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# A GENERALIZATION OF DYNKIN'S IDENTITY AND SOME APPLICATIONS

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Let $X(t)$ be a right continuous temporally homogeneous Markov process, $T_t$ the corresponding semigroup and $A$ the weak infinitesimal generator. Let $g(t)$ be absolutely continuous and $\tau$ a stopping time satisfying

$$E_x(\int_0^\tau |g(t)|\,dt) < \infty \qquad \text{and} \qquad E_x(\int_0^\tau |g'(t)|\,dt) < \infty .$$

Then for $f \in \mathscr{D}(A)$ with $f(X(t))$ right continuous the identity

$$E_x g(\tau)f(X(\tau)) - g(0)f(x) = E_x(\int_0^\tau g'(s)f(X(s))\,ds) + E_x(\int_0^\tau g(s)Af(X(s))\,ds)$$

is a simple generalization of Dynkin's identity ($g(t) \equiv 1$).

With further restrictions on $f$ and $\tau$ the following identity is obtained as a corollary:

$$E_x(f(X(\tau))) = f(x) + \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k!} E_x(\tau^k A^k f(X(\tau))) + \frac{(-1)^{n-1}}{(n-1)!} E_x(\int_0^\tau u^{n-1}A^n f(X(u))\,du) .$$

These identities are applied to processes with stationary independent increments to obtain a number of new and known results relating the moments of stopping times to the moments of the stopped processes.

**0. Introduction.** In [7] W. J. Hall proved identities of the form

$$E((X(\tau))^n) = \sum_{k=1}^{[n/2]} \sum_{l=0}^{n-2k} a_{k,l}^{(n)} E(\tau^k X(\tau)^l)$$

for processes with stationary independent increments. For $n = 2$ these identities are a simple consequence of a natural extension of Dynkin's Identity $E(f(X(\tau))) - f(x) = E(\int_0^\tau Af(X(s))\,ds)$. In attempting to apply this method of proof for $n > 2$, we were led to the identity

$$E_x(f(X(\tau))) = f(x) + \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{(k-1)!} E_x(\tau^k A^k f(X(\tau))) + \frac{(-1)^{n-1}}{(n-1)!} E_x(\int_0^\tau u^{n-1}A^n f(X(u))\,du)$$

(which we later discovered also appears in work of Has'minskii [8] and Helms [9]) and to natural extensions of it. As straight forward applications of this and related identities we are able to obtain the results of Hall [7] (under slightly weaker assumptions) as well as other results including an inequality of Burkholder and Gundy [1].

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In the next section we prove a generalization of Dynkin's identity and specialize it in Section 2 for a process with stationary independent increments with zero mean. In Section 3 we discuss the relation between the moments of this process and the associated canonical measure. Finally in Section 4 we give new proofs of Hall's results and some new results.

The referee pointed out that the identity in Theorem 4 may be obtained directly from the change of variable formula for semimartingales in [2] provided one knows that certain expectations are finite. Our techniques, however, depend on standard material in the theory of Markov processes and can also be applied to processes that do not have independent increments (see [10]).

**1. Generalization of Dynkin's identity.** Let $(\Omega, \mathbb{F}, P)$ be a probability space and let $\{X(t, \omega); t \geqq 0\}$ be a right continuous temporally homogeneous Markov process with a topological state space $E$. Let $B$ be the Banach space of bounded measurable real-valued functions on $E$ endowed with the supnorm. For each $t$ define the operator $T_t$ on $B$ by

$$(T_t f)(x) = E(f(X(t)) \mid X(0) = x) .$$

It is known [4] that $\{T_t; t \geqq 0\}$ is a semigroup of bounded operators on B, which is strongly continuous on some subspace. Let $A$ denote the weak infinitesimal generator of the semigroup $\{T_t\}$ defined as follows: Let $f \in B$ be such that

$$\sup_{t>0} \left\| \frac{(T_t f)(x) - f(x)}{t} \right\| < \infty \tag{1}$$

and

$$\lim_{t \downarrow 0} \frac{(T_t f)(x) - f(x)}{t} = g(x) \quad \text{exists} \tag{2}$$

for each $x$ in $E$ where $g \in B$. Denote this class of $f$'s as $\mathscr{D}(A)$. For $f \in \mathscr{D}(A)$ define $Af = g$.

