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LAST EXIT TIMES AND ADDITIVE FUNCTIONALS[1]

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The various objects examined in this paper arise in the study of last exit times and balayage of additive functionals for standard Markov processes. The most important results concern the characterization of the Laplace transform of an entrance law, the relationship between the last exit distribution from a set and the capacitary measure of the set, the characterization of projective sets and d-sets, and a last exit decomposition formula for finite sets F which expresses the distribution of X_t in terms of the last exit from F prior to t.

1. Introduction. This paper contains a number of variations on the theme that there is a close relationship between last exit times and the balayage of additive functionals. A number of these results (especially in Sections 2 and 3) are more or less known (at least implicitly). In particular, some of the ideas in Section 2 appear in a very general form in Azema [1]. Still it seemed worthwhile to give a systematic and explicit presentation of these ideas. We shall treat standard processes throughout this paper, but by making use of the Ray compactification many of the results, appropriately modified, extend to right continuous strong Markov processes.

In Section 2 we describe the balayage of an additive functional A on a set D in two steps. The first step amounts to sweeping the mass of the measure dA_t on an interval I during which the process avoids D onto the last time the process was in D prior to I. In the second step one takes the natural (or previsible) projection of the object obtained in the first step. Combining this with a change of variable one obtains the very useful formula (2.13), due originally to Azema [1]. (Since the first draft on this paper was written we have discovered that B. Maisonneuve uses a similar approach to balayage in his thesis "Systemes regeneratifs.") In Section 3 we study the distribution of the last exit place from a set D. Of special importance is the relationship (3.5), under duality hypotheses, between the capacitary measure of D and this last exit distribution. Related results have been obtained recently by Chung [6]. Such results go back, at least for Brownian motion, to McKean [9], and for general infinitely divisible processes to Port and Stone [15]. Section 4 is somewhat more technical in nature. We give some characterizations of projective sets and d-sets—concepts which have arisen in the study of the balayage of additive functionals. In Section 5 we present a last exit decomposition (5.21). Such decompositions for Markov chains

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were originally investigated by Chung. See [5] for a good survey of such results, and also [14] and [8] for recent developments. Our main contribution here is the method of attack. Our basic tools are Azema's formula and the simple change of variable (2.4), together with the analytic result on entrance laws in Section 6. Sections 3, 4, and 5 are completely independent of one another and may be read in any order. Finally in Section 6 we characterize the Laplace transform of an entrance law. The result is used in Section 5. Since this characterization may be of some independent interest, we have written Section 6 in such a manner that it is completely independent of the earlier sections of this paper. This result is due to Neveu [13] in the chain case.

We would like to thank R. M. Blumenthal for a conversation which put us on the right track to understanding the last exit decomposition in Section 5.

2. Balayage of additive functionals. We begin this section with some remarks on a change of variable formula which contains most forms ordinarily used in the theory of processes. See, for example, [10] VII, T. 12 and [2] V, (3.42). To avoid mentioning technical trivialities at subsequent points of the paper, we shall spell out carefully the relevant facts now.

Suppose r_t is a right continuous increasing function on $[0, \infty)$ taking values in $[0, \infty]$. Let $r_\infty = \lim_{t\to\infty} r_t$. It is not assumed that $r_0 = 0$. For $0 \leqq s \leqq \infty$, let $l_s = \sup\{t : r_t < s\}$, the sup of the empty set being defined to be 0. Let $L_s = \sup\{t : r_t \leqq s\}$. The following facts are standard:

(2.1) $l_0 = 0$, l_s is left continuous on $[0, \infty]$ with values in $[0, \infty]$; L_s is right continuous on $[0, \infty)$ with values in $[0, \infty]$; $l_s = L_s$ for all but countably many $s \in [0, \infty)$.

$$(2.2) \qquad \begin{aligned} r_t &= \inf\{s : l_s > t\} \\ &= \inf\{s : L_s > t\} \qquad \text{for all } t \in [0, \infty)\,, \end{aligned}$$

the inf of the emply set being defined to be ∞.

$$(2.3) \qquad \{s : 0 < l_s \leqq b\} = (r_0, r_b] \qquad \text{for } 0 < b < \infty\,.$$

Using (2.3) and the monotone class theorem, one sees that if m is a right continuous increasing function on $[0, \infty)$ and $m(\infty) = \lim_{t\to\infty} m(t)$, then for any positive Borel function g on $[0, \infty)$, one has

$$(2.4) \qquad \int_{(0,\infty)} g(t)\, dm(r_t) = \int_{(0,\infty)} g(l_s) 1_{\{0<l_s<\infty\}}\, dm(s)\,.$$

In the special case where m is continuous on $[0, \infty)$, the term l_s may be replaced by L_s in the right-hand integral, which can then be written $\int_{r_0}^{r_\infty} g(L_s)\, dm(s)$. However, note that $g(L_s)$ may fail to be defined at $s = r_\infty$.

Suppose, for the remainder of this section, that $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ is a standard Markov process with state space $(E, \mathscr{E})$. Let M be a fixed exact MF of X. For most applications, M will be of the form $M_t = 1_{[0, T\wedge\zeta)}(t)$ where T is the hitting time of some Borel set. Let $S = \inf\{t : M_t = 0\} \wedge \zeta$.

(2.5) DEFINITION. A process $\{A_t\}$ is called a raw additive functional (RAF) of (X, M) in case the following two conditions are satisfied almost surely:

(a) $t \to A_t$ is non-decreasing, right continuous, continuous at $t = S$, $A_0 = 0$, and A is constant on $[S, \infty)$;

(b) for each t and $s \geqq 0$, $A_{t+s} = A_t + M_t(A_s \circ \theta_t)$ on $\{t < \zeta\}$.

A raw additive functional of (X, M) which is adapted to $(\mathscr{F}_t)$ is just an additive functional (AF) of (X, M) in the usual sense. The term AF will be reserved exclusively for adapted processes. If A is a RAF of (X, M) and if, for some $\alpha \geqq 0$, A has finite α-potential $u_A{}^\alpha(x) = E^x \int_0^\infty e^{-\alpha t}\, dA_t$, there exists a unique NAF B of (X, M) such that $u_B{}^\beta = u_A{}^\beta$ for all $\beta \geqq \alpha$. Call B the *natural projection* of A. (Note that B is independent of α.)

It is an unfortunate fact that the general theory of processes cannot be applied routinely to the theory of standard processes because of the special role played by the stopping time ζ in the latter theory. We mention here only one aspect of a "general theory" of standard processes. Let **J** denote the stochastic interval $\{(t, \omega): 0 \leqq t < \zeta(\omega)\}$. A function Y defined on **J** will always be extended to $[0, \infty] \times \Omega$ by setting $Y = 0$ on $\{(t, \omega): t \geqq \zeta(\omega)\}$. The smallest of σ-field on **J** containing all adapted processes on **J** which are left continuous on $(0, \zeta)$ is called the previsible σ-field on **J**. An increasing process A on **J** is extended to $[0, \infty] \times \Omega$ by setting $A_t = A_{\zeta-}$ for $t \geqq \zeta$. To distinguish between the two conventions for extension to $[0, \infty] \times \Omega$, one should think of the first extension being for functions, the second being for measures. An obvious modification of the proof of Meyer's integration lemma (VII, T 17 of [10]) shows that if Y is a positive previsible process on **J** and if A and B are integrable (relative to all P^x) increasing processes on **J**, not necessarily adapted, such that for all $x \in E$, $E^x\{A_\infty - A_t \mid \mathscr{F}_t\} = E^x\{B_\infty - B_t \mid \mathscr{F}_t\}$ for all $t \geqq 0$, then for all $x \in E$

$$E^x \int_0^\infty Y_t\, dA_t = E^x \int_0^\infty Y_t\, dB_t\,. \tag{2.6}$$

Of course, both integrals in (2.6) are really over $[0, \zeta)$. Finally if A and B are RAF's of (X, M) with the same finite potential, that is, $E^x(A_\infty) = E^x(B_\infty) < \infty$ for all x, then the condition above (2.6) holds and, hence, so does (2.6).

