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MEASURE-INVARIANT SETS[1]

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Let μ be a probability measure on $(\mathscr{R}, \mathscr{B})$, where $\mathscr{R}$ is the real line and $\mathscr{B}$ the family of Borel sets on $\mathscr{R}$. A measurable set 'A' is called μ-invariant if $\mu(A+\theta) = \mu(A)\ \forall\,\theta,\ -\infty < \theta < \infty$. Let $\mathscr{A}(\mu)$ denote the family of all μ-invariant sets. Let $S(\mu)$ denote the set where the characteristic function of μ vanishes. In this paper we establish the following results concerning μ-invariant sets. (i) Suppose $S(\mu) \cap \overline{[S(\mu) \oplus S(\mu)]}$ is compact. Then A is μ-invariant implies $\mu(A) = 0, \frac{1}{2}, 1$. (ii) Fourier series representations are developed to study μ-invariant sets. (iii) Dependence of $\mathscr{A}(\mu * \nu)$ on $\mathscr{A}(\mu)$ and $\mathscr{A}(\nu)$ is examined and representations for $\mu * \nu$-invariant sets are derived. (iv) Dependence of $\mathscr{A}(\mu)$ on $S(\mu)$ is carefully examined. (v) A conjecture that $\mathscr{A}(\mu) \subset \mathscr{A}(\nu)$ implies that μ is a factor of ν is shown to be false.

1. Preliminaries. The main object of this paper is to present some results which we have recently obtained concerning measure-invariant sets for a translation parameter family of probability measures. Although our results have their origin in statistics, they are purely measure theoretic in nature. For reasons of simplicity, unless stated otherwise, we restrict our attention to probability measures on the real line.

The study of measure-invariant sets was initiated by Basu and Ghosh (1969). They proved a number of interesting results concerning these sets and posed a number of unsolved problems. Later Pathak and Rickert (1971) solved some of these problems. The problems that remain are most interesting and perhaps more difficult. We present a systematic account of these problems and solve some of them.

Let μ be a probability measure (p.m.) on $(\mathscr{R}, \mathscr{B})$, where $\mathscr{R}$ is the real line and $\mathscr{B}$ the family of Borel sets on $\mathscr{R}$. A measurable set A is called measure-invariant (μ-invariant) if $\mu(A+\theta) = \mu(A)\ \forall\,\theta,\ -\infty < \theta < \infty$. In terms of convolutions it is easily seen that A is μ invariant iff $(I_A - p) * \tilde{\mu} = 0$, where I_A is the indicator function of A, $p = \mu(A)$ and $\forall\, B \in \mathscr{B}$, $\tilde{\mu}(B) = \mu(-B)$. We denote by $\mathscr{A}(\mu)$ the family of all μ-invariant sets. A μ-invariant set 'A' is called "nontrivial" if $0 < \mu(A) < 1$. The p.m. μ is called weakly complete (incomplete) if it does not (does) possess a nontrivial μ-invariant set.

One of the interesting problems concerning μ-invariant sets is to investigate conditions under which a given p.m. μ is weakly incomplete. The problem is

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far from being solved satisfactorily. Basu and Ghosh (1969) and Pathak and Rickert (1971) have obtained a number of necessary conditions and sufficient conditions for the existence of nontrivial μ-invariant sets. We examine these conditions here and strengthen some of their results.

Let μ and ν be two given p.m.'s. It is then of interest to know if there exists any relationship among the families $\mathscr{A}(\mu)$, $\mathscr{A}(\nu)$ and $\mathscr{A}(\mu * \nu)$. We study this problem and obtain explicit expressions for $\mu * \nu$-invariant sets in certain special cases.

Suppose that μ is the uniform p.m. on [0, 1]. Basu and Ghosh showed that $\mathscr{A}(\mu)$, in this case, consists entirely of sets that are periodic of period one. If ν is any p.m. then it is easily seen that $\mathscr{A}(\mu) \subset \mathscr{A}(\mu * \nu)$. In a personal communication, D. Basu wondered if the converse relation is also valid, i.e. if every ν with $\mathscr{A}(\mu) \subset \mathscr{A}(\nu)$ is decomposable with μ as a factor (see also *Ann. Math. Statist.* **40** (1969) 173–174). We show that this is not true.

2. Existence of μ-invariant sets. One of the interesting problems concerning μ-invariant sets is to find conditions under which a given p.m. μ admits nontrivial μ-invariant sets. A satisfactory solution of this problem seems to depend on certain structural properties of the set $S(\mu) = \{t: \hat{\mu}(t) = 0\}$ where the characteristic function of μ vanishes, and involves the use of many interesting analogous results from harmonic analysis. It might therefore be worthwhile to first state some useful from harmonic analysis that will be used later in our work.

Let $\mathscr{F}$ be a family of functions in L_1. Define $S(\mathscr{F}) = \{t: \hat{f}(t) = 0 \ \forall f \in \mathscr{F}\}$, $\alpha(\mathscr{F}) = \{\varphi : \varphi \in L_\infty \text{ and } \varphi * f = 0 \text{ for all } f \in \mathscr{F}\}$ and $\beta(\mathscr{F}) = \{e(tx): t \in S(\mathscr{F})\}$, where $\hat{f}(t) = \int e(-tx)f(x)\,dx$ and $e(tx) = \exp(itx)$. The set $\alpha(\mathscr{F})$ of functions has been studied extensively in harmonic analysis (see Rudin (1962) pages 158–159 and 184–185). The set $S(\mathscr{F})$ is called an S-set if $\alpha(\mathscr{F})$ is the weak*-closed subspace spanned by $\beta(\mathscr{F})$. (Note that in general the former is only a subset of $\alpha(\mathscr{F})$). It is known that if the boundary of $S(\mathscr{F})$ is a countable set then $S(\mathscr{F})$ is an S-set. (The general problem of finding necessary and sufficient conditions for an arbitrary closed set to be an S-set is very difficult. For a thorough discussion of these and other related results we refer the reader to Rudin (1962).) The following lemma is now immediate.

LEMMA 2.1. *Let $f \in L_1$, $S(f) = \{t: \hat{f}(t) = 0\}$ and $\beta(f) = \{e(tx): t \in S(f)\}$. Let $\varphi \in L_\infty$ and suppose that $S(f)$ is an S-set. Then $\varphi * f = 0$ iff φ belongs to the weak*-closed subspace spanned by $\beta(f)$.*

The lemma simply means that if $\varphi * f = 0$ and if $S(f)$ is an S-set, then φ is the weak*-limit of trigonometric polynomials of the kind $\sum a_j e(t_j x)$, where each $t_j \in S(f)$. The lemma plays an important role in the study of μ-invariant sets.

