta))^{\gamma/2}$$

for each $0 \leqq \beta < \alpha$ and $0 < \gamma \leqq 2$, and all $n \geqq 1/(\alpha - \beta)$.

Corollary 1 follows directly from Lemma 2.

PROOF OF LEMMA 2. Since $E||\ ||^\gamma \leqq (E||\ ||^2)^{\gamma/2}$, it suffices to prove the lemma for $\gamma = 2$. After direct computaion, involving the assumption that (R_i, T_i), $i = 1, 2, \cdots$, are independent and identically distributed, we have

$$\begin{aligned} E||\sum_{i=j+1}^{k} R_0 \cdots R_{i-1} T_i||^2 \\ &= (k-j)E||T_1||^2 \\ &\quad + 2 \sum_{i=1}^{k-j-1} E((R_0 \cdots R_{i-1} T_i)' R_0 \cdots R_i \sum_{h=i+1}^{k-j} (ER_1)^{h-i-1} ET_1) . \end{aligned}$$

By Lemma 1 we can write

$$\sum_{h=i+1}^{k-j} (ER_1)^{h-i-1} = (I - (ER_1)^{k-j-i})(I - ER_1)^{-1} ,$$

and hence, using norm inequalities and the fact that $||ER_1|| \leqq 1$, we obtain

$$E||\sum_{i=j+1}^{k} R_0 \cdots R_{i-1} T_i||^2 \leqq (k-j)E||T_1||^2(1 + 4d^{\frac{1}{2}}||(I - ER_1)^{-1}||) .$$

LEMMA 3. *There is a constant K such that*

$$P[\max_{\alpha n+1 \leqq k \leqq (\alpha+\delta)n} ||n^{-\frac{1}{2}} \sum_{i=\alpha n+1}^{k} R_0 \cdots R_{i-1} T_i|| \geqq \varepsilon] \leqq K\delta^{1+\nu/2}\varepsilon^{-2-\nu}$$

for each $\alpha \geqq 0$, $\varepsilon > 0$ and $\delta > 0$, and all $n \geqq \max\{\delta^{-1}, c\varepsilon^{-2}\}$, where c is a positive constant (here we assume $\nu \leqq 2$, as we certainly may do).

COROLLARY 2.

$$\lim_{\delta \downarrow 0} \delta^{-1} \limsup_{n\to\infty} \int_{[||n^{-\frac{1}{2}} \sum_{i=\alpha n+1}^{(\alpha+\delta)n} R_0 \cdots R_{i-1} T_i|| \geqq \varepsilon]} ||n^{-\frac{1}{2}} \sum_{i=\alpha n+1}^{(\alpha+\delta)n} R_0 \cdots R_{i-1} T_i||^2 \, dP = 0$$

for all $\alpha \geqq 0$ and $\varepsilon > 0$.

PROOF OF COROLLARY 2. It follows from the identity (see [1] page 223)

$$\int_{[\|n^{-\frac12}\sum_{i=\alpha n+1}^{(\alpha+\delta)n} R_0\cdots R_{i-1}T_i\|\geq\varepsilon]} \|n^{-\frac12}\sum_{i=\alpha n+1}^{(\alpha+\delta)n} R_0\cdots R_{i-1}T_i\|^2\,dP$$
$$= \varepsilon^2 P[\|n^{-\frac12}\sum_{i=\alpha n+1}^{(\alpha+\delta)n} R_0\cdots R_{i-1}T_i\|\geq\varepsilon]$$
$$+ \int_{\varepsilon^2}^{\infty} P[\|n^{-\frac12}\sum_{i=\alpha n+1}^{(\alpha+\delta)n} R_0\cdots R_{i-1}T_i\|\geq t^{\frac12}]\,dt$$

and Lemma 3.

PROOF OF LEMMA 3. We have (see [1] page 88, for the second inequality)

$$\begin{aligned} &P[\max_{\alpha n+1\leq k\leq(\alpha+\delta)n} \|n^{-\frac12}\textstyle\sum_{i=\alpha n+1}^{k} R_0\cdots R_{i-1}T_i\|\geq\varepsilon] \\ \text{(a)}\qquad &\leq P[\max_{1\leq k\leq \delta n+1} \|n^{-\frac12}\textstyle\sum_{i=1}^{k} R_0\cdots R_{i-1}T_i\|\geq\varepsilon] \\ &\leq P[\max_{1\leq k\leq \delta n+1} \min\{\|n^{-\frac12}\textstyle\sum_{i=1}^{k} R_0\cdots R_{i-1}T_i\|, \\ &\qquad \|n^{-\frac12}\textstyle\sum_{i=k+1}^{\delta n+1} R_0\cdots R_{i-1}T_i\|\}\geq\varepsilon/4] + P[\max_{1\leq k\leq\delta n+1}\|T_k\|\geq n^{\frac12}\varepsilon/4]\,. \end{aligned}$$