In the remainder of this paper the equality of two processes means that they are indistinguishable.

Fix now a set $D \in \mathscr{E}^n$ where $\mathscr{E}^n$ denotes the nearly Borel subsets of E. Let $T = T_D = \inf\{t > 0: X_t \in D\}$. The process $T^t = t + T_D \circ \theta_t = \inf\{s > t: X_s \in D\}$ is increasing and right continuous. Define $L^s = \inf\{t: T^t > s\}$ and $l^s = \sup\{t: T^t < s\}$. Then (L^s) is right continuous, (l^s) is left continuous, and $l^s = L^{s-}$ if $0 < s < \infty$. If D^f denotes the fine closure of D, then one also has $L^s = \sup\{t \leqq s: X_t \in D^f\}$ and $l^s = \sup\{t < s: X_t \in D\}$. If D is replaced by D^f throughout, then the processes (T^t), (L^t), and (l^t) are unchanged, and so in many situations there is no loss of generality in assuming that D is finely closed. The behavior of the above defined processes under shifting is summarized by the following formulas, where $a^+ = a \vee 0$:

(2.7) $$T^t \circ \theta_u = T^{t+u} - u\,;$$

(2.8) $$L^s \circ \theta_u = (L^{s+u} - u)^+\,;$$

(2.9) $$l^s \circ \theta_u = (l^{s+u} - u)^+\,.$$

We frequently write $T(t)$ for T^t, $L(s)$ for L^s, and so on.

For fixed M and D as above, if A is a RAF of (X, M) having finite potential $u_A(x) = E^x A_\infty$, the *raw balayage* of A on D is the process $\tilde{A}_D$ defined by

(2.10) $$\tilde{A}_D(t) = A_{T(t)} - A_T\,.$$

It is easy to see, using (2.7), that $\tilde{A}_D$ is a RAF of (X, M). The action of (2.10) can be expressed by saying that the measure $d\tilde{A}_D$ on $(0, \infty)$ is obtained by placing the mass of dA on the interval $(L^t, T^t]$, during which X is not in D, at the point L^t, provided $L^t > 0$. Even if A is adapted, $\tilde{A}_D$ is generally not adapted. The potential of $\tilde{A}_D$ is given by

$$u_{\tilde{A}_D}(x) = E^x \tilde{A}_D(\infty) = E^x(A_\infty - A_T) = Q_D u_A(x)$$

where $\{Q_t\}$ is the semigroup for (X, M). It follows that the natural projection, A_D, of $\tilde{A}_D$ is a NAF of (X, M) having potential $Q_D u_A(x)$, and hence is the balayage of A on D as defined by Motoo. See [2] V. 4. The α-balayage, $A_D{}^\alpha$ of A on D can be obtained using the above procedure as follows: if A is a RAF of (X, M) having finite α-potential $u_A{}^\alpha(x) = E^x \int_0^\infty e^{-\alpha t}\, dA_t$, then $A_t{}^{(\alpha)} = \int_0^t e^{-\alpha s}\, dA_s$ is a RAF of (X, M^α), where $M_t{}^\alpha = M_t e^{-\alpha t}$, and has finite potential. Its balayage $A_D{}^{(\alpha)}$ on D is NAF of (X, M^α) so $A_D{}^\alpha(t) = \int_0^t e^{\alpha s}\, dA_D{}^{(\alpha)}(s)$ is a NAF of (X, M) having α-potential $u^\alpha_{A_D{}^\alpha}(x) = Q_D{}^\alpha u_A{}^\alpha(x)$.

As an application of the construction of balayage given above, we prove a formula, first obtained by Azema [1] for Hunt processes, which will be used in Section 5.

(2.11) PROPOSITION. *Let A be a* RAF *of X having finite α-potential. Let $D \in \mathscr{E}^n$, $L(s) = L^s = \sup\{t \leqq s : X_t \in D^f\}$, $l(s) = l^s = \sup\{t < s : X_t \in D\}$, and let B^α be the α-balayage of A on D. Then if Y is a bounded or positive previsible process on* **J** *one has, for all $x \in E$,*

(2.12) $$E^x \int_0^\infty Y_t e^{-\alpha t}\, dB_t{}^\alpha = E^x \int_0^\infty Y_{l(s)} 1_{\{0<l(s)<\zeta\}} e^{-\alpha s}\, dA_s\,.$$

In particular, if A is continuous and $0 < u < \infty$, taking $Y_t = f(X_{t-}) 1_{(0,u]}(t) 1_{(0,\zeta)}(t)$, one has for all $x \in E$

(2.13) $$E^x \int_{(0,u]} f(X_t) e^{-\alpha t}\, dB_t{}^\alpha = E^x \int_0^\infty f(X_{L(s)-}) 1_{\{0<L(s)\leqq u\}} e^{-\alpha s}\, dA_s\,.$$

PROOF. Let $B_t = \int_0^t e^{-\alpha s}\, dB_s{}^\alpha$ and $A_t{}^\alpha = \int_0^t e^{-\alpha s}\, dA_s$. Then A^α is a RAF of $(X, e^{-\alpha t})$ and B is its balayage on D. If $\tilde{A}^\alpha$ is the raw balayage of A^α on D one has, by (2.6),

(2.14) $$E^x \int_0^\infty Y_t\, dB_t = E^x \int_0^\infty Y_t\, d\tilde{A}_t{}^\alpha = E^x \int_0^\infty Y_t\, dA^\alpha_{T(t)}\,.$$

Applying (2.4) with $m(t) = A_t^\alpha$, $r_t = T(t)$, we find

$$E^x \int_0^\infty Y_t \, dA^\alpha_{T(t)} = E^x \int_0^\infty Y_{l(s)} 1_{\{0<l(s)<\infty\}} \, dA_s^\alpha$$
$$= E^x \int_0^\infty Y_{l(s)} 1_{\{0<l(s)<\zeta\}} \, dA_s^\alpha .$$

Then (2.12) follows from (2.14), and (2.13) holds since the special choice of Y just above (2.13) is previsible on **J**.

The following result will be used in Section 5.

(2.15) LEMMA. *Let $G = G^r \subset D \cap D^r$ and let A be a continuous* RAF *of X having a finite α-potential. Suppose also that A_D^α is continuous. Then $A_D^\alpha(t \wedge T_G)$ is the α-balayage on D of the* RAF, *$A_{t\wedge T_G}$ of (X, T_G).*

PROOF. The conditions on G and D guarantee that $P_G^\alpha P_D^\alpha = P_G^\alpha$ and $P_D^\alpha P_G^\alpha = P_G^\alpha$ ([2] page 63). Let $\{Q_t\}$ be the semigroup for the subprocess (X, T_G). The RAF $A_{t\wedge T_G}$ has α-potential $v^\alpha(x) = E^x \int_0^{T_G} e^{-\alpha t} \, dA_t$ so its α-balayage on D has α-potential

$$Q_D^\alpha v^\alpha(x) = E^x\{e^{-\alpha T_D} v^\alpha(X_{T_D}); T_D < T_G\}$$
$$= E^x\{e^{-\alpha T_D} E^{X_{T_D}} \int_0^{T_G} e^{-\alpha t} \, dA_t; T_D < T_G\}$$
$$= E^x\{\int_{(T_D, T_G]} e^{-\alpha t} \, dA_t; T_D < T_G\} .$$

Since $T_D \leqq T_G$ we obtain

$$Q_D^\alpha v^\alpha(x) = E^x \int_{(T_D, T_G]} e^{-\alpha t} \, dA_t$$
$$= P_D^\alpha u_A^\alpha(x) - P_G^\alpha u_A^\alpha(x) .$$

On the other hand, the α-potential of $A_D^\alpha(t \wedge T_G)$ is

$$E^x \int_0^{T_G} e^{-\alpha t} \, dA_D^\alpha(t) = u^\alpha_{A_D^\alpha}(x) - P_G^\alpha u^\alpha_{A_D^\alpha}(x)$$
$$= P_D^\alpha u_A^\alpha(x) - P_G^\alpha P_D^\alpha u_A^\alpha(x) .$$

The equality of these α-potentials proves the statement of the lemma.