Before we state our results concerning μ-invariant sets it is worthwhile to briefly describe the results obtained previously. Let μ be a p.m. Let $S(\mu)$ be the set where the characteristic function of μ vanishes. Suppose that $S(\mu)$ is finite. In this case Basu and Ghosh showed that μ is weakly complete. Later Pathak and

Rickert showed that if $S(\mu)$ is compact then also μ is weakly complete. The following unpublished result is due to Basu and Blum. Let X be a random variable with p.m. μ and suppose that $S(\mu) = \{\pm\pi, \pm 2\pi, \cdots\}$. Then a set A is μ-invariant iff A is essentially periodic of period 1. Further the fractional part (X) of X has the uniform distribution over $[0, 1]$ and the p.m. μ is necessarily absolutely continuous. Results which we have obtained are generally of a negative nature. They furnish structural conditions on $S(\mu)$ under which the p.m. μ does not admit nontrivial μ-invariant sets.

THEOREM 2.1. *Let μ be a* p.m. *Let $S = \{t : \hat{\mu}(t) = 0\}$ be an S-set and suppose that $S \cap \overline{(S \oplus S)} = K$, where K is a compact S-set. Then A is μ-invariant implies $\mu(A) = 0, \frac{1}{2}$ or 1.*

PROOF. Consider the p.m. μ and let A be a μ-invariant set with $\mu(A) = p$. If μ is not absolutely continuous, we convolve μ with an absolutely continuous p.m. ν with $S(\nu) = \emptyset$. It then follows from a result of Basu and Ghosh that A is $\mu * \nu$-invariant with $\mu * \nu(A) = \mu(A)$. Further $S(\mu * \nu) = S(\mu)$. We therefore may and do assume without loss of generality that μ is absolutely continuous. Let $d\mu/dx = f$. It is easily seen that $(I_A - p) * \tilde{f} = 0$, where $\tilde{f}(x) = f(-x)\ \forall x$. So $(I_A - p)$ belongs to the weak*-closed subspace spanned by $\beta(\tilde{f}) = \beta(f) = \{e(tx) : t \in S(f)\}$. A little consideration will now show that $(1 - 2p)I_A + p^2 = (I_A - p)(I_A - p)$ belongs to the weak*-closed subspace spanned by $B = \{e(tx) : t \in \overline{S \oplus S}\}$. Consequently $(1 - 2p)I_A$ belongs to the weak*-closed subspace spanned by $\{e(tx) : t \in K \cup \{0\}\}$. (Note this last assertion assumes that K is an S-set.) Since $K \cup \{0\}$ is compact, it follows from a theorem of Pathak and Rickert that $(1 - 2p)I_A$ is essentially a constant. This implies $p = 0, \frac{1}{2}$ or 1. ∎

To demonstrate that the above theorem cannot be improved any further, let us consider the following example. Let μ be the p.m. which assigns probability $\frac{1}{2}$ to each of the points $x = \pm\frac{1}{2}$. Then $\hat{\mu}(t) = \cos(t/2)$ so that $S = \{\pm\pi, \pm 3\pi, \cdots\}$ and $S \oplus S = \{0, \pm\pi, \pm 2\pi, \cdots\}$ so that $S \cap \overline{(S \oplus S)} = \emptyset$. It can be easily seen that this p.m. does indeed possess μ-invariant sets of measures $0, \frac{1}{2}$, and 1. The set $A = \bigcup_{-\infty}^{\infty} (2n, 2n + 1]$ is, for example, a μ-invariant set with $\mu(A) = \frac{1}{2}$.

In order to obtain other results similar to that of the preceding theorem it is necessary to develop Fourier series type results for μ-invariant sets. We were unable to find such results in the literature. Aside from their application to our work, it is hoped that these results will be found interesting in their own right.

3. Fourier series for μ-invariant sets. Throughout this section we work with a given $f \in L^1$ and assume that $S(f)$ is a discrete set; such an $S(f)$ is automatically an S-set. We derive here Fourier series for functions in $\alpha(f) = \{\varphi : \varphi \in L_\infty$ and $\varphi * f = 0\}$. We first establish a number of lemmas that are needed for this development.

LEMMA 3.1. *Let $f \in L_1$. Let $u \in S(f)'$ and consider $B = \{e(tx) : t \in S(f)\} \cup \{e(ux)\}$. Then the weak*-closed subspace spanned by B is given by $W(B) = \{\psi : \psi = \varphi + ce(ux) : \varphi \in \alpha(f), |c| < \infty\}$.*

PROOF. This is immediate on noting that if $\{\psi_n = \varphi_n + c_n e(ux) : n \geqq 1\}$ is a w^*-convergent sequence in $W(B)$. Then $\lim \int \psi_n \tilde{f}(x)\, dx = \lim c_n \hat{f}(u)$ exists so that $\lim c_n = c$ exists. Hence $\{\varphi_n : n \geqq 1\}$ converges in weak*-sense to some $\varphi \in L_\infty$. Thus $w^*\text{-}\lim [\varphi_n(x) + c_n e(ux)] = \varphi(x) + ce(ux)$. □

LEMMA 3.2. *Let $\varphi \in \alpha(f)$. Let u be a given real number. Then φ admits the following unique representation*

$$\varphi(x) = \varphi_1(x) + c(u)e(ux)$$

where φ_1 belongs to the weak-closed subspace spanned by $\{e(tx) : t \in S(f) \cap \{u\}'\}$.*

PROOF. In view of the preceding lemma, it suffices to prove the uniqueness of the representation. Since the set $S(f)$ is discrete, $e(ux)$ does not belong to the weak*-subspace spanned by $\{e(tx) : t \in S(f) \cap \{u\}'\}$. The uniqueness of the representation follows from this observation. □

The following definition is thus unambiguous.

DEFINITION. The coefficient $c(u)$ of $e(ux)$, in the preceding representation of $\varphi(x)$ is called the Fourier coefficient of φ. In the sequel we denote it by $\hat{\varphi}(u)$. If $\phi \in \alpha(f)$ then $\hat{\phi}(u) = 0$ for u not in $S(f)$. Although the above definition tells us what $\hat{\varphi}(u)$ is, it does not provide us a way of actually calculating it. The following lemma does.