For $f \in \mathscr{D}(A)$

$$(T_t f)(x) - f(x) = \int_0^t (T_u Af)(x)\, du . \tag{3}$$

Using $E_x$ for the expectation operator with $X(0) = x$ this becomes

$$E_x f(X(t)) - f(x) = E_x\left(\int_0^t Af(X(u))\, du\right) . \tag{4}$$

This plus the Markov property implies that

$$Y_1(t) = f(X(t)) - f(X(0)) - \int_0^t Af(X(u))\, du \tag{5}$$

is a martingale.

Now if $\tau$ is a stopping time with $E_x(\tau) < \infty$ then since $|Y_1(\tau)| \leqq 2\|f\| + \|Af\|\tau$, applying Doob's optional sampling theorem [3] one readily gets the following identity first observed by Dynkin.

THEOREM 1. *(Dynkin* [4]*). If $f \in \mathscr{D}(A)$ and $f(X(t))$ is right continuous in $t$* a.s. *then for any stopping time $\tau$ with $E_x(\tau) < \infty$ one has $E_x Y_1(X(\tau)) = 0$ or alternatively*

$$E_x f(X(\tau)) = f(x) + E_x\left(\int_0^\tau Af(X(u))\, du\right) . \tag{6}$$

Next, let $g(t)$ be a real-valued, absolutely continuous function. The following semigroup identity is straightforward. For $f \in \mathscr{D}(A)$

$$(7)\qquad g(t)(T(t-u)f)(x) - g(u)f(x) = \int_u^t g'(s)(T(s-u)f)(x)\,ds + \int_u^t g(s)(T(s-u)Af)(x)\,ds\,.$$

If $g(t) \equiv 1$, and $u = 0$ then (7) becomes (3). This, as before, implies that

$$(8)\qquad Y_2(t) = g(t)f(X(t)) - g(0)f(X(0)) - \int_0^t g'(s)f(X(s))\,ds - \int_0^t g(s)Af(X(s))\,ds$$

is a martingale.

Once again applying Doob's optional sampling theorem [2] we get the following generalization of Dynkin's identity.

THEOREM 2. *Let $f \in \mathscr{D}(A)$ and $f(X(t))$ be right continuous in $t$. Let $g(t)$ be absolutely continuous in $t$. If $\tau$ is a stopping time such that*

$$E_x\left(\int_0^\tau |g(s)|\,ds\right) < \infty \qquad \text{and} \qquad E_x\left(\int_0^\tau |g'(s)|\,ds\right) < \infty$$

*then*

$$(9)\qquad E_x g(\tau)f(X(\tau)) - g(0)f(x) = E_x\left(\int_0^\tau g'(s)f(X(s))\,ds\right) + E_x\left(\int_0^\tau g(s)Af(X(s))\,ds\right).$$

We can use Theorems 1 and 2 to obtain a further generalization of Theorem 1.

THEOREM 3. *For $n \geqq 1$ let $f, Af, \cdots, A^{n-1}f \in \mathscr{D}(A)$ and $f(X(t)), Af(X(t)), \cdots, A^{n-1}f(X(t))$ be right continuous in $t$. Let $\tau$ be a stopping time such that $E_x(\tau^n) < \infty$. Then*

$$(10)\qquad E_x f(X(\tau)) = f(x) + \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k!} E_x(\tau^k A^k f(X(\tau))) + \frac{(-1)^{n-1}}{(n-1)!} E_x\left(\int_0^\tau s^{n-1}A^n f(X(s))\,ds\right).$$

PROOF. Take $g(t) = t^k$ and $f$ to be $A^k f$ in Theorem 2. We get for $k \leqq n-1$

$$(11)\qquad E_x\tau^k A^k f(X(\tau)) = kE_x\left(\int_0^\tau s^{k-1}A^k f(X(s))\,ds\right) + E_x\left(\int_0^\tau s^k A^{k+1}f(X(s))\,ds\right).$$

Setting

$$(12)\qquad a_k = \frac{(-1)^k}{k!} E_x\left(\int_0^\tau s^k A^{k+1}f(X(s))\,ds\right)$$

we get

$$(13)\qquad a_k - a_{k-1} = \frac{(-1)^k}{k!} E_x(\tau^k A^k f(X(\tau))) \equiv b_k\,.$$