3. Last exit distributions. If D is a nearly Borel subset of E, we define $L = L_D = \sup\{t: X_t \in D\}$ to be the "last exit" time from D. Clearly $L \leqq \zeta$. In the notation of Section 2, $L_D = L^\zeta = l^\zeta$. In this section we shall study the distribution of $X(L-)$ and the joint distribution of L and $X(L-)$. The results are a special case of (2.13), but in order to emphasize their essential simplicity we shall give a direct proof. Moreover, in order not to be overburdened by technicalities, we shall not give the most general possible results. Our results are related to those obtained recently by Chung [6].

(3.1) DEFINITION. Let D be a nearly Borel subset of E. Then D is transient if (i) $L_D < \infty$ almost surely and (ii) $P_D 1$ is a natural potential. We say that D is transient on $[0, \zeta)$ if $L_D < \zeta$ almost surely.

By the definition of natural potential condition (ii) above is equivalent to the condition that for each x if $\{T_n\}$ is an increasing sequence of stopping times with limit $T \geqq \zeta$ almost surely P^x, then $P^x[T_n + T_D \circ \theta_{T_n} < \zeta] \to 0$ as $n \to \infty$. It is immediate that if D is transient on $[0, \zeta)$, then D is transient. Here is a simple sufficient condition that a set be transient.

(3.2) PROPOSITION. *Suppose that there exists a nonnegative function f with Uf finite and $D \subset \{Uf \geqq 1\}$. Then D is transient. If, in addition, there exists a sequence $\{\zeta_n\}$ of stopping times increasing to ζ strictly from below, then D is transient on $[0, \zeta)$.*

PROOF. To see that $P_D 1$ is a natural potential, let $\{T_n\}$ be an increasing sequence of stopping times with $\lim T_n \geqq \zeta$ almost surely P^x. Then

$$P_{T_n} Uf(x) \geqq P_{T_n + T_D \circ \theta_{T_n}} Uf(x) \geqq P^x[T_n + T_D \circ \theta_{T_n} < \zeta]$$

and $P_{T_n} Uf(x) \to 0$ as $n \to \infty$ since $Uf(x) < \infty$ and $\lim T_n \geqq \zeta$. Now if $L = L_D$, then $\{L = \infty\} \subset \{n + T_D \circ \theta_n < \infty\} = \{n + T_D \circ \theta_n < \zeta\}$ for every n and so by the above argument $L < \infty$ almost surely. Under the additional assumption, $\{L = \zeta\} \subset \{\zeta_n + T_D \circ \theta_{\zeta_n} < \zeta\}$ for all n, and as above one has $L < \zeta$ almost surely.

Observe that if for some $\alpha \geqq 0$, $U^\alpha f$ is lower semicontinuous for $f \in \mathbf{C}_K^+$ and $x \to U(x, K)$ is finite for all compact subsets K of E, then any $D \in \mathscr{E}^n$ with compact closure in E satisfies the condition in the first sentence of (3.2).

The following condition on X will be useful in the sequel:

(LL) The trajectories $t \to X_t$ have left limits in E_Δ on $(0, \infty)$ almost surely.

Since $t \to X_t$ has left limits on $(0, \zeta)$ and (ζ, ∞), the thrust of (LL) is that $X_{\zeta-}$ exists in E_Δ almost surely on $\{\zeta < \infty\}$. It is known (I-9.15 of [2] or XIV-T17 of [12]) that a Hunt process satisfies (LL). Also if ζ is totally inaccessible, then (LL) holds (XIV-T23 of [12]).

If D is transient, then $P_D 1$ is the potential of a unique NAF, $A_D = A$, that is, $P_D 1 = u_A$. Note that $u_A(x) = P^x[T_D < \infty] = P^x[L > 0]$. We come now to the main result of this section.

(3.3) PROPOSITION. *Let D be transient, $L = L_D$, and $A = A_D$. Assume, in addition, either that* (LL) *holds or that D is transient on $[0, \zeta)$. Then for each $x \in E$, $s > 0$, and Borel (or universally measurable) $\Gamma \subset E$,*

(i) $P^x[X(L-) \in \Gamma, L > 0] = U_A(x, \Gamma)$
(ii) $P^x[X(L-) \in \Gamma; L > s] = \int P_s(x, dy) U_A(y, \Gamma)$
(iii) $E^x\{e^{-\alpha L}; X(L-) \in \Gamma; L > 0\} = U_A^\alpha(x, \Gamma)$.

Here $U_A(x, \cdot)$, respectively $U_A^\alpha(x, \cdot)$, $\alpha > 0$, is the potential kernel, respectively α-potential kernel, associated with the NAF, *A.*

PROOF. Fix $x \in E$ and consider the increasing process $B_t = I_{\{0 < L \leqq t\}}$. In general B is not adapted, but since $L < \infty$

$$\begin{aligned} E^x\{B_\infty - B_t \mid \mathscr{F}_t\} &= P^x[L > t \mid \mathscr{F}_t] \\ &= P^x[L \circ \theta_t > 0 \mid \mathscr{F}_t] \\ &= u_A(X_t) \,. \end{aligned}$$

On the other hand

$$E^x\{A_\infty - A_t \mid \mathscr{F}_t\} = E^x\{A_\infty \circ \theta_t \mid \mathscr{F}_t\} = u_A(X_t) .$$

Consequently from Meyer's integration lemma

$$E^x \int_0^\infty Y_t\, dA_t = E^x \int_0^\infty Y_t\, dB_t = E^x\{Y_L;\, L > 0\} \tag{3.4}$$

for every nonnegative previsible process Y. So far we have only used the fact that D is transient. If (LL) holds, then letting $Y_t = I_\Gamma(X_{t-})$, $Y_t = I_\Gamma(X_{t-})I_{(s,\infty)}(t)$, and $Y_t = e^{-\alpha t}I_\Gamma(X_{t-})$ and using the fact that A is natural, (3.4) reduces to (i), (ii), and (iii) of (3.3) respectively. On the other hand if $L < \zeta$ almost surely B does not charge ζ, and so by the extension of Meyer's lemma mentioned in (2.6), we may apply (3.4) to the above three processes Y defined only for $0 < t < \zeta$ to obtain the conclusion of Proposition 3.3 when D is transient on $[0, \zeta)$. This establishes (3.3).

Suppose D is transient and (LL) holds. Applying (3.3i) with $\Gamma = E$

$$P^x[X(L-) \in E,\, L > 0] = U_A(x, E) = u_A(x) = P^x[L > 0] ,$$

that is, $X(L-) \in E$ almost surely on $\{L > 0\}$.

In the remainder of this section we assume that the process X has a dual process $\hat{X}$ relative to a σ-finite measure ξ as in Section VI-1 of [2]. We refer the reader to [2] for notation and terminology.