For the last term in (a) we have

$$\begin{aligned} P[\max_{1\leq k\leq\delta n+1}\|T_k\|\geq n^{\frac12}\varepsilon/4] &\leq 1-(1-P[\|T_1\|\geq n^{\frac12}\varepsilon/4])^{\delta n+1} \\ &\leq 1-[1-(n^{\frac12}\varepsilon/4)^{-2-\nu}E\|T_1\|^{2+\nu}]^{\delta n+1}\,, \end{aligned}$$

the last inequality holding, by Chebyshev's inequality, for all sufficiently large values of $n\varepsilon^2$. Using the inequality $1-(1-x)^a \leq ax$ for $0\leq x\leq 1$ and $a\geq 1$, it follows that

$$P[\max_{1\leq k\leq\delta n+1}\|T_k\|\geq n^{\frac12}\varepsilon/4] \leq 4^{2+\nu}E\|T_1\|^{2+\nu}\varepsilon^{-2-\nu}n^{-1-\nu/2}(\delta n+1)$$

for all sufficiently large values of $n\varepsilon^2$, and hence there is a constant K_1 such that

$$\text{(b)}\qquad P[\max_{1\leq k\leq\delta n+1}\|T_k\|\geq n^{\frac12}\varepsilon/4] \leq K_1\delta^{1+\nu/2}\varepsilon^{-2-\nu}$$

for all $n\geq\max\{\delta^{-1}, c\varepsilon^{-2}\}$, where c is a positive constant.

For any $0\leq i<j<k\leq\delta n+1$ we have

$$\begin{aligned} &P[\|n^{-\frac12}\textstyle\sum_{h=i+1}^{j} R_0\cdots R_{h-1}T_h\|\geq\varepsilon/4,\ \|n^{-\frac12}\textstyle\sum_{h=j+1}^{k} R_0\cdots R_{h-1}T_h\|\geq\varepsilon/4] \\ \text{(c)}\qquad &\leq P[\|n^{-\frac12}\textstyle\sum_{h=i+1}^{j-1} R_i\cdots R_{h-1}T_h\|\geq\varepsilon/8, \\ &\qquad \|n^{-\frac12}\textstyle\sum_{h=j+1}^{k} R_j\cdots R_{h-1}T_h\|\geq\varepsilon/4] + P[\|T_1\|\geq n^{\frac12}\varepsilon/8]\,. \end{aligned}$$

Using independence, Chebyshev's inequality and Lemma 2 on the first term on the right of (c) (note that $1+\nu/2\leq 2$), we see that there are constants K_2 and K_3 such that

$$\begin{aligned} &P[\|n^{-\frac12}\textstyle\sum_{h=i+1}^{j-1} R_i\cdots R_{h-1}T_h\|\geq\varepsilon/8,\ \|n^{-\frac12}\textstyle\sum_{h=j+1}^{k} R_j\cdots R_{h-1}T_h\|\geq\varepsilon/4] \\ \text{(d)}\qquad &\leq (8(n^{\frac12}\varepsilon)^{-1})^{1+\nu/2}(K_2(j-1-i))^{(1+\nu/2)/2}(4(n^{\frac12}\varepsilon)^{-1})^{1+\nu/2}(K_2(k-j))^{(1+\nu/2)/2} \\ &\leq (n^{\frac12}\varepsilon)^{-2-\nu}(K_3(k-i))^{1+\nu/2}\,. \end{aligned}$$

By Chebyshev's inequality there is a constant K_4 such that

$$\text{(e)}\qquad P[\|T_1\|\geq n^{\frac12}\varepsilon/8] \leq K_4(n^{\frac12}\varepsilon)^{-2-\nu}\,.$$