LEMMA 3.3. *Let $g \in L^1$ be such that $\hat{g}(u) = 1$ and $S(g) \supset S(f) \cap \{u\}'$. Then $\varphi * g(0) = \int \varphi(x)\tilde{g}(x)\, dx = \hat{\varphi}(u)$.*

PROOF. It is clear that φ admits the representation

$$\varphi(x) = \varphi_g(x) + \hat{\varphi}(u)e(ux)$$

where $\varphi_g \in \alpha(g)$.

The lemma is now immediate. □

LEMMA 3.4. *Let $\varphi \in \alpha(f)$ and suppose that $\hat{\varphi}(u) = 0\ \forall u \in S(f)$. Then $\varphi = 0$ a.e.*

PROOF. For each $u \in S(f)$ let $g(\cdot, u) \in L_1$ with $S(g(\cdot, u)) = S(f) \cap \{u\}'$. Since $S(f)$ is discrete, $S(f) \cap \{u\}'$ is a closed set. Consequently such a $g(\cdot, u)$ exists. By Lemma 3.2 we can write $\varphi(x) = \varphi_g(x) + \hat{\varphi}(u)e(ux) = \varphi_g(x)$, where $\varphi_g(x)$ belongs to the weak*-closed subspace of $\beta(g(\cdot, u)) = \{e(tx) : t \in S(g)\}$. Consequently $\varphi * g(\cdot, u) = \varphi_g * g(\cdot, u) = 0$. Thus $\varphi \in \alpha(G)$, where $G = \{g(\cdot, u) : u \in S\} \cup \{f\}$. Since $S(G) = \emptyset$, it follows that $\alpha(G) = 0$ so $\varphi = 0$ a.e. □

We thus have established the following uniqueness lemma.

LEMMA 3.5. *Let $f \in L_1$ and $\varphi \in \alpha(f)$. Then φ is the only member of $\alpha(f)$ with Fourier coefficients $\{\hat{\varphi}(u) : u \in S(f)\}$. We formally write $\varphi(x) = \sum_{u \in S(f)} \hat{\varphi}(u)e(ux)$.*

LEMMA 3.6. *Let $\varphi \in \alpha(f)$ and suppose that there exists a sequence of trigonometric* *functions* $\{\sum c_n(u)e(ux): n \geqq 1\}$, *where the sum* $\sum$ *runs over finitely many* $u \in S(f)$, *which converges in the weak*-sense to* φ. *Then* $\forall u \in S(f)$, *we have* $\lim c_n(u) = \hat{\varphi}(u)$.

PROOF. Let $g \in L_1$ be such that $\hat{g}(u) = 1$ and $S(g) \supset S(f) \cap \{u\}'$. Then

$$\begin{aligned}\lim_n c_n(u) &= \lim c_n(u) \int e(ux)\tilde{g}(x)\,dx \\ &= \lim \left[\int \{\textstyle\sum_{t\in S(f)} c_n(t)e(tx)\}\tilde{g}(x)\,dx\right] = \int \varphi(x)\tilde{g}(x)\,dx = \hat{\varphi}(u)\end{aligned}$$

by Lemma 3.3. □

LEMMA 3.7. *Let* $f \in L_1$. *Let* $\varphi \in \alpha(f)$. *Let* $\{u_1, u_2, \cdots, u_n\} \subset S$. *Let* $g \in L_1$ *be real valued and nonnegative*, $\hat{g}(0) = 1$, $S(g) = (-\varepsilon, \varepsilon)'$, *where* $\varepsilon > 0$ *is chosen so that* $\forall k$, $1 \leqq k \leqq n$, $S(f) \subset [S(g) \oplus u_k] \cup \{u_k\}$. *Then*

$$\textstyle\sum_1^n |\hat{\varphi}(u_k)|^2 \leqq \int |\varphi(x)|^2\tilde{g}(x)\,dx\,.$$

PROOF. Let $\psi(x) = \varphi(x) - \sum_1^n \hat{\varphi}(u_k)e(u_k x)$ and suppose $g \in L_1$ has the properties described in the lemma. Then

$$\begin{aligned}0 \leqq \int |\psi(x)|^2\tilde{g}(x)\,dx &= \int |\varphi(x)|^2\tilde{g}(x)\,dx + \textstyle\sum_1^n |\hat{\varphi}(u_k)|^2 \int \tilde{g}(x)\,dx \\ &\quad - \textstyle\sum_1^n \hat{\varphi}(u_k) \int e(u_k x)\overline{\varphi(x)}\tilde{g}(x)\,dx \\ &\quad - \textstyle\sum_1^n \bar{\hat{\varphi}}(u_k) \int e(-u_k x)\varphi(x)\tilde{g}(x)\,dx \\ &\quad + \textstyle\sum_{j\neq k} \hat{\varphi}(u_j)\overline{\hat{\varphi}(u_k)} \int e((u_j - u_k)x)\tilde{g}(x)\,dx \\ &= \int |\varphi(x)|^2\tilde{g}(x)\,dx + \textstyle\sum_1^n |\hat{\varphi}(u_k)|^2 - 2\sum_1^n |\hat{\varphi}(u_k)|^2 \\ &= \int |\varphi(x)|^2\tilde{g}(x)\,dx - \textstyle\sum_1^n \hat{\varphi}(u_k)|^2\,.\end{aligned}$$

Consequently

$$\textstyle\sum_1^n |\hat{\varphi}(u_k)|^2 \leqq \int |\varphi(x)|^2\tilde{g}(x)\,dx\,. \qquad □$$

COROLLARY.

$$\textstyle\sum_1^n |\hat{\varphi}(u_k)|^2 \leqq [||\varphi||_\infty]^2\,.$$

We now use the above results to prove the following theorem.

THEOREM 3.1. *Let* μ *be a* p.m. *Let* $S = \{t\colon \hat{\mu}(t) = 0\}$ *and* $S \oplus S$ *be discrete sets and* $S \cap (S \oplus S) = \emptyset$. *Suppose that for each* $u, v, w \in S$, *the equation* $2v = u + w$ *implies* $u = v = w$. *Then* A *is* μ*-invariant implies* $\mu(A) = 0$ *or* 1.