Suppose now that $D \in \mathscr{E}^n$ and that there exists a (necessarily unique) NAF, A with $P_D 1 = u_A$. Then by a theorem of Revuz [16] there exists a σ-finite measure π_D on E such that $U_A f(x) = \int u(x, y) f(y) \pi_D(dy)$. In particular $P_D 1 = U\pi_D$, and π_D is called the *capacitary measure* of D. Note that no Feller conditions on the resolvent or coresolvent are being assumed. The argument at the top of page 287 of [2] shows that π_D is carried by $D \cup {}^rD$ where ${}^rD$ is the set of points coregular for D. If there exists a nonnegative integrable function f such that $\bar{D} \subset \{f\hat{U} \geqq 1\}$, then π_D is finite because

$$\begin{aligned}\infty > \int f(x) P_D 1(x)\, dx &= \int f(x) U\pi_D(x)\, dx \\ &= \int f\hat{U}\, d\pi_D \geqq \pi_D(\bar{D}) = \pi_D(E) .\end{aligned}$$

Now suppose that the hypotheses of Proposition 3.3 hold. Then, in the present situation, (3.3i) may be written

$$P^x[X(L-) \in dy,\, L > 0] = u(x, y)\pi_D(dy) . \tag{3.5}$$

This gives a nice probabilistic meaning to the capacitary measure π_D. In particular, $X(L-) \in D \cup {}^rD$ almost surely on $\{L > 0\}$. Also it follows from (3.3ii) that for a bounded nonnegative f and $t > 0$

$$E^x\{f(X_{L-});\, L > t\} = P_t U(f\pi_D)(x) ,$$

and so, since $U(f\pi_D) \leqq \|f\| P_D 1 \leqq \|f\|$,

$$E^x\{f(X_{L-});\, 0 < L \leqq t\} = U(f\pi_D)(x) - P_t U(f\pi_D)(x) . \tag{3.6}$$

Integrating (3.6) over a set F of finite measure,

$$\int_F E^x\{f(X_{L-}); 0 < L \leqq t\}\,dx = \int (I_F \hat{U} - I_F \hat{P}_t \hat{U}) f\,d\pi_D$$
$$= \int [\textstyle\int_0^t \hat{P}_s(F, x)\,ds] f(x) \pi_D(dx)\,,$$

and letting $F \uparrow E$ through a sequence of sets of finite measure we have

$$\int E^x\{f(X_{L-}); 0 < L \leqq t\}\,dx = \int [\textstyle\int_0^t \hat{P}_s(E, x)\,ds] f(x) \pi_D(dx)\,.$$

In particular, if $\hat{X}$ is Markovian on E, that is, $\hat{P}_t(E, x) = 1$ for all t and x, we obtain the amusing formula

$$\int E^x\{f(X_{L-}); 0 < L \leqq t\}\,dx = t \int f\,d\pi_D\,. \tag{3.7}$$

Throughout this section we have assumed the parameter $\alpha = 0$. There are, of course, similar results for $\alpha > 0$. We leave their explicit formulation to the interested reader. Finally the aficionado of processes will doubtless have noticed that, by considering the accessible and totally inaccessible parts of ζ, the conclusions of Proposition 3.3 and the ensuing discussion (especially (3.5)) hold for every transient set D provided that X is special standard. We shall spare the reader the details.

4. Projective sets and *d*-sets. Recall that a nearly Borel set D is called projective if for every $\alpha > 0$, the α-balayage on D of any CAF is also a CAF. See [2] V. 4. Recall also that a nearly Borel set D is called a d-set if

$$P^x\{X_{T_n} \in D \text{ for all } n,\ X_T \notin D,\ T < \zeta\} = 0 \tag{4.1}$$

for all $x \in E$ whenever $\{T_n\}$ is an increasing sequence of stopping times with limit T. See [7]. The object of this section is to investigate the relationship between the definitions given above and to obtain alternate characterizations of such sets.

We make the following notational convention about random sets: a subset $\mathbf{H}$ of $[0, \infty] \times \Omega$ is always denoted by a boldface letter, and its ω-sections $H = H(\omega)$ are denoted by the corresponding letter, usually with ω suppressed.

For any set $F \in \mathscr{E}^n$, define $\mathbf{Z}_F$ to be the subset of $[0, \infty) \times \Omega$ determined by

$$\mathbf{Z}_F = \{(t, \omega) : t > 0,\ X_t(\omega) \in F\}\,. \tag{4.2}$$

In conformity with the above convention, $Z_F = \{t > 0 : X_t(\omega) \in F\}$. Denote by $\bar{Z}_F$ the closure of Z_F in $(0, \zeta)$ and let $\bar{\mathbf{Z}}_F = \{(t, \omega) : t \in \bar{Z}_F(\omega)\}$. If F is finely closed, $\bar{Z}_F - Z_F$ is a.s. countable. Recall that for a standard process X, a stopping time T is accessible on $\{T < \zeta\}$ if and only if $X_T = X_{T-}$ a.s. on $\{T < \zeta\}$ ([7]).

(4.3) PROPOSITION. *Let D be a finely closed nearly Borel subset of E. Then D is projective if and only if*

(4.4) $\quad \bar{\mathbf{Z}}_D - \mathbf{Z}_{D^r}$ *contains the graph of no accessible stopping time.*

Also, D is a d-set if and only if

(4.5) $\quad \bar{\mathbf{Z}}_D - \mathbf{Z}_D$ *contains the graph of no accessible stopping time.*

PROOF. It is known ([2] V. (4.3)) that D is projective if and only if $T^t = t + T_D \circ \theta_t$ is q.l.c. in the sense that

(4.6) $$P^x\{\lim_n (T_n + T_D \circ \theta_{T_n}) \neq T + T_D \circ \theta_T, \lim_n T_n + T_D \circ \theta_{T_n} < \zeta\} = 0$$

for all $x \in E$ whenever $\{T_n\}$ is an increasing sequence of stopping times with limit T.

Assume that (4.4) holds. If $\{T_n\}$ is an increasing sequence of stopping times with limit T, define $R_n = T_n + T_D \circ \theta_{T_n} (\leqq T + T_D \circ \theta_T)$ and let $R = \lim_n R_n$. Certainly $T \leqq R \leqq T + T_D \circ \theta_T$. Since $X_t \notin D$ on $(T, T + T_D \circ \theta_T)$ and $X_{R_n} \in D$ on $\{R_n < \zeta\}$, we see that $R = T$ on $\Lambda = \{R < T + T_D \circ \theta_T, R < \zeta\}$. But $\Lambda = \{R = T\} \cap \{T < T + T_D \circ \theta_T; T < \zeta\} \in \mathscr{F}_T$ and $T_n < T$ for all n on Λ so T is accessible on Λ. However $R_n \uparrow T$ on Λ so $T \in \bar{Z}_D$. By (4.4), $T \in Z_{D^r}$ a.s. on Λ, hence $T = T + T_D \circ \theta_T$ on Λ, so Λ is null. By (4.6), D must be projective.

Assume now that D is projective. Suppose there exists a stopping time T which is accessible on $\{T < \zeta\}$ such that $T(\omega) \in \bar{Z}_D(\omega) - Z_{D^r}(\omega)$ for $\omega \in \Gamma$, where Γ is not null. There exists an initial measure μ, a subset Γ_0 of Γ with $P^\mu(\Gamma_0) > 0$, and an increasing sequence $\{T_n\}$ of stopping times such that $\lim_n T_n = T$ and $T_n < T$ for all n a.s. P^μ on Γ_0. Let $R_n = T_n + T_D \circ \theta_{T_n}$. Then $R_n \leqq T$ for all n a.s. P^μ on Γ_0 because $T \in \bar{Z}_D$. But $R_n \uparrow T + T_D \circ \theta_T$ a.s. P^μ on Γ_0 by (4.6), so $T_D \circ \theta_T = 0$. Thus $X_T \in D^r$ a.s. P^μ on Γ_0, contradicting $T \in \bar{Z}_D - Z_{D^r}$. Hence (4.4) holds.

Turning to the characterization of a d-set, assume firstly that D is a d-set. If T is accessible on $\{T < \zeta\}$ and $\Lambda = \{T \in \bar{Z}_D - Z_D\}$ is not null, choose μ, $\Lambda_0 \subset \Lambda$ and $\{T_n\}$ such that $P_\mu(\Lambda_0) > 0$, $\{T_n\}$ increases, $\lim_n T_n = T$ on Λ_0 and $T_n < T$ for all n on Λ_0. Since $X_T \notin D$ on Λ_0 but $\sup\{t < T: X_t \in D\} = T$ on Λ_0, we have $T_n \leqq R_n = T_n + T_D \circ \theta_{T_n} < T$ a.s. P^μ on Λ_0, for all n. But since D is a d-set, (4.1) says $X_T \in D$ a.s. on Λ_0. Thus Λ is null and (4.5) holds.