It follows from (c), (d) and (e) that there is a constant K_5 such that

$$\begin{aligned} &P[\|n^{-\frac12}\textstyle\sum_{h=i+1}^{j} R_0\cdots R_{h-1}T_h\|\geq\varepsilon/4,\ \|n^{-\frac12}\textstyle\sum_{h=j+1}^{k} R_0\cdots R_{h-1}T_h\|\geq\varepsilon/4] \\ &\qquad\leq (n^{\frac12}\varepsilon)^{-2-\nu}((k-i)K_5)^{1+\nu/2} \end{aligned}$$

for all $\varepsilon > 0$ and all n, and therefore by Theorem 12.1 of [1] there is a constant K_6 such that for all $\varepsilon > 0$ and all n

$$\text{(f)} \quad P[\max_{1 \leqq k \leqq \delta n+1} \min \{||n^{-\frac{1}{2}} \textstyle\sum_{i=1}^{k} R_0 \cdots R_{i-1} T_i||, \\ ||n^{-\frac{1}{2}} \textstyle\sum_{i=k+1}^{\delta n+1} R_0 \cdots R_{i-1} T_i||\} \geqq \varepsilon/4] \\ \leqq (n^{\frac{1}{2}}\varepsilon)^{-2-\nu}(K_6(\delta n + 1))^{1+\nu/2} \leqq K_7 \delta^{1+\nu/2}\varepsilon^{-2-\nu} ,$$

the last inequality holding for all $n \geqq \delta^{-1}$ and some constant K_7.

The lemma is established by (a), (b) and (f).

In the following lemmas $\mathscr{F}_{\alpha n}$ will denote the Borel field generated by R_{i-1}, T_i, $1 \leqq i \leqq \alpha n$, for $\alpha n \geqq 1$, and $\mathscr{F}_{\alpha n} = \{\phi, \Omega\}$ for $0 \leqq \alpha n < 1$.

LEMMA 4.

$$\limsup_{n\to\infty} E||E[n^{-\frac{1}{2}} \textstyle\sum_{i=\alpha n+1}^{(\alpha+\delta)n} R_0 \cdots R_{i-1} T_i \,|\, \mathscr{F}_{\alpha n}]|| = 0$$

for all $\alpha \geqq 0$ *and* $\delta > 0$.

We omit the proof of Lemma 4 because we will give that of Lemma 7, which runs along the same lines but involves additional material.

For the proof of Lemma 7 we need the following result.

LEMMA 5. *If* $R_1, R_2, \cdots$ *are independent and identically distributed irreducible random orthogonal matrices of order* d, *and* S *is a random symmetric matrix of order* d, *then*

$$n^{-1} \textstyle\sum_{i=1}^{n} E[R_1 \cdots R_i S(R_1 \cdots R_i)' \,|\, S] \to (d^{-1} \operatorname{tr} S)I \quad \textit{everywhere}$$

as $n \to \infty$, *and if* $E||S||^p < \infty$ *for given* $p > 0$, *then*

$$E||n^{-1} \textstyle\sum_{i=1}^{n} E[R_1 \cdots R_i S(R_1 \cdots R_i)' \,|\, S] - (d^{-1} \operatorname{tr} S)I||^p \to 0$$

as $n \to \infty$.

PROOF. Proving the first statement for each point of Ω is the same as proving it for S constant. Let

$$\varphi(n) = n^{-1} \textstyle\sum_{i=1}^{n} E(R_1 \cdots R_i S(R_1 \cdots R_i)') .$$

$\{\varphi(n)\}$ is uniformly bounded, hence every subsequence has a convergent subsequence, and therefore it suffices to show that each convergent subsequence has the limit $(d^{-1} \operatorname{tr} S)I$; so let us assume that for a subsequence $\{n_k\}$ of $\{n\}$, $\varphi(n_k) \to M$ as $k \to \infty$. We can write

$$\varphi(n) = n^{-1} \textstyle\sum_{i=2}^{n} E(R_1 E(R_2 \cdots R_i S(R_2 \cdots R_i)')R_1') + n^{-1}E(R_1 S R_1') \\ = E(R_1 \varphi(n) R_1') - n^{-1}E(R_1 \cdots R_{n+1} S(R_1 \cdots R_{n+1})') + n^{-1}E(R_1 S R_1') ;$$

hence $\varphi(n_k) \to M$ as $k \to \infty$ implies that the matrix M is symmetric and satisfies the equation $M = E(R_1 M R_1')$, and therefore it follows from Lemma 6 that $M = mI$ for some constant m; but $m = d^{-1} \operatorname{tr} M = d^{-1} \lim_{k\to\infty} \operatorname{tr} \varphi(n_k) = d^{-1} \operatorname{tr} S$.