PROOF. Let A be μ-invariant with $\mu(A) = p$, $0 < p < 1$. We assume without loss of generality that μ is absolutely continuous with density f. Then A is μ-invariant iff $(I_A - p) \in \alpha(f^{\sim}) = \alpha(f)$, where $f^{\sim}(x) = f(-x)$ $\forall x$. Since $S(f) = S(f^{\sim}) = S$, it follows that $(I_A - p)$ is in the weak*-closed subspace of $\beta(f) = \{e(tx)\colon t \in S\}$. Let $v \in S$. Then $2v \notin S$. Therefore there is a suitable $g \in L_1$ such that $\hat{g}(2v) = 1$ and $\hat{g}(t) = 0$ for $|t - 2v| > \delta$ where δ is chosen so that $(2v - \delta, 2v + \delta)$ does not include any elements of S or $(S \oplus S) \cap \{2v\}'$. Since $2v \notin S$, it is easily seen that $\hat{I}_A(2v) = 0$ so that from Lemma 3.3

$$0 = \int I_A(x)g(x)\,dx = (1 - p)^{-1} \int I_A(x)(I_A(x) - p)g(x)\,dx\,.$$

Now let $\{\sum_{u\in S} c_n(u)e(ux)\colon n \geqq 1\}$ be a sequence of trigonometric polynomials which converges in the weak*-sense to $(I_A - p)$. Then the above equation yields

$$0 = \lim_{n\to\infty} \int I_A(x)(\textstyle\sum_{u\in S} c_n(u)e(ux))g(x)\,dx$$
$$= \lim_{n\to\infty} \textstyle\sum_{u\in S} c_n(u) \int I_A(x)e(ux)g(x)\,dx\,.$$

Now for each u, $S(g(x)e(ux)) = (2v - u - \delta, 2v - u + \delta)'$. Further a $w \in S$ also belongs to $(2v - u - \delta, 2v - u + \delta)$ iff $2v - \delta < u + w < 2v + \delta$. Since $u + w \in S \oplus S$, it follows from our construction that $2v = u + w$. Now by our hypothesis it follows that $u = v = w$. Consequently if $u \neq v$, $S(g(x)e(ux)) \supset S(f) \cup \{0\}$ and if $u = v$, $S(g(x)e(ux)) \supset S(f) \cap \{v\}^{\sim}$. This implies from Lemma 3.3 that $\int I_A(x)e(ux)g(x)\,dx = 0$ if $u \neq v$ and $= \hat{I}_A(v)$ if $u = v$. On substituting these values in the earlier equation we get

$$0 = \lim_{n\to\infty} c_n(v)\hat{I}_A(v) = [\hat{I}_A(v)]^2$$

by Lemma 3.6.

Consequently $\hat{I}_A(v) = 0\ \forall v \in S$ so that $(I_A - p) = 0$ a.e. Consequently $p = 0$ or 1. This is a contradiction. Hence A is a trivial set. □

A method similar to the above can be used to prove the following stronger result.

THEOREM 3.2. *Let μ be the given* p.m. *Let $S = \{t : \hat{\mu}(t) = 0\}$ be discrete and the limit points of $\overline{(S \oplus S)}$ form a compact set. Let K be a compact set such that $S \cap \overline{S \oplus S} \subset K$ and for each $v \in S \cap K'$ and for each $u, w \in S$, the equation $2v = u + w$ implies $u = v = w$. Then A is μ-invariant implies $\mu(A) = 0$ or 1.*

The two preceding theorems have some interesting applications. Roughly speaking they assert that if there are large or uneven gaps in the elements of the set $S(\mu)$ of a p.m. μ then μ possesses no nontrivial μ-invariant sets. We consider several illustrative examples.

EXAMPLE 1. Let μ be a p.m. on $[0, 2\pi]$ with density

$$f(x) = \text{const.}\left[\sum_{n=1}^{\infty} \frac{\cos nx}{2^n} - \sum_{n=1}^{\infty} \frac{\cos 3^n x}{2^{3n}}\right].$$

Thus $S(\mu) = \{\pm 3^n : n = 1, 2, \cdots\}$. It is easily seen that $S(\mu)$ satisfies the hypotheses of Theorem 3.1. Consequently it has no nontrivial μ-invariant sets.

EXAMPLE 2. Let μ be a p.m. with density $f(x) = |x|$ if $|x| \leqq 1$ and $= 0$ otherwise. It can be shown that the characteristic function of μ vanishes at t iff $t = \tan(t/2)$. These zeroes are of the form $t = 2n\pi + \varepsilon_n$, where ε_n goes monotonically to $\pm\pi$ as n approaches $\pm\infty$. It can be shown that these zeroes satisfy the hypotheses of Theorem 3.2. So μ does not have any nontrivial μ-invariant sets. It is perhaps worthwhile mentioning here that more generally if the p.m. μ has a density of the form $f(x) = a_0 + a_1 x + \cdots + a_k x^k$, $0 \leqq x \leqq 1$ and $= 0$ otherwise, similar techniques can be used to show that these p.m.'s have no nontrivial μ-invariant sets.

4. Convolution and measure-invariant sets. Let μ_1 and μ_2 be given probability measures. Basu and Ghosh showed that every μ-invariant set A is also $\mu_1 * \mu_2$-invariant, and $\mu_1 * \mu_2(A) = \mu_1(A)$. In this section we study the structural properties of $\mu_1 * \mu_2$-invariant sets. Although it is easy to see that $\mathscr{A}(\mu_1) \cup \mathscr{A}(\mu_2) \subset \mathscr{A}(\mu_1 * \mu_2)$, we do not yet know the exact relationship between $\mathscr{A}(\mu_1 * \mu_2)$, and $\mathscr{A}(\mu_1)$ and $\mathscr{A}(\mu_2)$ in general situations. There are, however, a variety of cases in which we can obtain an explicit structure for $\mu_1 * \mu_2$-invariant sets. These special cases are the object of our study here. We need the following preliminary results for this purpose.

LEMMA 4.1. *Let $p_1, \cdots, p_k$ be k positive real numbers. For every $g \in L_\infty$ define $\Delta_i g(x) = g(x + p_i) - g(x)$. Let $\mathscr{U}$ denote the family of all functions of the form $h = g_1 + g_2 + \cdots + g_k$, where g_i is such that $\Delta_i g_i = 0$ a.e., $1 \leqq i \leqq k$. The set $\mathscr{U}$ is weak*-closed and an $h \in U$* iff $\Delta_1 \Delta_2 \cdots \Delta_k(h) = 0$ a.e.