Finally, suppose (4.5) holds. If T_n increases to T then T is accessible on $\Gamma = \{X_{T_n} \in D$ for all n, $T_n < T < \zeta$ for all $n\}$. By (4.5), $T \notin \bar{Z}_D - Z_D$ a.s. on Γ, so $\{X_{T_n} \in D$ for all n, $X_T \notin D$, $T < \zeta\}$ is null, hence D is a d-set.

(4.7) COROLLARY. *If $D \in \mathscr{E}^n$ and D is finely closed, then if D is projective, D is a d-set.*

(4.8) COROLLARY. *If D and F are finely closed, nearly Borel, projective sets, then $D \cup F$ is projective.*

PROOF. $\bar{\mathbf{Z}}_{D \cup F} - \mathbf{Z}_{(D \cup F)^r} \subset (\bar{\mathbf{Z}}_D - \mathbf{Z}_{D^r}) \cup (\bar{\mathbf{Z}}_F - \mathbf{Z}_{F^r})$.

(4.9) COROLLARY. *If $D \in \mathscr{E}^n$ and if D is a finely perfect d-set, then D is projective.*

Finally, we give a characterization of projectivity which is similar to the characterization of d-sets given by (4.1).

(4.10) PROPOSITION. *Let $D \in \mathscr{E}^n$ and suppose D is finely closed. Then D is* *projective if and only if both of the following conditions are satisfied.*

(4.11) *whenever* $\{S_n\}$ *is an increasing sequence of stopping times with limit* S, $X_S \in D^r$ a.s. *on* $\{S_n < S$ *for all* n, $S < \zeta$, $X_{S_n} \in D$ *for all* $n\}$;

(4.12) $X_{T_D} \neq X_{T_D-}$ a.s. *on* $\{X_{T_D} \in D - D^r, T_D < \zeta\}$

PROOF. Suppose firstly that D is projective. If $S_n \uparrow S$, then S is accessible on $\Lambda = \{S_n < S$ for all n, $S < \zeta$, $X_{S_n} \in D$ for all $n\}$. By (4.4), since $S \in \bar{Z}_D$ on Λ, $S \in Z_{D^r}$ a.s. on Λ, so (4.11) holds. To obtain (4.12), notice that on $\{X_{T_D} \in D - D^r, X_{T_D} = X_{T_D-}, T_D < \zeta\} = \Gamma$, T_D is accessible and $T_D \in \bar{Z}_D - Z_{D^r}$ so Γ is null.

Suppose now that D satisfies (4.11) and (4.12). Assume that D is not projective. Then there exists an accessible S such that $[S] \subset \bar{\mathbf{Z}}_D - \mathbf{Z}_{D^r}$ with positive probability. Choose an initial measure μ, a set $\Gamma \subset \{S \in \bar{Z}_D - Z_{D^r}\}$ with $P^\mu(\Gamma) > 0$ and $\{S_n\}$ such that $\lim S_n = S$ and $S_n < S$ for all n on Γ. Let $L_0 = \Gamma \cap \{\sup[t < S\colon X_t \in D] = S\}$ and $\Gamma_1 = \Gamma - \Gamma_0$. Almost surely P^μ on Γ_0, $R_n = S_n + T_D \circ \theta_{S_n} < S$ and $\lim_n R_n = S$, hence by (4.11), $X_S \in D^r$ on Γ_0. This contradicts the assumption $S \in \bar{Z}_D - Z_{D^r}$ on Γ, so $P^\mu(\Gamma_0) = 0$. However, a.s. P^μ on Γ_1, $R_n = S_n + T_D \circ \theta_{S_n} = S$ for sufficiently large n, whereas (4.12) implies that $X_{R_n} \neq X_{R_n-}$ a.s. on $\{X_{R_n} \in D - D^r\}$, and so $X_S \neq X_{S-}$ a.s. on $\{X_S \in D - D^r\}$. This contradicts the assumed accessibility of S on Γ so $P^\mu(\Gamma_1) = 0$. It follows that D must be projective.

5. A last exit decomposition. In this section we apply some of the preceding techniques to obtain a "last exit" decomposition. Our result contains the recent work of Pittenger [14], but our methods are quite different from his.

We begin by summarizing some results on local times. See [2]. Let b be a fixed point in E that is regular for itself. Let A be a fixed version of the local time at b and let τ be the right continuous inverse of A. Then τ_t is a subordinator relative to P^b (τ may jump to infinity), and $E^b(e^{-\alpha\tau(t)}) = e^{-tg(\alpha)}$ where

(5.1) $$g(\alpha) = \gamma\alpha + \int (1 - e^{-\alpha u})\nu(du)\,.$$

In (5.1), $\gamma \geqq 0$ and ν is a measure on $(0, \infty]$ such that $\int (u \wedge 1)\nu(du) < \infty$. The possible mass of ν at infinity corresponds to the possible jump of τ to infinity. It is easy to check the standard fact that $\gamma = \lim_{\alpha\to\infty} g(\alpha)/\alpha$. Moreover

(5.2) $$E^b \int_0^\infty e^{-\alpha t}\, dA_t = E^b \int_0^\infty e^{-\alpha\tau(t)}\, dt = g(\alpha)^{-1}\,.$$

The following lemma contains a well-known fact about subordinators. Unfortunately we do not know a precise reference for it in the literature. Therefore we shall give a proof in the spirit of this paper that we hope will amuse the reader.

(5.3) LEMMA. *Let* $\tau(t)$ *be a strictly increasing subordinator on a probability space* $(\Omega, \mathscr{F}, P)$ *with* $\tau_0 = 0$ *and* $E(e^{-\alpha\tau(t)}) = e^{-tg(\alpha)}$ *where* g *is given by* (5.1). *Let* A_t *be* *the (continuous) inverse of τ and let Q denote the range of τ. Then almost surely*

$$\int_0^t I_Q(s)\, ds = \gamma A_t \tag{5.4}$$

for all t. Of course, the integral on the left in (5.4) is just the Lebesgue measure of $Q \cap [0, t]$.

PROOF. Throughout the argument ω is fixed subject only to the requirements that $t \to \tau_t(\omega)$ is right continuous, strictly increasing, vanishes at $t = 0$ and for $t > 0$

$$\tau_t(\omega) = \gamma t + \sum_{s \leq t} [\tau_s(\omega) - \tau_{s-}(\omega)]\,. \tag{5.5}$$

We now suppress ω in our notation. Let K be the set of continuity points of $t \to \tau_t$. Then the complement of K in $[0, \infty)$ is countable and one easily checks that $\{t\colon A_t \in K\} \subset Q$ and that the difference of these two sets is countable. Bearing these facts in mind and using (2.4) we have

$$\begin{aligned}\alpha \int_0^\infty e^{-\alpha t} I_Q(t)\, dt &= -\int_0^\infty I_Q(t)\, de^{-\alpha t} \\ &= -\int_0^\infty I_K(A_t)\, de^{-\alpha t} = -\int_0^\infty I_K(t)\, de^{-\alpha \tau_t}\,.\end{aligned}$$

But from (5.5) we obtain

$$-de^{-\alpha\tau_t} = \alpha\gamma e^{-\alpha\tau_t}\, dt + dJ(t)$$

where dJ is a purely discrete measure putting all of its mass on the set of discontinuities of $t \to \tau_t$. See [11], for example. Consequently

$$\int_0^\infty e^{-\alpha t} I_Q(t)\, dt = \gamma \int_0^\infty e^{-\alpha\tau_t}\, dt = \gamma \int_0^\infty e^{-\alpha t}\, dA_t\,,$$

and inverting Laplace transforms this yields (5.4) since both sides of (5.4) are continuous in t and vanish at $t = 0$.