By Theorem 1

$$E_x f(X(\tau)) - f(x) = a_0$$

which by (13) is

$$-\left[\sum_{k=1}^{n-1} b_k + a_{n-1}\right] = \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k!} E_x(\tau^k A^k f(X(\tau))) + \frac{(-1)^{n-1}}{(n-1)!} E_x\left(\int_0^\tau s^{n-1}A^n f(X(s))\,ds\right)$$

proving (10). □

REMARK 1. It is worth noting that one can again give a martingale proof of Theorem 3 using the semigroup identity

$$T_{t-s}f - f = \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k!} (t^k T_{t-s} A^k f - s^k A^k f) - \frac{1}{(n-1)!} \int_0^t u^{n-1} T_{u-s} A^n f\, du \, . \tag{14}$$

REMARK 2. Theorem 3 appears implicitly in the work of Hasminskii [8] and Helms [9] who used it to give a probabilistic representation of solutions of some differential equations.

**2. Processes with stationary independent increments and their generators.** Let $\{X(t); t \geqq 0\}$ be a right continuous process with stationary independent increments. The general form of the infinitesimal generator for this process is given by (see Feller [5])

$$Af(x) = \int_{|y|<1} \frac{f(x+y) - f(x) - yf'(x)}{y^2} \Omega(dy) + \int_{|y|\geqq 1} \frac{f(x+y) - f(x) - f'(x)}{y^2} \Omega(dy) + bf'(x) \tag{15}$$

where $b$ is a constant, the integrand is defined as $\frac{1}{2}f''(x)$ at $y = 0$, and $\Omega$ is a measure on $R$, $\int (1 + y^2)^{-1}\Omega(dy) < \infty$ and is defined for all bounded $f$ which are twice continuously differentiable functions with bounded derivatives—of two orders. We take the Banach space on which the semigroup $T_t$ associated with $X(t)$ acts to be the space of all bounded continuous functions having limits as $x \to \pm\infty$. If

$$\int \frac{1}{1+|y|} \Omega(dy) < \infty \tag{16}$$

then we may take $A$ to be of the form

$$Af(x) = \int_{-\infty}^{+\infty} \frac{f(x+y) - f(x) - yf'(x)}{y^2} \Omega(dy) + bf'(x) \, . \tag{17}$$

Also, it is known that $E|X(t)| < \infty$ for $t > 0$ if and only if (16) holds. If $EX(t) = 0$ then $b$ becomes zero in (17). *From now on we shall assume* (16) *holds and take* $b = 0$ *in* (17). Thus $\{X(t); t \geqq 0\}$ is a process with stationary independent increments with zero mean. We shall always assume that this process has right continuous sample paths. The generator $A$ now simply becomes

$$Af(x) = \int_{-\infty}^{+\infty} \int_0^1 \int_0^z f''(x + yu)\, du\, dz\, \Omega(dy) \, . \tag{18}$$

The right side of (18) is well defined for all $f$ such that

$$\sup_{0\leqq u\leqq 1} \int |f''(x + uy)|\Omega(dy) < \infty \, . \tag{19}$$

We may extend the operator $A$ to all $f$'s satisfying (19). We shall denote this extension by $\mathscr{A}$. We know that Theorems 2 and 3 are valid for $f$'s in the domain of $A$. We can now extend them to a larger class of $f$'s. Of course, we have to use the extended operator $\mathscr{A}$ on the right side. The precise result is

THEOREM 4. *Let $\varphi$ be a nonnegative twice continuously differentiable convex function on $R$. Let $g$ be continuously differentiable and nondecreasing. Let $\tau$ be a stopping time such that $E_x(\tau) < \infty$. Then*

$$E_x g(\tau)\varphi(X(\tau)) - g(0)\varphi(x) = E_x\left(\textstyle\int_0^\tau g'(s)\varphi(X(s))\,ds\right) + E_x\left(\textstyle\int_0^\tau g(s)\mathscr{A}\varphi(X(s))\,ds\right) \tag{20}$$

*where equality holds even when one side is infinite.*

PROOF. Let us assume first that $g$ and $g'$ are bounded and there exists $k < \infty$ such that $\varphi$ is linear for $|x| \geqq k$. This implies that $\varphi''$ has compact support and $\varphi'$ is bounded. There exists a sequence of bounded, twice continuously functions $\varphi_n$ such that

$$\sup_{n,x}|\varphi_n'(x)| < \infty\,, \qquad \sup_{n,x}|\varphi_n''(x)| \equiv M < \infty$$

and $\varphi_n \uparrow \varphi$. Theorem 2 is valid for each $\varphi_n$ and the $A\varphi_n$ are uniformly bounded in $n$. By monotone convergence on the left and dominated convergence on the right we obtain (20).