PROOF. This is based on induction. The lemma is evident for $k = 1$. Now suppose that the lemma holds for $(k - 1)$ so that $\Delta_1 \Delta_2 \cdots \Delta_{k-1} g = 0$ a.e. iff $g = g_1 + \cdots + g_{k-1}$ where $\Delta_i g_i = 0$ a.e., $1 \leqq i \leqq (k - 1)$. Now let $g \in L_\infty$ and suppose that $\Delta_1 \Delta_2 \cdots \Delta_k g = 0$ a.e. By induction it follows that $\Delta_k g = g_1(x) + \cdots + g_{k-1}(x)$ where $g_j \in L_\infty$ and $\Delta_j g_j = 0$ a.e., $1 \leqq j \leqq (k - 1)$. Consequently $g(x + p_k) = g(x) + g_1(x) + \cdots + g_{k-1}(x)$ so that

$$g(x) - [\textstyle\sum_{r=1}^{N} g(x + rp_k)]/(N + 1) = \sum_{j=1}^{(k-1)} h_{j,N}(x)\,, \quad \text{say,}$$

where $h_j \in L_\infty$ and $\Delta_j h_j = 0$ a.e.

It is easy to see that $||\sum_{r=1}^{N} g(x + rp_k)/(N + 1)||_\infty \leqq ||g||_\infty$ and $||\sum_{j=1}^{(k-1)} h_{j,N}(x)||_\infty \leqq ||g_1(x) + \cdots + g_{(k-1)}(x)||_\infty$. Consequently there exists a subsequence $\{N_s\}$ such that $\sum_{r=1}^{N_s} g(x + rp_k)/(N_s + 1)$ and $\{\sum_{j=1}^{(k-1)} h_{j,N_s}(x)\}$ are w^*-convergent. Clearly the sequence $\{\sum_{r=1}^{N_s} g(x + rp_k)/(N_s + 1)\}$ converges to a function h_k which satisfies $\Delta_k h_k = 0$ a.e. and, by our induction hypothesis, $\sum_{j=1}^{k-1} h_{j,N_s}(x)$ to a function of the form $h_1(x) + h_2(x) + \cdots + h_{k-1}(x)$, where $\Delta_j h_j = 0$ a.e. Hence $g(x) = h_1(x) + \cdots h_{k-1}(x) + h_k(x)$. That the set $\mathscr{U}$ of elements $g \in L_\infty$ which satisfy $\Delta_1 \Delta_2 \cdots \Delta_k\, g = 0$ a.e. is weak*-closed is easy to see. □

A similar technique can be used to establish the following corollary.

COROLLARY. *Let $p_1, \cdots, p_k$ be k positive real numbers and $q_1, \cdots, q_k$ be k real numbers. For every $g \in L_\infty$, define $\Delta_r^*(g) = g(x + p_r)e(-p_r q_r) - g(x)$. Let $\mathscr{V}$ denote the family of all functions of the form*:

$$h(x) = e(q_1 x)g_1 + \cdots + e(q_k x)g_k$$

where $g_i \in L_\infty$ is such that $\Delta_i(g_i) = g_i(x + p_i) - g_i(x) = 0$ a.e. The set $\mathscr{V}$ is weak-closed and an $h \in V$* iff $\Delta_1^* \Delta_2^* \cdots \Delta_k^*(h) = 0$ a.e.

LEMMA 4.2. *Let A and B be two Lebesgue measurable sets of positive Lebesgue measure and suppose that $A \subset (0, p_1]$ and $B \subset (0, p_2]$ where p_1 and p_2 are such that p_1/p_2 is irrational. Let $A^* = \bigcup_{-\infty}^{\infty} (A \oplus mp_1)$ and $B^* = \bigcup_{-\infty}^{\infty} (B \oplus np_2)$. Then the set $A^* \cap B^*$ has positive Lebesgue measure.*

PROOF. Let λ denote Lebesgue measure. Since $\lambda(A) > 0$ and $\lambda(B) > 0$, it follows that there exist intervals I and J such the $\lambda(A \cap I) > (\frac{3}{4})\lambda(I)$ and $\lambda(B \cap J) > (\frac{3}{4})(J)$. We may assume that $\lambda(I) = \lambda(J) = d > 0$. Consequently there is a 'θ' such that $I + \theta = J$. Also there exist integers m and n such that $|mp_1 - np_2 - \theta| < d/4$ so that

$$\lambda((I + mp_1) \cap (J + np_2)) = \lambda((I + mp_1 - np_2) \cap J) > (3d/4) .$$

Now

$$\begin{aligned}
&\lambda(A^* \cap B^*) \\
&\quad \geqq \lambda((A + mp_1) \cap (B + np_2)) \\
&\quad \geqq \lambda((A \cap I + mp_1) \cap (B \cap J + np_2)) \\
&\quad \geqq \lambda((I + mp_1) \cap (J + np_2)) - \lambda((A' \cap I + mp_1) \cup (B' \cap J + np_2)) \\
&\quad \geqq \lambda((I + mp_1) \cap (J + np_2)) - \lambda((A' \cap I + mp_1) - \lambda(B' \cap J + np_2)) \\
&\quad > \frac{3d}{4} - \lambda(I + mp_1) - \lambda(J + np_2) + \lambda(A \cap I + mp_1) + \lambda(B \cap J + np_2) \\
&\quad > \frac{3d}{4} - 2d + \frac{3d}{4} + \frac{3d}{4} > 0 .
\end{aligned}$$

□

DEFINITION. Let $p_1, \cdots, p_k$ be k positive real numbers. We say that the set $\{p_i : 1 < i < k\}$ is *rationally independent* if for every set of integers $\{m_i : 0 \leqq i \leqq k\}$ the sum $m_0 + m_1 p_1 + \cdots m_k p_k$ is never zero unless $m_1 = m_2 = \cdots = m_k = 0$.

LEMMA 4.3. *Let $g_1, \cdots, g_k$ be 'k' L_∞-functions. Let g_i be periodic of period p_i, $1 \leqq i \leqq k$. Suppose that the set $\{p_i : 1 \leqq i \leqq k\}$ is rationally independent. Then the essential range* (e.r.) *of $h = g_1 + \cdots + g_k$ is given as follows.*

$$\text{e.r. } (h) = \text{e.r. } (g_1) \oplus \text{e.r. } (g_2) \oplus \cdots \oplus \text{e.r. } (g_k) .$$

PROOF. It is easy to see that e.r. $(h) \subset$ e.r. $(g_1) \oplus \cdots \oplus$ e.r. (g_k). To prove the converse we assume for simplicity that $k = 2$. It now suffices to show that if $a \in$ e.r. (g_1) and $b \in$ e.r. (g_2), then $a + b \in$ e.r. $(g_1 + g_2)$. Let 0_1 be an open set containing a and 0_2, an open set containing b. Then $\lambda(g_1^{-1}(0_1)) > 0$ and $\lambda(g_2^{-1}(0_2)) > 0$. It is clear that $g_i^{-1}(0_i)$ is a periodic set of period p_i $(i = 1, 2)$. Consequently by Lemma 4.2, $\lambda(g_1^{-1}(0_1) \cap g_2^{-1}(0_2)) > 0$. It can now be seen that this implies that $(a + b)$ is in the essential range of $g_1 + g_2$. □