If τ is the subordinator corresponding to the local time at b as in the discussion above Lemma 5.3, then Q differs from $\{t\colon X_t = b\}$ by a countable set and so (5.4) yields almost surely P^b

$$\int_0^t I_b(X_s)\, ds = \gamma A_t\,. \tag{5.6}$$

where I_b is the indicator of $\{b\}$. Taking α-potentials we obtain

$$E^b \int_0^\infty e^{-\alpha t} I_b(X_t)\, dt = \gamma / g(\alpha)\,. \tag{5.7}$$

In particular, it follows that $\gamma > 0$ if and only if $\{b\}$ has positive potential. But then (5.6) holds almost surely P^x for all x. This is clear if $\gamma = 0$. If $\gamma > 0$ the left side of (5.6) is a nonzero CAF of X with fine support $\{b\}$ and so is a multiple of A. Hence (5.6) must hold.

REMARK. One consequence of (5.6) is that if $\gamma > 0$, then $t \to E^x(A_t)$ is absolutely continuous with bounded density, $\gamma^{-1}P^x[X_t = b]$.

We now come to the main business of this section. Let F be a fixed finite subset of E such that each point $b \in F$ is regular for itself. Let A^b be a fixed version of the local time at b for each $b \in F$. If $h \in b\mathscr{E}_+$ (the bounded nonnegative

Borel functions on E), define

$$B_t = B_t(h) = \int_0^t h(X_s)\, ds .$$

Clearly F is projective. Therefore if for each $\alpha > 0$, $B^\alpha = B^\alpha(h)$ denotes the α-balayage of B on F, then B^α is a CAF with fine support contained in F and with $u^\alpha_{B^\alpha} = P_F{}^\alpha U^\alpha h$. Consequently for each $\alpha > 0$

$$B^\alpha(h) = \sum_{a\in F} C_a{}^\alpha(h) A^a \tag{5.8}$$

for appropriate constants $C_a{}^\alpha(h)$. Let us now fix a point $b \in F$. Applying Azema's formula (2.13) with $f = I_b$, we obtain for each $u > 0$

$$\begin{aligned} J(x, \alpha) &= \int_0^\infty E^x\{h(X_t);\, X_{L^t-} = b;\, 0 < L^t \leqq u\} e^{-\alpha t}\, dt \\ &= E^x \int_0^u e^{-\alpha t} I_b(X_t)\, dB_t{}^\alpha \\ &= C_b{}^\alpha(h) E^x \int_0^u e^{-\alpha t}\, dA_t{}^b , \end{aligned} \tag{5.9}$$

since (5.8) implies that $I_b(X_t)\, dB_t{}^\alpha = C_b{}^\alpha(h)\, dA_t{}^b$. Of course, in (5.9), $L^t = \sup\{s \leqq t\colon X_s \in F\}$—the last exit from F before t.

In order to evaluate the coefficient $C_b{}^\alpha(h)$ we set $G = F - \{b\}$ and let (W^α) be the resolvent of the process (X, T_G), that is

$$W^\alpha f(x) = E^x \int_0^{T_G} e^{-\alpha t} f(X_t)\, dt .$$

Now $t \to B_{t\wedge T_G}$ is a CAF of (X, T_G) and by (2.15) its α-balayage on F is just

$$B^\alpha_{t\wedge T_G} = \sum_{a\in F} C_a{}^\alpha(h) A^a_{t\wedge T_G} = C_b{}^\alpha(h) A^b_{t\wedge T_G} \tag{5.10}$$

because $A^a_{T_G} = 0$ for all $a \in G = F - \{b\}$. Since $t \to B^\alpha_{t\wedge T_G}$ is the α-balayage of $t \to B_{t\wedge T_G}$ on F relative to the process (X, T_G) its α-potential is given by $E^x\{e^{-\alpha T_F} W^\alpha h(X_{T_F});\, T_F < T_G\}$, and so taking α-potentials in (5.10) we find

$$W^\alpha h(b) = C^\alpha(h) E^b \int_0^\infty e^{-\alpha t}\, dA_{t\wedge T_G} \tag{5.11}$$

where we have written $C^\alpha(h) = C_b{}^\alpha(h)$ and $A = A^b$. (We shall continue to suppress these b's for the next several paragraphs). But $t \to A_{t\wedge T_G}$ is a local time at b for the process (X, T_G) and so its inverse τ_t is a subordinator relative to P^b. If $g(\alpha) = g_b(\alpha)$ is the subordinator exponent of (τ, P^b), then (5.11) becomes

$$C^\alpha(h) = g(\alpha) W^\alpha h(b) . \tag{5.12}$$

It is clear from (5.12) that $C^\alpha(h)$ is a measure in h which is carried by $E - G$. If c^α is the restriction of C^α to $E - F$, that is, for any $h \in b\mathscr{E}_+$

$$c^\alpha(h) = g(\alpha) W^\alpha(hI_{E-F})(b) , \tag{5.13}$$

then $C^\alpha(h) = c^\alpha(h) + g(\alpha)h(b) W^\alpha I_b(b)$. Now applying (5.7) to the process (X, T_G) we see that $W^\alpha I_b(b) = \gamma/g(\alpha)$, and so

$$C^\alpha(h) = \gamma h(b) + c^\alpha(h) \tag{5.14}$$

where γ is a nonnegative constant depending only on b and c^α is the finite measure on $E - F$ given by (5.13).

We next claim that there exists a unique entrance law $\eta_t(\cdot) = \eta_t^b(\cdot)$ for the process (X, T_F) such that

$$c^\alpha(h) = \int_0^\infty e^{-\alpha t}\eta_t(h)\, dt\,. \tag{5.15}$$

We refer the reader to Section 6 for the definition and basic properties of an entrance law. However, we point out that for each $t > 0$, $\eta_t(\cdot)$ is a finite measure on $E - F$ and that $t \to \eta_t(h)$ is right continuous if h is bounded and continuous. In light of Theorem 6.9, proved in the next section, the existence of (η_t) satisfying (5.15) will follow once we establish the following two properties of c^α:

$$\lim_{\alpha\to\infty} c^\alpha(1) = 0\,; \tag{5.16}$$

$$c^\alpha(h) - c^\beta(h) = (\beta - \alpha)c^\alpha(V^\beta h)\,, \tag{5.17}$$

where V^β is the resolvent of (X, T_F). Since $W^\alpha 1 = W^\alpha I_b + W^\alpha I_{E-F}$, $\alpha W^\alpha 1(b) \to 1$ as $\alpha \to \infty$, and $g(\alpha)/\alpha \to \gamma$ as $\alpha \to \infty$, (5.16) is clear if $\gamma = 0$. If $\gamma > 0$, (5.7) implies that $\alpha W^\alpha I_b(b) \to 1$ as $\alpha \to \infty$ and so (5.16) holds in this case also.