The second statement follows from the first by the dominated convergence theorem, since

$$||n^{-1} \textstyle\sum_{i=1}^{n} E[R_1 \cdots R_i S(R_1 \cdots R_i)' \,|\, S]||^p \leqq (d^{\frac{1}{2}}||S||)^p .$$

LEMMA 6. *If R is an irreducible random orthogonal matrix of order d and S is a random symmetric matrix of order d satisfying the equation $E[RSR' \mid S] = S$ almost surely, then $S = (d^{-1} \operatorname{tr} S)I$ almost surely.*

PROOF. The result is obvious for $d = 1$, so we assume $d \geqq 2$. It suffices to obtain the conclusion for each point of Ω for which the condition holds; thus we may suppose that S is a constant matrix satisfying the equation $E(RSR') = S$. There is an orthogonal matrix Q and a diagonal matrix L such that $QSQ' = L$, and therefore L satisfies $E(BLB') = L$, where $B = QRQ'$, and it is easy to see that B is an irreducible random orthogonal matrix. Denoting $L = (l_i \delta_{ij})$, where $(\delta_{ij}) = I$, and $B = (b_{ij})$, we see that the numbers l_i satisfy the equations $l_i \delta_{ij} = \sum_k l_k E(b_{ik} b_{jk})$ and in particular, for $j = i$, $l_i = \sum_k l_k Eb_{ik}^2$, $i = 1, \cdots, d$, and we may suppose $l_1 \leqq l_2 \leqq \cdots \leqq l_d$. Assuming that $l_1 = \cdots = l_u < l_{u+1} \leqq \cdots \leqq l_d$ with $u < d$ we obtain $l_i \sum_{k=u+1}^d Eb_{ik}^2 = \sum_{k=u+1}^d l_k Eb_{ik}^2$, $i = 1, \cdots, u$, where we have used $\sum_k Eb_{ik}^2 = 1$; this clearly implies that $Eb_{ik}^2 = 0$, $i = 1, \cdots, u$, $k = u+1, \cdots, d$, and hence $b_{ik} = 0$ almost surely $i = 1, \cdots, u$, $k = u + 1, \cdots, d$, and therefore the nontrivial subspace of points of R^d whose ith coordinates are zero for $i = 1, \cdots, u$ is almost surely invariant under B; but B is irreducible. Therefore $l_i = l$, $i = 1, \cdots, d$, for some constant l, and we conclude that this is the unique solution to $l_i \delta_{ij} = \sum_k l_k E(b_{ik} b_{jk})$. We have shown that $L = lI$, and the lemma follows from this.

LEMMA 7.

$$\limsup_{n\to\infty} E\|E[n^{-\frac{1}{2}} \textstyle\sum_{i=\alpha n+1}^{(\alpha+\delta)n} R_0 \cdots R_{i-1} T_i \cdot (n^{-\frac{1}{2}} \sum_{i=\alpha n+1}^{(\alpha+\delta)n} R_0 \cdots R_{i-1} T_i)' \mid \mathscr{F}_{\alpha n}] - \delta\sigma^2 I\| = 0$$

for all $\alpha \geqq 0$ and $\delta > 0$.

PROOF. First we note that

$$\operatorname{tr}[(I - ER_1)^{-1} ET_1 E(T_1' R_1)] = \operatorname{tr} E[R_1(I - ER_1)^{-1}(ET_1)T_1'],$$

and hence

$$\sigma^2 = d^{-1}\{E\|T_1\|^2 + 2 \operatorname{tr} E[R_1(I - ER_1)^{-1}(ET_1)T_1']\}.$$

Let

$$F_n = E[n^{-\frac{1}{2}} \textstyle\sum_{i=\alpha n+1}^{(\alpha+\delta)n} R_0 \cdots R_{i-1} T_i (n^{-\frac{1}{2}} \sum_{j=\alpha n+1}^{(\alpha+\delta)n} R_0 \cdots R_{j-1} T_j)' \mid \mathscr{F}_{\alpha n}].$$