LEMMA 4.4. *Let $f \in L_1$ and $S(f) \subset \bigcup_{j=1}^{k} H_j$, where $\forall j$, $1 \leqq j \leqq k$, $H_j = \{q_j + 2n\pi/p_j : n = 0, \pm 1, \cdots\}$ and $p_1, \cdots, p_k$ are k positive real numbers and $q_1, \cdots, q_k$ real numbers. Let $g \in L_\infty$ be such that $g * f = 0$. Then $g = g_1 + \cdots + g_k$, where $e(-q_j x) g_j(x)$ is almost everywhere a periodic function of period p_j.*

PROOF. It is clear that g is in weak*-subspace spanned by $\{e(tx) : t \in \cup H_j\}$. The lemma now follows easily from the corollary in Lemma 4.1. □

It is perhaps worth mentioning at this point that the results we have developed in this and the preceding section have close connections with properties of mean periodic and almost periodic functions. In a subsequent paper, we intend to study the connection between some of our results and the general tools of harmonic analysis.

We now use the preceding results to obtain the structure of μ-invariant sets in several cases.

THEOREM 4.1. *Let* $\mu_1, \cdots, \mu_k$ *be* k p.m.'*s. Let* $\mu = \mu_1 * \cdots * \mu_k$. *Let* $S(\mu_j) \subset \{q_j + 2n\pi/p_j : n = 0, \pm 1, \cdots\}$, $1 \leqq j \leqq k$. *Then every* μ*-invariant set* A *admits the following representation*:

$$I_A(x) = c + e(q_1 x)g_1(x) + \cdots + e(q_k x)g_k(x)$$

where $c = \mu(A)$ *and the* g_j *is almost everywhere a periodic function of period* p_j, $1 \leqq j \leqq k$.

PROOF. We assume without loss of generality that μ is absolutely continuous with $d\mu/dx = g$. If A is a μ-invariant set with $\mu(A) = c$ then $(I_A - c) * \tilde{f} = 0$, where $\tilde{f}(x) = f(-x)\ \forall x$. So $(I_A - c)$ belongs to $\alpha(\tilde{f}) = \alpha(f)$. By Lemma 2.1, $(I_A - c)$ is, therefore, in the weak*-closed subspace spanned by $\beta(f) = \{e(tx) : t \in \bigcup_{j=1}^k H_j\}$, where $H_j = \{q_j + 2n\pi/p_j : n = 0, \pm 1, \cdots\}$. The result of the theorem now follows from Lemma 4.4. □

REMARK. The above theorem furnishes, at least in some special cases, the precise structure of μ-invariant sets when μ is the convolution of several measures. It now follows that if $\mu = \mu_1 * \cdots * \mu_n$, then $\mathscr{A}(\mu)$ can be expected to be a much bigger family than $\bigcup \mathscr{A}(\mu_j)$. The following example illustrates this point.

EXAMPLE. Let α be a positive irrational number. Let μ_1 be the uniform p.m. on $[0, 1]$, μ_2, the uniform p.m. on $[0, 1/\alpha]$ and μ_3, the uniform p.m. on $[0, 1/(1 + \alpha)]$. It is easily seen that $\mathscr{A}(\mu_1)$ consists entirely of sets periodic of period 1, $\mathscr{A}(\mu_2)$ consists of sets periodic of period $(1/\alpha)$ and $\mathscr{A}(\mu_2)$ consists of sets periodic of period $1/(1 + \alpha)$. Now consider the set A whose indicator function is given by $I_A(x) = \langle x \rangle + \langle \alpha x \rangle - \langle (1 + \alpha)x \rangle$, where $\langle y \rangle = y - [y]$, $[y]$ being the greatest integer less than y. It follows from Theorem 4.1. that A is μ-invariant, where $\mu = \mu_1 * \mu_2 * \mu_3$. It is also easily seen that I_A is not a periodic function and so is not μ_i-invariant for any $i = 1, 2, 3$. This show that $\mathscr{A}(\mu)$ is strictly bigger than $\bigcup_{r=1}^3 \mathscr{A}(\mu_i)$.

An interesting problem is therefore to ask when $A(\mu_1 * \mu_2 * \cdots * \mu_k) = \bigcup_{i=1}^k A(\mu_i)$. It seems very difficult to provide a satisfactory solution to this problem in a completely general setting. As the following theorem shows there are, nonetheless, special situations in which this does indeed happen.

THEOREM 4.2. *Let* $\mu_1, \cdots, \mu_k$ *be* k p.m.'*s. Let* $\mu = \mu_1 * \cdots * \mu_k$. *Let* $S(\mu_j) \subset \{2n\pi/p_j : n = \pm 1, \pm 2, \cdots\}$, $1 \leqq j \leqq k$, *and suppose the numbers* $\{p_j : 1 \leqq j \leqq k\}$ *are rationally independent. Then* $A(\mu) = \bigcup_{i=1}^k A(\mu_i)$.

PROOF. Let A be a μ-invariant set. It follows from Theorem 4.1. that I_A

admits the representation $I_A = g_1 + \cdots + g_k$, where g_j is essentially periodic of period p_j. Since $\{p_j\colon 1 \leqq j \leqq k\}$ is rationally independent, it follows from Lemma 4.3 that e.r. $(I_A) =$ e.r. $(g_1) \oplus \cdots \oplus$ e.r. (g_k). Hence $I_A \doteq g_j$ for some j. It is now easily seen that this last equality implies that A must be μ_j-invariant so $A \in \mathscr{A}(\mu_j)$. Thus $\mathscr{A}(\mu) = \bigcup_{j=1}^{k} A(\mu_j)$. □

5. Almost μ-invariant and S-sets. In an earlier paper Pathak and Rickert (1971) noted that the existence of μ-invariant sets depends more on the structural properties of $S(\mu)$, the set where the characteristic function of μ vanishes, rather than the p.m. μ itself. In this section, we explore the dependence of $\mathscr{A}(\mu)$ on $S(\mu)$. We have mentioned in Section 2 some of the main results that have been obtained in this direction. Some related results that are known are as follows. Let μ_1 and μ_2 be two absolutely continuous p.m.'s with $S(\mu_1) = S(\mu_2)$. Then Rickert and Pathak showed that $\mathscr{A}(\mu_1) = \mathscr{A}(\mu_2)$ and $A \in \mathscr{A}(\mu_i)$ implies $\mu_1(A) = \mu_2(A)$. Now suppose that μ_2 is an absolutely continuous p.m. and μ_1 is any p.m. In this case they showed that $\mathscr{A}(\mu_1) \subset \mathscr{A}(\mu_2)$. If μ_1 and μ_2 are any two p.m.'s, we do not yet know the exact relationsphip between $\mathscr{A}(\mu_1)$ and $\mathscr{A}(\mu_2)$. As an attempt in this direction we establish a few results in some special cases. We first state and prove an unpublished result noted by Basu and Blum.