We come finally to the verification of (5.17). We may assume that $h \in b\mathscr{E}_+$ and vanishes on F. Writing $T_F(t) = T_F{}^t = t + T_F \circ \theta_t$ for typographical convenience and noting that $t \leqq T_F(t) \leqq T_G$ if $t < T_G$, we have for $\beta \neq \alpha$

$$\begin{aligned} W^\alpha V^\beta h(b) &= E^b \int_0^{T_G} e^{-\alpha t} E^{X(t)} \int_0^{T_F} e^{-\beta s} h(X_s)\, ds\, dt \\ &= E^b \int_0^{T_G} e^{(\beta-\alpha)t} \int_t^{T_F(t)} e^{-\beta s} h(X_s)\, ds\, dt \\ &= E^b \int_0^{T_G} e^{(\beta-\alpha)t} \int_t^{T_G} e^{-\beta s} h(X_s)\, ds\, dt \\ &\qquad - E^b \int_0^{T_G} e^{(\beta-\alpha)t} \int_{T_F(t)}^{T_G} e^{-\beta s} h(X_s)\, ds\, dt \\ &= J_1 - J_2\,. \end{aligned}$$

But

$$\begin{aligned} J_1 &= E^b \int_0^{T_G} e^{-\beta s} h(X_s) \int_0^s e^{(\beta-\alpha)t}\, dt\, ds \\ &= (\beta - \alpha)^{-1}[W^\alpha h(b) - W^\beta h(b)]\,. \end{aligned}$$

Since $T_G = T_F(t) + T_G \circ \theta_{T_F(t)}$ if $t < T_G$ and $\{t < T_G\} \in \mathscr{F}_t \subset \mathscr{F}_{T_F(t)}$ we have

$$\begin{aligned} J_2 &= E^b \int_0^{T_G} e^{(\beta-\alpha)t} e^{-\beta T_F(t)} E^{X(T_F(t))} \int_0^{T_G} e^{-\beta s} h(X_s)\, ds\, dt \\ &= W^\beta h(b) E^b \int_0^{T_G} e^{(\beta-\alpha)t} e^{-\beta T_F(t)} I_{\{X(T_F(t))=b\}}\, dt \end{aligned}$$

where we have used the facts that $X(T_F(t)) \in F$ and $W^\alpha h(x) = 0$ for all $x \in G = F - \{b\}$. Let $L = \sup\{t < T_G;\, X_t = b\}$ and $T = T_{\{b\}}$. Then on $\{t < T_G\}$ one has $\{X(T_F(t)) = b\} = \{t < L\}$, and on $\{t < L\}$, $T_F(t) = T(t) = t + T \circ \theta_t$. Therefore $J_2 = \rho(\alpha, \beta) W^\beta h(b)$ where

$$\rho(\alpha, \beta) = E^b \int_0^L e^{(\beta-\alpha)t} e^{-\beta T(t)}\, dt\,.$$

Let $a_t = A_{t\wedge T_G}$ so that a_t is a local time at b for (X, T_G) and τ_t is the (right continuous) inverse of a. Observe that $\tau(a_t) = T(t)$ if $t < L$ and that $\tau(a_t) = \infty$ if $t \geqq L$. Therefore using (2.4),

$$\begin{aligned} (\beta - \alpha)\rho(\alpha, \beta) &= (\beta - \alpha) E^b \int_0^\infty e^{(\beta-\alpha)t} e^{-\beta\tau(a_t)}\, dt \\ &= E^b \int_0^\infty e^{-\beta\tau(a_t)}\, de^{(\beta-\alpha)t} \\ &= E^b \int_0^\infty e^{-\beta\tau(t)}\, de^{(\beta-\alpha)\tau(t)} \\ &= -1 - E^b \int_0^\infty e^{(\beta-\alpha)\tau(t-)}\, de^{-\beta\tau(t)}\,. \end{aligned}$$

But

$$E^b\{e^{-\beta\tau(t+s)} - e^{-\beta\tau(s)} \mid \mathscr{F}_s\} = e^{-\beta\tau(s)}[e^{-tg(\beta)} - 1]$$
$$= -E^b\{\textstyle\int_s^{s+t} g(\beta)e^{-\beta\tau(u)}\,du \mid \mathscr{F}_s\},$$

and so by Meyer's integration lemma, (VII T17 of [7]),

$$(\beta - \alpha)\rho(\alpha, \beta) = -1 + g(\beta)E^b \textstyle\int_0^\infty e^{(\beta-\alpha)\tau(t-)}e^{-\beta\tau(t)}\,dt$$
$$= -1 + \frac{g(\beta)}{g(\alpha)}.$$

Combining this with the previously calculated values of J_1 and J_2 yields

$$(\beta - \alpha)W^\alpha V^\beta h(b) = W^\alpha h(b) - \frac{g(\beta)}{g(\alpha)} W^\beta h(b),$$

and multiplying by $g(\alpha)$ we obtain (5.17).

For each $x \in E$ and $b \in F$ let

$$(5.18) \qquad m_b^x(t) = E^x(A_t^b); \qquad m_b(t) = m_b^b(t)$$

where A^b is our fixed version of the local time at b (for the full process X). Each m_b^x is continuous, increasing, and vanishes at $t = 0$. Combining (5.15), (5.14), (5.9), and the remark following (5.7), and inverting the resulting Laplace transform we obtain for each fixed $u > 0$

$$(5.19) \qquad E^x\{h(X_t);\, X_{L^t-} = b;\, 0 < L^t \leqq u\}$$
$$= h(b)I_{[0,u]}(t)P^x[X_t = b] + \textstyle\int_0^{t\wedge u} \eta_{t-s}^b(h)\,dm_b^x(s)$$

almost everywhere (Lebesgue) in t. Suppose that h is a bounded nonnegative continuous function and fix $u > 0$. We next show that (5.19) holds identically in $t > u$. Firstly if $t > u$ the right side of (5.19) reduces to $\int_0^u \eta_{t-s}^b(h)\,dm_b^x(s)$ which is clearly right continuous in t on (u, ∞). (Recall that $s \to \eta_s^b(h)$ is right continuous and bounded on $[\varepsilon, \infty)$ for each $\varepsilon > 0$.) On the other hand from the properties of F it follows that if $L^t \leqq u < t$, then $X_s \notin F$ on $[t, t + \delta)$ for some $\delta > 0$. (We omit the qualifying phrase "almost surely" in such statements.) Consequently $L^s = L^t$ for $t \leqq s < t + \delta$, and so the left side of (5.19) is right continuous in t on (u, ∞). Therefore for each fixed u, t with $0 < u < t$ we have

$$E^x\{h(X_t);\, X_{L^t-} = b;\, 0 < L^t \leqq u\} = \textstyle\int_0^u \eta_{t-s}^b(h)\,dm_b^x(s).$$

If we fix $t > 0$ and let $u \uparrow t$ in this formula we obtain

$$(5.20) \qquad E^x\{h(X_t);\, X_{L^t-} = b;\, 0 < L^t < t\} = \textstyle\int_0^t \eta_{t-s}^b(h)\,dm_b^x(s)$$

identically for $t > 0$. But both sides of (5.20) are measures in h, and so (5.20) must hold for all $h \in b\mathscr{E}$ and $t > 0$. Note that for a fixed $t > 0$, $X_t = X_{t-}$ if $t < \zeta$ and if $X_t = b$ then $L^t = t$. Therefore for each $t > 0$

$$E^x\{h(X_t);\, X_{L^t-} = b;\, 0 < L^t = t\} = h(b)P^x[X_t = b].$$

Combining this with (5.20) and noting that $L^t \leqq t$ we see that (5.19) holds for all positive values of t and u.

If we now sum (5.19) with $u = t$ for $b \in F$ and observe that $X_{L^t-} \in F$ when $L^t > 0$ and that $\{L^t = 0\} = \{T_F < t\}$, we obtain the "last exit" decomposition

$$(5.21) \qquad E^x[h(X_t)] = E^x[h(X_t); t < T_F\} + E^x\{h(X_t); X_t \in F\} + \textstyle\sum_{b \in F} \int_0^t \eta_{t-s}^b(h)\, dm_b^x(s)$$

which holds for all $t > 0$ and $h \in b\mathscr{E}$. If we define for $b \in F$ and $s \geqq 0$

$$f_s(x, b) = P^x[X_{T_F} = b, T_F > s]\,,$$

then it is easy to check that for each $b \in F$, $(s, x) \to f_s(x, b)$ is an exit law for (Q_t), that is, $Q_t f_s(\cdot, b) = f_{t+s}(\cdot, b)$. Moreover, using the strong Markov property, (I-8.16) of [2], one finds for each $b \in F$

$$(5.22) \qquad m_b^x(s) = -\textstyle\sum_{a \in F} \int_0^s m_b^a(s-u) d_u f_u(x, a)\,.$$

Combining this with (5.21) we obtain the first entrance—last exit decomposition

$$(5.23) \qquad P_t h(x) = Q_t h(x) + \textstyle\sum_{b \in F} P_t(x, \{b\}) h(b) - \textstyle\sum_{a,b \in F} \int_0^t d_u f_u(x, a) \int_u^t \eta_{t-s}^b(h)\, d_s m_b^a(s-u)\,,$$

valid for $t > 0$ and $h \in b\mathscr{E}$; a formula which has a transparent probabilistic interpretation. Finally if E is countable, it follows from (II-12-Th. 4) of [4] that $u \to f_u(x, a)$ has a continuous derivative, and then, from (5.22), so does $s \to m_b^x(s)$.