Using our distribution assumptions and properties of conditional expectation, and denoting $b(n) = [(\alpha + \delta)n] - [\alpha n] - 1$, we obtain, after long and tedious but straightforward computations,

$$\begin{aligned} F_n = {} & E[R_0 \cdots R_{\alpha n}\{n^{-1} \textstyle\sum_{i=1}^{b(n)} E(R_1 \cdots R_i E(T_1 T_1')(R_1 \cdots R_i)')\}(R_0 \cdots R_{\alpha n})' \mid \mathscr{F}_{\alpha n}] \\ & + E[R_0 \cdots R_{\alpha n}\{n^{-1} \textstyle\sum_{i=1}^{b(n)-1} E(R_1 \cdots R_i(E(T_1(ET_1')(I - ER_1')^{-1}R_1') \\ & + E(R_1(I - ER_1)^{-1}(ET_1)T_1'))(R_1 \cdots R_i)')\}(R_0 \cdots R_{\alpha n})' \mid \mathscr{F}_{\alpha n}] \\ & + U_n - V_n, \end{aligned}$$

where

$$\begin{aligned} U_n = {} & n^{-1}E[R_0 \cdots R_{\alpha n}\{T_{\alpha n+1} T'_{\alpha n+1} + T_{\alpha n+1}(ET_1')(I - (ER_1')^{b(n)})(I - ER_1')^{-1}R'_{\alpha n+1} \\ & + R_{\alpha n+1}(I - (ER_1)^{b(n)})(I - ER_1)^{-1}(ET_1)T'_{\alpha n+1}\}(R_0 \cdots R_{\alpha n})' \mid \mathscr{F}_{\alpha n}] \end{aligned}$$

and

$$V_n = E[R_0 \cdots R_{\alpha n}\{n^{-1} \sum_{i=1}^{b(n)-1} E(R_1 \cdots R_i(T_{i+1}(ET_1')(ER_1')^{b(n)-i}(I - ER_1')^{-1}R'_{i+1} + R_{i+1}(ER_1)^{b(n)-i}(I - ER_1)^{-1}(ET_1)T'_{i+1})(R_1 \cdots R_i)')\}(R_0 \cdots R_{\alpha n})' \mid \mathscr{F}_{\alpha n}] .$$

(U_n contains the terms having i or j equal to $\alpha n + 1$; the first term of F_n is the sum of the terms with $i = j$; the sum of the terms with $i \neq j$ is equal to the following expression plus its transpose:

$$E[R_0 \cdots R_{\alpha n}\{n^{-1} \sum_{i=\alpha n+2}^{(\alpha+\delta)n-1} E(R_{\alpha n+1} \cdots R_{i-1} R_i \sum_{j=i+1}^{(\alpha+\delta)n} (ER_1)^{j-i-1}(ET_1)T_i' \times (R_{\alpha n+1} \cdots R_{i-1})')\}(R_0 \cdots R_{\alpha n})' \mid \mathscr{F}_{\alpha n}] ,$$

where the sum on j can be written as

$$\sum_{j=i+1}^{(\alpha+\delta)n} (ER_1)^{j-i-1} = (I - (ER_1)^{(\alpha+\delta)n-i})(I - ER_1)^{-1} ;$$

the second term of F_n is the sum of the terms with $i \neq j$ having the factor $(I - ER_1)^{-1}$, plus its transpose, and V_n is the sum of those having the factors $(ER_1)^{(\alpha+\delta)n-i}(I - ER_1)^{-1}$, plus its transpose.)

Using properties of the norm and conditional expectation, we have

$$\begin{aligned} E\|F_n - \delta\sigma^2 I\| &\leq d^{\frac{1}{2}}\|n^{-1} \sum_{i=1}^{b(n)} E(R_1 \cdots R_i E(T_1 T_1')(R_1 \cdots R_i)') - \delta d^{-1}E\|T_1\|^2 I\| \\ &\quad + d^{\frac{1}{2}}\|n^{-1} \sum_{i=1}^{b(n)-1} E(R_1 \cdots R_i(E[T_1(ET_1')(I - ER_1')^{-1}R_1'] \\ &\quad + E[R_1(I - ER_1)^{-1}(ET_1)T_1'])(R_1 \cdots R_i)') \\ &\quad - \delta d^{-1}\, 2 \operatorname{tr} E[R_1(I - ER_1)^{-1}(ET_1)T_1']I\| + E\|U_n\| + E\|V_n\| . \end{aligned}$$