LEMMA 5.1. *Let μ be a* p.m. *with $S(\mu) \supset \{cn : n = \pm 1, \pm 2, \cdots\}$ where $c > 0$. Then μ is absolutely continuous.*

PROOF. Without loss of generality we let $c = 1$. Define for each Borel set $B \subset [0, 2\pi)$, $\mu^*(B) = \sum_{-\infty}^{\infty} \mu(B + 2k\pi)$. Then μ^* is p.m. on $[0, 2\pi)$. Further $\hat{\mu}^*(n) = \hat{\mu}(n) = 0 \;\forall n = \pm 1, \pm 2, \cdots$. Thus μ^* is the uniform p.m. on $[0, 2\pi)$. The absolute continuity of μ now easily follows from that of μ^*. □

An immediate consequence of the preceding lemma is the following theorem.

THEOREM 5.1. *Let μ be a* p.m. *with $S(\mu) = \{cn : n = \pm 1, \pm 2, \cdots\}$. Then μ possesses invariant sets of all sizes and a set A is μ-invariant* iff *A is essentially periodic of period $2\pi/c$.*

PROOF. Let ν be the uniform p.m. on $[0, 2\pi/c)$. It follows from a lemma of Basu and Ghosh ((1969) Lemma 5, page 163) that ν possesses invariant sets of all sizes and a set A is ν-invariant iff A is essentially periodic of period $2\pi/c$. Now if μ is any p.m. with $S(\mu) = \{cn : n = \pm 1, \pm 2, \cdots\}$, it follows from Lemma 5.1 that μ is absolutely continuous. Also $S(\mu) = S(\nu)$. Thus from our remark at the beginning of this section, it follows that $\mathscr{A}(\mu) = \mathscr{A}(\nu)$. □

To obtain results similar to that of the preceding theorem in cases where the underlying measure μ is not necessarily absolutely continuous we have found it necessary to introduce the following definitions and terms. We say that a set A is *almost μ-invariant* if there is a p, $0 \leqq p \leqq 1$, such that $\mu(A + \theta) = p$ for almost $(\lambda)\theta$, where λ denotes the Lebesgue measure. For a given p.m. μ, we denote by $\mathscr{B}(\mu)$, the family of all almost μ-invariant sets. If μ is an absolutely continuous p.m. then the notion of almost μ-invariance coincides with μ-invariance and $\mathscr{A}(\mu) = \mathscr{B}(\mu)$. It is easily seen that a set B is almost μ-invariant with $\mu(B + \theta) = p$ a.e. iff $(I_B - p) * \tilde{\mu} = 0$ a.e., where $\tilde{\mu}$ is such that for each Borel set A, $\tilde{\mu}(A) = \mu(-A)$. With respect to almost μ-invariance we have

LEMMA 5.2. *Let* μ_1 *and* μ_2 *be two* p.m.'*s and suppose that* $B \in \mathscr{B}(\mu_1) \cap \mathscr{B}(\mu_2)$. *Then* $\mu_1(B + \theta) = \mu_2(B + \theta)$ a.e.

PROOF. Let $\mu_i(B + \theta) = p_i$ a.e. Then $(I_B - p_i) * \tilde{\mu}_i = 0$ so that $(I_B - p_i) * \tilde{\mu}_1 * \tilde{\mu}_2 * \lambda = 0$ where λ is any absolutely continuous p.m. Hence $p_1 = p_2 = \tilde{\mu}_1 * \tilde{\mu}_2 * \lambda(B)$. □

LEMMA 5.3. *Let* μ_1 *and* μ_2 *be two* p.m.'*s with* $S(\mu_1) = S(\mu_2) = S$ *and suppose that* S *is an* S*-set. Then* $\mathscr{B}(\mu_1) = \mathscr{B}(\mu_2)$.

PROOF. Let λ be an absolutely continuous p.m. with $S(\lambda) = \emptyset$. Then $\mu_i * \lambda$ $(i = 1, 2)$ is an absolutely continuous p.m. Also $S(\mu * \lambda) = S(\mu_2 * \lambda) = S$. So $\mathscr{A}(\mu_1 * \lambda) = \mathscr{A}(\mu_2 * \lambda)$. Now let B be an almost μ_1-invariant set with $\mu_1(B + \theta) = p$ a.e. Then $(I_B - p) * \tilde{\mu}_1 = 0$ a.e. so that $(I_B - p) * \tilde{\mu}_1 * \tilde{\lambda} = 0$. Thus $B \in \mathscr{A}(\mu_1 * \lambda) = \mathscr{A}(\mu_2 * \lambda)$. Consequently $(I_B - p) * \tilde{\mu}_2 * \tilde{\lambda} = 0$. Since $S(\lambda) = \emptyset$, $\beta(\lambda) = \{0\}$. This implies $(I_B - p) * \tilde{\mu}_2 = 0$ a.e. □

COROLLARY 1. *If the* p.m. μ_1, *in the lemma, is absolutely continuous then* $\mathscr{A}(\mu_1) = \mathscr{B}(\mu_2)$.

The above results will now be used to study the nature of μ-invariant sets in terms of structural properties of sets where characteristic functions of given p.m.'s vanish. Unless stated otherwise we denote the set of nonzero integers by Z.

THEOREM 5.2. *Let* μ *be a* p.m. *with* $S(\mu) = Z \cap \{nZ\}'$, *where* n *is a positive integer. Let* A *be a* μ*-invariant set. Then* $\mu(A) = k/n$ *for some* k, $0 \leqq k \leqq n$.

PROOF. Let A be μ-invariant with $\mu(A) = p$. It is easily seen that $S(\mu) = \bigcup_{j=1}^{(n-1)} H_j$, where $H_j = \{j + nk : k = 0, \pm 1, \cdots\}$, $1 \leqq j \leqq (n - 1)$. It now follows as in the proof of Theorem 4.1 that A admits the representation:

$$I_A(x) = p + e(x)g_1(x) + \cdots + e((n-1)x)g_{n-1}(x) ,$$

where the g_i is a periodic function of period $2\pi/n$.