Next let $g$ and $\varphi$ satisfy the hypothesis of the theorem. It is not difficult to see that there exist continuously differentiable $g_n$ such that $g_n$ and $g_n'$ are bounded, $g_n \uparrow g$, $g_n' \uparrow g'$ and there exist $\varphi_n$ nonnegative, convex and ultimately linear such that $\varphi_n \uparrow \varphi$, $\varphi_n'' \uparrow \varphi''$. Thus $\mathscr{A}\varphi_n \uparrow \mathscr{A}\varphi$. Now by the monotone convergence theorem and the first part of the proof we get (20) with the interpretation that if one side is infinite so is the other.

REMARK 1. Both sides of (20) are linear in $g$ and $\varphi$. Hence, when all the quantities involved are finite then (20) will be valid for linear combinations of $\varphi$'s and $g$'s i.e. in much greater generality.

REMARK 2. As a simple application of Theorem 4 consider the case $\Omega(R) < \infty$ $g(t) = 1$ and $\varphi(x) = x^2$. Then $\mathscr{A}\varphi$ is well defined and by Theorem 4 we get

COROLLARY 1. *If $E_x(\tau) < \infty$ then*

$$E_x X^2(\tau) = \Omega(R)E_x(\tau)\,. \tag{21}$$

REMARK 3. The hypothesis $E_x(\tau) < \infty$ is indispensable. In fact, if $\{X(t)\}$ is standard Brownian motion and $\tau$ is the first time the process exceeds one then $E_0(\tau) = \infty$ whereas by continuity of paths $X^2(\tau) = 1$ w.p. 1. and thus has all the moments.

REMARK 4. One can extend the result of Theorem 3 to a larger class of $f$'s in a similar fashion. We note that

$$\sup_{0\leqq t_i\leqq 1,\, i=1,2,\cdots,k} \int\int\int |\varphi^{(2k)}(x + t_1y_1 + t_2y_2 + \cdots + t_ky_k)|\Omega(dy_1)\Omega(dy_2)\Omega(dy_k) < \infty$$

for $k = 1, 2, \cdots, n$ implies $\mathscr{A}\varphi, \mathscr{A}^2\varphi, \cdots, \mathscr{A}^n\varphi$ are all well defined.

**3. Moments of a processes with stationary independent increments and the associated canonical measure $\Omega$.** The results of this section are part of the oral tradition in this area and in fact, are even derivable for some cases from results of Feller [6] regarding the relation between the tails of $X(t)$ and $\Omega$. We present them here for the sake of completeness as well as due to the fact that they are not readily available. Also the power of the semigroup approach is illustrated here very clearly.

THEOREM 5. *Let $\{X(t);\ t \geqq 0\}$ be a process with stationary independent increments with zero mean. Let $\Omega$ be the associated measure in the sense of* (18). *Then, for $\alpha > 1$*

$$E|X(t)|^\alpha < \infty \qquad \text{iff} \quad \int_{|y|\geqq 1} |y|^{\alpha-2}\Omega(dy) < \infty\ . \tag{22}$$

PROOF. For $\alpha \geqq 2$, taking $g \equiv 1$ and $\varphi(x) = |x|^\alpha$, Theorem 4 implies

$$\begin{aligned} E_x|X(t)|^\alpha &- |x|^\alpha \\ &= \alpha\cdot(\alpha-1)E_x\Big(\int_0^t \int_{-\infty}^{+\infty} \int_0^1 \int_0^z |X(s)+yu|^{\alpha-2}\,du\,dz\,\Omega(dy)\,ds\Big)\ . \end{aligned} \tag{23}$$