Since $b(n)n^{-1} \to \delta$, the first two terms converge to zero as $n \to \infty$ by Lemma 5 (with S constant). Again by properties of the norm and conditional expectation, it is clear that there is a constant K such that

$$E\|U_n\| \leq Kn^{-1}E\|T_1\|^2 ,$$

and hence $E\|U_n\| \to 0$ as $n \to \infty$. Similarly,

$$\begin{aligned} E\|V_n\| &\leq K\|n^{-1} \sum_{i=1}^{b(n)-1} E(R_1 \cdots R_{i+1}(ER_1)^{b(n)-i}(I - ER_1)^{-1}(ET_1)(R_1 \cdots R_i T_{i+1})')\| \\ &= K\|n^{-1}E \sum_{i=1}^{b(n)-1} R_1 \cdots R_{b(n)+1}(I - ER_1)^{-1}(ET_1)(R_1 \cdots R_i T_{i+1})'\| , \end{aligned}$$

and hence there is a constant L such that

$$E\|V_n\| \leq LE\|n^{-1} \sum_{i=1}^{b(n)-1} R_1 \cdots R_i T_{i+1}\| ,$$

and then $E\|V_n\| \to 0$ as $n \to \infty$ by Corollary 1.

The proof is complete.

Proof of Theorem. Corollaries 1 and 2 and Lemmas 4 and 7 imply the conditions of Rosén's theorem in [4], which follow by using the basic inequalities and the fact that the norms in [4] and ours are equivalent. Therefore the conclusion of the theorem is obtained under the assumption that $E\|T_1\|^{2+\nu} < \infty$ for some $\nu > 0$.

To prove the theorem in general let $T_i^{(\delta)} = T_i e^{-\delta||T_i||}$, $\delta > 0$, and define

$$T^{(\delta)}(n) = \sum_{i=1}^n R_0 \cdots R_{i-1} T_i^{(\delta)} .$$

Clearly $E||T_1^{(\delta)}||^{2+\nu} < \infty$ for $\nu > 0$, and hence by the special case of the theorem already proved $n^{-\frac{1}{2}}T^{(\delta)}(n)$ converges weakly to $N(\delta)$ as $n \to \infty$, where $N(\delta)$ is the normal distribution with mean zero and covariance matrix $\sigma^2(\delta)I$, with

$$\sigma^2(\delta) = d^{-1}\{E||T_1^{(\delta)}||^2 + 2 \operatorname{tr} [(I - ER_1)^{-1}ET_1^{(\delta)}E(T_1^{(\delta)\prime}R_1)]\} .$$

By the dominated convergence theorem $\sigma^2(\delta) \to \sigma^2$ as $\delta \to 0$, and therefore $N(\delta)$ converges weakly as $\delta \to 0$ to the normal distribution with mean zero and covariance matrix $\sigma^2 I$. The theorem will be established if we show

$$\lim_{\delta\to0} \limsup_{n\to\infty} P[||n^{-\frac{1}{2}}T(n) - n^{-\frac{1}{2}}T^{(\delta)}(n)|| \geqq \varepsilon] = 0$$

for each $\varepsilon > 0$ (see [1], Theorem 4.2). By Chebyshev's inequality and Lemma 2 we have

$$\begin{aligned} P[||n^{-\frac{1}{2}}T(n) - n^{-\frac{1}{2}}T^{(\delta)}(n)|| \geqq \varepsilon] \\ &= P[||n^{-\frac{1}{2}} \textstyle\sum_{i=1}^n R_0 \cdots R_{i-1} T_i(1 - e^{-\delta||T_i||})|| \geqq \varepsilon] \\ &\leqq \varepsilon^{-2}KE||T_1(1 - e^{-\delta||T_1||})||^2 , \end{aligned}$$

whence the desired result follows by the dominated convergence theorem.

The joint convergence of $R(n)$ and $n^{-\frac{1}{2}}T(n)$ can be proved by using the same observation made by Tutubalin in [6] for his special case; that is, noticing that for any fixed k the vector $S(n, k) = n^{-\frac{1}{2}} \sum_{i=1}^{n-k} R_0 \cdots R_{i-1} T_i$ has the same weak limit as $n^{-\frac{1}{2}}T(n)$, and that the conditional distribution of $R(n)$ given fixed $R_1, \cdots, R_{n-k}$ (and consequently given fixed $S(n, k)$) is close to the Haar distribution on $G(S_1)$ for large k (because of the translation invariance of the Haar measure).

To obtain the asymptotic distribution of $x\overline{g(n)}$, note that every subsequence of the sequence $\{R(n)(X + T(n))n^{-\frac{1}{2}}\}$ has a weakly convergent subsequence with limit HN, where H is the Haar distribution on $G(S_1)$ and is independent of N; but HN is distributed as N (see [1], Theorem 2.3).

Note. Lemmas 1 and 6, and a similar version of Lemma 5 appear originally in [3], but are included here to make this paper self-contained.

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