It is now elementary to see that $\sum_0^{n-1} I_A(x + 2j\pi/n) = np$ a.e. Since the left side of the above equation equals k, $0 \leqq k \leqq n$, it follows that $p = k/n$, $0 \leqq k \leqq n$. □

REMARKS. Any further strengthening of the above theorem without any added restrictions on the set $S(\mu)$ does not seem possible. For example if μ is a p.m. which assigns probability $1/n$ to each of the points of the set $E = \{2k\pi : 0 \leqq k \leqq n\}$ then

$S(\mu) = Z \cap \{nZ\}^{\sim}$. It can be seen that this μ does indeed possess μ-invariant sets of all sizes k/n, $0 \leqq k \leqq n$.

It is easy to see that the set $S(\mu)$ considered in the above theorem can be written as a union of cosets of the subgroup nZ as $S(\mu) = \{nZ + 1\} \cup \cdots \cup \{nZ + (n-1)\}$. The next theorem is perhaps more interesting than the preceding one; it asserts that if $S(\mu)$ is the union of a fewer number of cosets of $\{nZ\}$ than those in Theorem 5.2 then μ is weakly complete.

THEOREM 5.3. *Let n be a prime number. Let μ be a* p.m. *and suppose that for some k, $1 < k \leqq (n-1)$, $S(\mu) \subset Z \cap \{nZ\}' \cap \{nZ + k\}'$. Then μ is weakly complete.*

PROOF. We assume without loss of generality that $S(\mu) \subset Z \cap \{nZ\}' \cap \{nZ + 1\}'$, and set $z_0 = e(2\pi/n)$. Let A be a μ-invariant set with $\mu(A) = p$. It follows from Theorem 4.1 that A admits the following representation:

$$I_A(x) = p + e(2x)g_2(x) + \cdots + e((n-1)x)g_{n-1}(x)$$

where the g_j is a periodic function of period $(2\pi/n)$.

This representation now yields that

$$\begin{aligned}(E - z_0^2)(E - z_0^3) \cdots (E - z_0^{n-1})I_A(x) &= p(1 - z_0^2) \cdots (1 - z_0^{n-1}) \\ &= np/(1 - z_0) \quad \text{a.e.}\end{aligned}$$

where 'E' denotes the following increment operator $Ef(x) = f(x + 2\pi/n)$. Simplification of the above equation yields

$$\begin{aligned}I_A(x + 2(n-2)\pi/n) + (1 + z_0)I_A(x + 2(n-3)\pi/n) + \cdots \\ + (1 + z_0 + \cdots + z_0^{n-2})I_A(x) = np/(1 - z_0) \quad \text{a.e.}\end{aligned}$$

Consequently there exist x and y, $y \neq x + 2\pi/n$, such that

$$\begin{aligned}\textstyle\sum_{k=1}^{(n-2)} (1 + z_0 + \cdots + z_0^k)[I_A(x + 2(n-2-k)\pi/n) \\ - I_A(y + 2(n-2-k)\pi/n)] = 0\,.\end{aligned}$$

So z_0 is a root of a polynomial of degree $(n-2)$ with integral coefficient. This is impossible since z_0 is a root of the irreducible polynomial $P(z) = 1 + z + \cdots + z^{n-1}$ of degree $(n-1)$. (see, e.g. Herstein (1964) page 122). Hence $\mu(A) = p = 0$ or 1. □

The following corollaries can be established in a similar manner.

COROLLARY 1. *Let μ be a* p.m. *with $S(\mu) \subset Z$. Suppose for some $n \geqq 1$, $S(\mu) \cap \{nZ\}$ consists of finitely many elements. Then A is μ-invariant implies $\mu(A) = k/n$, $0 \leqq k \leqq n$.*

COROLLARY 2. *Let n be a prime number. Let μ be a* p.m. *with $S(\mu) \subset Z$. Suppose that for some k, $1 \leqq k \leqq (n-1)$, $S(\mu) \cap [\{nZ\} \cup \{nZ + k\}]$ consists of finitely many elements. Then μ is weakly complete.*

It seems to us that the preceding approach to Theorem 5.3 can be particularly useful in a variety of situations, when $S(\mu) \subset Z$, in deciding if a given p.m. μ is weakly complete. For example let μ be a p.m. which assigns probability p_k to $\{k\}$, $0 \leqq k \leqq n$, $\sum p_k = 1$. Further suppose that $P(z) = \sum p_k z^k$ is an irreducible polynomial. It can then be shown in a similar fashion that if $P(z)$ has a root z_0 with $|z_0| \neq 1$, then μ is weakly complete (this last condition holds if $p_0 \neq p_n$).

In view of Theorem 5.3 and its corollaries it is perhaps tempting to conjecture that if a p.m. μ is weakly incomplete then $S(\mu) \cup \{0\}$ must contain either a closed subgroup of the form cZ for some $c > 0$ or a set of the form $cZ \cap \{cuZ\}'$ for some $n \geqq 1$. Although closed subgroups and their cosets do seem to play an important role in weak incompleteness of p.m.'s, we cannot settle the above conjecture at the present time.

6. A conjecture. If μ is the uniform p.m. on [0, 1] then $\mathscr{A}(\mu)$ consists of all sets that are periodic of period one. It is easily seen that if ν is a decomposable p.m. with μ as a factor then $\mathscr{A}(\mu) \subset \mathscr{A}(\nu)$. Now let ν be a given p.m. with $\mathscr{A}(\mu) \subset \mathscr{A}(\nu)$. In a personal communication D. Basu raised the following question concerning the p.m. ν: is every such ν a decomposable p.m. with μ as a factor? We show here that this is not true in general. Consider the p.m. ν whose density function is given as follows:

$$\begin{aligned} f(x) &= 1 \qquad \text{if} \quad n + \textstyle\sum_{k=0}^{(n-1)} \frac{1}{2^k} \leqq x < n + \sum_{k=0}^{n} \frac{1}{2^k}, \qquad n = 1, 2, \cdots \\ &= 0 \qquad \text{elsewhere.} \end{aligned}$$

It is easily seen that $S(\nu) \supset \{2n\pi : n = \pm 1, \pm 2, \cdots\}$ so $A(\nu)$ contains all periodic sets of period one. It can be shown that the p.m. ν does not admit a decomposition of the form $\nu = \mu * \gamma$ with γ a finite signed measure.

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