For $p \geqq 0$, $|a+b|^p \leqq c(|a|^p + |b|^p)$ where $c = 2^{p-1}$ if $p > 1$ and 1 if $0 \leqq p \leqq 1$. Consequently

$$E_x|X(t)|^\alpha \leqq |x|^\alpha + \int_0^t (c_1 E_x|X(s)|^{\alpha-2} + c_2)\,ds \tag{24}$$

where $c_2 = 2^{\alpha-2}\int |y|^{\alpha-2}\Omega(dy)$. If $\int |y|^{\alpha-2}\Omega(dy) < \infty$, it follows easily that $E_x|X(t)|^\alpha < \infty$. Conversely, if $E|X(t)|^\alpha < \infty$ then $E_x|X(s)|^{\alpha-2} < \infty$ for $s \leqq t$ and from (23) we get $\int_{-\infty}^{+\infty}\int_0^1\int_0^z E_x|X(s)+yu|^{\alpha-2}\,du\,dz\,\Omega(dy) < \infty$. Since

$$|yu|^{\alpha-2} \leqq c(|yu + X(s)|^{\alpha-2} + |X(s)|^{\alpha-2})$$

we have $\int_{-\infty}^{+\infty}\int_0^1\int_0^z |yu|^{\alpha-2}\,du\,dz\,\Omega(dy) < \infty$, and hence $\int |y|^{\alpha-2}\Omega(dy) < \infty$.

Turning now to the case $1 < \alpha < 2$ we first note that for $\varphi(x) = |x|^\alpha$, $\mathscr{A}\varphi$ is not well defined. To avoid this difficulty define

$$\begin{aligned} \psi(x) &= 1 && \text{if} \quad |x| \leqq 1 \\ &= |x|^{\alpha-2} && \text{if} \quad |x| > 1 \end{aligned}$$

and

$$\varphi(x) = \int_0^x \int_0^y \psi(t)\,dt\,dy\ .$$

Then $\varphi(0) = \varphi'(0) = 0$, $\varphi''(x) \geqq 0$ and decreasing as $x \to \infty$. It is easily checked that

$$\begin{aligned} |\varphi(x+y) - \varphi(x) - y\varphi'(x)| &= \int_0^{|y|}\int_0^z \varphi''(w+x)\,dw\,dz \\ &\leqq 2\int_0^{|y|}\int_0^{z/2} \varphi''(w)\,dw\,dz \quad (\text{since } \varphi'' \text{ is } \downarrow) \\ &\leqq 4\varphi\Big(\frac{|y|}{2}\Big)\ . \end{aligned} \tag{25}$$

Consequently, if $\int |y|^{\alpha-2}\Omega(dy) < \infty$

$$\mathscr{A}\varphi(x) \leqq 4\int \frac{\varphi(|y|/2)}{y^2}\,\Omega(dy) \equiv K < \infty\ ,$$

and hence

$$E_x\varphi(X(t)) = \varphi(x) + E_x(\textstyle\int_0^t \mathscr{A}\varphi(X(s))\,ds) \tag{26}$$
$$\leqq 1 + Kt < \infty .$$

Since $\varphi(x) \sim (\alpha(\alpha-1))^{-1}|x|^\alpha$ as $|x| \to \infty$ this proves that $E|X(t)|^\alpha < \infty$. The converse is proved in the same way as before using the estimate

$$\varphi''(x+y) \geqq |x+y|^{\alpha-2} \qquad \text{for } x \geqq 0, y \geqq 1 . \tag{27}$$

By a similar method one can establish the following result on moment generating functions.

THEOREM 6. *For $\theta > 0$, $\int e^{\theta y}\Omega(dy) < \infty$ implies $E_x e^{\theta X(t)} < \infty$.*

PROOF. Let $\psi_N(x) = \theta^2 e^{\theta(x \wedge N)}$ and

$$\begin{aligned}\varphi_N(x) = \textstyle\int_{-\infty}^x \int_{-\infty}^y \psi_N(t)\,dt\,dy &= e^{\theta x} & x \leqq N\\ &= e^{\theta N}\left(1 + \theta(x-N) + \theta^2\frac{(x-N)^2}{2}\right) & x > N .\end{aligned}$$

Then $\varphi_N$ is convex and

$$E_x\varphi_N(X(t)) = \varphi(x) + E_x(\textstyle\int_0^t \mathscr{A}\varphi_N(X(s))\,ds) . \tag{28}$$

But $\mathscr{A}\varphi_N(x) = \int_{-\infty}^{+\infty}\int_0^1\int_0^z \psi_N(x+wy)\,dw\,dz\,\Omega(dy)$ and since

$$\psi_N(x+wy) \leqq e^{\theta wy}\psi_N(x) \leqq \theta^2 e^{\theta wy}\varphi_N(x)$$

$E_x\mathscr{A}\varphi_N(X(s)) \leqq \theta^2 a(\theta)E_x\varphi_N(X(s))$ where $a(\theta) = \sup_{0\leqq w\leqq 1}\int e^{\theta wy}\Omega(dy)$. Thus from (28) we get

$$E_x\varphi_N(X(t)) \leqq \varphi(x) + \theta^2 a(\theta)\textstyle\int_0^t E_x\varphi_N(X(s))\,ds .$$

It can be checked that $E_x(\varphi_N(X(t))) < \infty$, so Gronwall's inequality implies

$$E_x\varphi_N(X(t)) \leqq Ke^{\gamma t} \tag{29}$$

where $K$ and $\gamma$ depend on $\theta$ but not $N$. If we let $N \to \infty$, the theorem follows by Fatou's lemma.

**4. Application to moments of $X(\tau)$.** In this section we obtain Hall's [6] results on $E(X(\tau)^n)$ where $\tau$ is a stopping time for the process $\{X(t); t \geqq 0\}$. We remind the reader that $X(\cdot)$ is a process with stationary independent increments with zero mean. We shall assume without loss of generality that $X(0) = 0$ a.s.

THEOREM 7. *Let $\tau$ be a stopping time for $X(\cdot)$. Then the following hold:*

(i) *For $n \geqq 2$ there exist constants $c$ and $C$ (finite and positive) depending on $n$ and $\Omega$ such that if $E(\tau) < \infty$ and $\int |y|^{n-2}\Omega(dy) < \infty$ then*

$$cE\tau^{n/2} \leqq E|X(\tau)|^n \leqq C\max\{E(\tau), E(\tau^{n/2})\} . \tag{30}$$

(ii) *If $n \geqq 2$, $E(\tau^{n/2}) < \infty$, $\int |y|^{n-2}\Omega(dy) < \infty$ then there exist constants $a_{k,l}$ independent of $\tau$ but depending on $n$ and $\Omega$ such that*

$$E(X(\tau)^n) = \textstyle\sum_{l\leqq n-2k,\,k\leqq[n/2]} a_{k,l}^{(n)} E(\tau^k X(\tau)^l) . \tag{31}$$

PROOF. Taking $\varphi(x) = |x|^n$ and $g(t) \equiv 1$ in Theorem 4 we get

$$E(|X(\tau)|^n) = n(n-1)E(\textstyle\int_0^\tau \int_R \int_0^1 \int_0^z |X(s) + wy|^{n-2}\, dw\, dz\, \Omega(dy)\, ds) .$$

But, $|a + b|^{n-2} \leqq c(|a|^{n-2} + |b|^{n-2})$ and hence

$$E(|X(\tau)|^n) \leqq c_1 E(\tau) + c_2 E(\textstyle\int_0^\tau |X(s)|^{n-2}\, ds) \tag{32}$$

where we used the fact $\int |y|^{n-2}\Omega(dy) < \infty$. Clearly,

$$\begin{aligned} E(\textstyle\int_0^\tau |X(s)|^{n-2}\, ds) &\leqq E(\tau \sup_{s \leqq \tau} |X(s)|^{n-2}) \\ &\leqq (E(\tau^{n/2}))^{2/n} E((\sup_{s\leqq\tau} |X(s)|)^n)^{(n-2)/n} . \end{aligned}$$

Let $\tau$ be a bounded stopping time. It follows from (32) that $E(|X(\tau)|^n) < \infty$. By Doob's result on submartingales (note that for $n \geqq 1$, $|X(s)|^n$ is a submartingale) [2]

$$E(\sup_{s\leqq\tau} |X(s)|^n) \leqq \left(\frac{n}{n-1}\right)^n (E|X(\tau)|^n) \qquad \text{for } n > 1 . \tag{33}$$

From (32) we now have

$$E|X(\tau)|^n \leqq c_1 E(\tau) + c_2 (E(\tau^{n/2}))^{2/n} (E|X(\tau)|^n)^{(n-2)/n} \tag{34}$$

and hence

$$(E|X(\tau)|^n) \leqq \left[\frac{c_1 E(\tau)}{(E|X(\tau)|^n)^{(n-2)/n}} + c_2 (E(\tau^{n/2}))^{2/n}\right]^{n/2} . \tag{35}$$

Now drop the assumption that $\tau$ is bounded. Taking $\tau \wedge N$ and letting $N \to \infty$ in (35) yields the finiteness of $E(|X(\tau)|^n)$ as well as (34) and (35) for all stopping times with $E(\tau^{n/2}) < \infty$. Now either $E\tau < (E\tau^{n/2})^{2/n}(E|X(\tau)|^n)^{(n-2)/n}$ or the other way around. In the former case we see from (34) that

$$E|X(\tau)|^n \leqq (c_1 + c_2)^{n/2}(E(\tau^{n/2})) .$$

In the latter case we get

$$E|X(\tau)|^n \leqq (c_1 + c_2)E(\tau) .$$

This proves the right half of (30).

We now turn to the left half of (30).

Let $g(s) = (s \wedge k)^{(n/2)-1}$. Then by Theorem 4 and the nonnegativity of $g'$

$$E(g(\tau)X(\tau)^2) \geqq cE(\textstyle\int_0^\tau g(s)\, ds) \tag{36}$$

where $c = \Omega(R)/2$. By the Hölder inequality this yields

$$\begin{aligned} (Eg(\tau)^{n/(n-2)})^{(n-2)/n}(E|X(\tau)|^n)^{2/n} &\geqq cE(\textstyle\int_0^\tau (s \wedge k)^{n/2-1}\, ds) \\ (E(\tau \wedge k)^{n/2})^{(n-2)/n}(E|X(\tau)|^n)^{2/n} &\geqq cE(\textstyle\int_0^{\tau\wedge k} s^{n/2-1}\, ds) \\ &\geqq \frac{2c}{n}(E(\tau \wedge k)^{n/2}) \end{aligned}$$

or

$$\left(\frac{2c}{n}\right)^{n/2} E(\tau \wedge k)^{n/2} \leqq E|X(\tau)|^n . \tag{37}$$

Letting $k \to \infty$ we get (30).

In view of Remark 4 following Theorem 4 the following result is valid. Let $\varphi = \varphi_1 - \varphi_2$ where $\varphi_1$ and $\varphi_2$ are nonnegative with $\varphi_i^{(2\nu)} \geqq 0$ for $\nu = 0, 1, \cdots, [n/2] + 1$.

Then provided the quantities involved are all finite one has

$$E_x\varphi(X(\tau)) = \varphi(x) + \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k!} E_x(\tau^k \mathscr{A}^k \varphi(X(\tau))) + \frac{(-1)^{n-1}}{(n-1)!} E_x(\textstyle\int_0^\tau s^{n-1}\mathscr{A}^n\varphi(X(s))\, ds)\,. \tag{38}$$

For the function $\varphi(x) = x^n$ for $n$ even (38) follows directly from Theorem 4. For $\varphi(x) = x^n$ for odd $n \geqq 2$, (38) holds if we set

$$\begin{aligned} \varphi_1 &= x^n \quad \text{if } x > 0 & \varphi_2 &= 0 \quad \text{if } x > 0 \\ &= 0 \quad \text{if } x \leqq 0 & &= -x^n \quad \text{if } x \leqq 0 \quad \text{and } \varphi = \varphi_1 - \varphi_2\,. \end{aligned}$$

Given an $n$, $\mathscr{A}^k\varphi$ will be zero for $k > [n/2]$. This is clear from (18). Also $\mathscr{A}^k\varphi(x)$ is a polynomial in $x$ of degree at most $n - 2k$. So to prove (31) it is enough to check the finiteness of $E_x(\tau^k|X(\tau)|^l)$ for $k \leqq [n/2]$ and $l \leqq n - 2k$. But by Holder's inequality

$$E_x\tau^k|X(\tau)|^l \leqq (E_x\tau^{n/2})^{2k/n}(E_x|X(\tau)|^{ln/(n-2k)})^{(n-2k)/n}\,,$$

since $l \leqq n - 2k$ and $E_x|X(\tau)|^n < \infty$ by (i).

REMARK 1. In the case of the standard Browninan motion the constant $c_1$ in (32) in zero and we can improve (30) to read

$$cE_x\tau^{n/2} \leqq E_x|X(\tau)|^n \leqq CE_x\tau^{n/2}$$

for $n > 2$. This result is known and is due to Burkholder and Gundy [1] who employ very different methods.

REMARK 2. For processes other than the Brownian motion the right side of (30) cannot be improved to $CE_x\tau^{n/2}$ for $n > 2$. In fact, if that were to be the case then $\lim\sup_{t\to 0} t^{-n/2}E_x|X(t)|^n \leqq C < \infty$. But for $n > 2$ this implies the continuity of the sample paths and hence makes the process a Brownian motion.

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