6. **Entrance laws.** In this section we shall prove the result characterizing the Laplace transform of an entrance law that was used in Section 5. Since this result is purely analytic in nature and of some independent interest, we shall formulate it as a theorem about semigroups of kernels and give a proof that is independent of the preceding sections of this paper. Thus this section may be read without reference to the earlier portions of this paper.

We fix a locally compact space E with a countable base and, as usual, let $\mathscr{E}(\mathscr{E}^*)$ denote the σ-algebra of Borel (universally measurable) subsets of E. We let $\mathbf{C}_K$, $\mathbf{C}_0$ and $\mathbf{B}$ denote respectively, the spaces of continuous functions with compact support, continuous functions vanishing at infinity, and bounded universally measurable functions. All functions are real valued and $\|\cdot\|$ denotes the supremum norm. Let $\{P_t; t \geqq 0\}$ be a semigroup of sub-Markovian kernels on the measure space $(E, \mathscr{E}^*)$. That is for each $t \geqq 0$, P_t is a linear positive map from $\mathbf{B}$ to $\mathbf{B}$ with $P_t 1 \leqq 1$ such that for each $x \in E$, $f \to P_t f(x)$ is measure denoted by $P_t(x, dy)$, and satisfying $P_{t+s} = P_t P_s$ for all $t, s \geqq 0$. *We do not assume that P_t maps Borel functions into Borel functions nor do we assume that $P_0 = I$.* We make the following regularity assumption on (P_t):

(6.1) For each $f \in \mathbf{C}_K$ and $x \in E$, $t \to P_t f(x)$ is right continuous on $[0, \infty)$.

Clearly (6.1) then holds for any $f \in \mathbf{C}_0$. We assume that $\{P_t\colon t \geqq 0\}$ is a given semigroup of kernels satisfying (6.1) throughout the remainder of this section.

(6.2) DEFINITION. A family $\{\eta_t; t > 0\}$ of finite measures on $(E, \mathscr{E}^*)$ is an entrance law (for (P_t)) provided

$$\eta_t P_s = \eta_{t+s} \tag{6.3}$$

for all $t > 0$ and $s \geqq 0$.

It follows from (6.1) and (6.3) that $t \to \eta_t(f)$ is right continuous on $(0, \infty)$ for each $f \in \mathbf{C}_0$. Since $P_t 1 \leqq 1$, (6.3) implies that $t \to \eta_t(1)$ is decreasing on $(0, \infty)$. Consequently, since it is the increasing limit of a sequence of right continuous functions, it must be right continuous. Note that $\lim_{t\downarrow 0} \eta_t(1)$ exists but may be infinite. However, if (P_t) is Markovian, that is, $P_t 1 = 1$ for all t, then (6.3) implies that $t \to \eta_t(1)$ is constant. It is a standard argument to show that $t \to \eta_t(f)$ is universally measurable on $(0, \infty)$ for each $f \in \mathbf{B}$. We say that an entrance law (η_t) is *locally integrable* provided

$$\textstyle\int_0^1 \eta_t(1)\, dt < \infty\ . \tag{6.4}$$

For each $\alpha > 0$ we define

$$U^\alpha f(x) = \textstyle\int_0^\infty e^{-\alpha t} P_t f(x)\, dt \tag{6.5}$$

for $f \in \mathbf{B}$. Then $\{U^\alpha; \alpha > 0\}$ is the resolvent of (P_t). If $f \in \mathbf{B}^+$ we define $U = U^0$ by (6.5) with $\alpha = 0$. Of course, one needs to show that there is enough joint measurability in $(t, x) \to P_t f(x)$ so that (6.5) is defined, $U^\alpha\colon \mathbf{B} \to \mathbf{B}$ for $\alpha > 0$, and that one can use Fubini's theorem without fear. These are standard facts—see, for example, the discussion on page 114 of [2]. It is evident that for each $\alpha \geqq 0$, U^α is a kernel on $(E, \mathscr{E}^*)$ which we denote by $U^\alpha(x, dy)$.

Suppose that (η_t) is a locally integrable entrance law. Then for each $\alpha > 0$

$$c^\alpha(f) = \textstyle\int_0^\infty e^{-\alpha t} \eta_t(f)\, dt \tag{6.6}$$

defines a finite measure on $(E, \mathscr{E}^*)$. It is immediate from the dominated convergence theorem that

$$\lim_{\alpha\to\infty} c^\alpha(1) = 0\ . \tag{6.7}$$

Moreover, a straightforward computation shows that for $f \in \mathbf{B}$ and $\alpha, \beta > 0$

$$c^\alpha(f) - c^\beta(f) = (\beta - \alpha) c^\alpha(U^\beta f)\ . \tag{6.8}$$

The object of this section is to prove that (6.7) and (6.8) characterize the Laplace transform of a locally integrable entrance law.

(6.9) THEOREM. *Suppose that $\{c^\alpha; \alpha > 0\}$ is a family of finite measures satisfying* (6.7) *and* (6.8). *Then there exists a unique locally integrable entrance law (η_t) such that* (6.6) *holds for all $f \in \mathbf{B}$.*

PROOF. First observe that (6.8) implies that $\alpha \to c^\alpha(1)$ is decreasing and continuous on $(0, \infty)$. Hence $\lim_{\alpha\downarrow 0} c^\alpha(1)$ exists but may be infinite. We first prove Theorem 6.9 under the additional assumption that $\lim_{\alpha\downarrow 0} c^\alpha(1) < \infty$. This will represent the main work of the proof, the reduction of the general situation to this special case being simple. Thus until further notice this special assumption is in force.

If $f \in \mathbf{B}^+$ a straightforward induction argument using (6.8) and the resolvent equation shows that

$$\frac{d^n}{d\alpha^n}[c^\alpha(f)] = (-1)^n n!\, c^\alpha[(U^\alpha)^n f]\,. \tag{6.10}$$

Consequenly $\alpha \to c^\alpha(f)$ is completely monotonic, and hence by the Bernstein–Hausdorff–Widder theorem there exists a unique increasing right continuous function $B_t(f)$ on $[0, \infty)$ such that $B_0(f) = 0$ and

$$c^\alpha(f) = \int_0^\infty e^{-\alpha t}\, dB_t(f) + b(f) \tag{6.11}$$

where $b(f) \geqq 0$. Since $c^\alpha(f) \leqq ||f||c^\alpha(1)$, it is immediate from (6.7) that $b(f) = 0$. Also $\lim_{\alpha\downarrow 0} c^\alpha(1) < \infty$ implies that $B_\infty(f) < \infty$. If $f \in \mathbf{B}$, writing $f = f^+ - f^-$, we obtain a (unique) right continuous function $B_t(f)$ of bounded variation on $[0, \infty)$ with $B_0(f) = 0$ such that

$$c^\alpha(f) = \int_0^\infty e^{-\alpha t}\, dB_t(f)\,. \tag{6.12}$$

It follows from the uniqueness theorem for Laplace transforms that for each t, B_t is a positive, linear, bounded operator on $\mathbf{B}$. Moreover, if (f_n) is a sequence in $\mathbf{B}^+$ decreasing to zero, then for each $\alpha > 0$, $c^\alpha(f_n) \downarrow 0$. But then by the continuity theorem for Laplace transforms $dB_t(f_n) \to 0$ weakly as measures on $[0, \infty)$, and hence $B_t(f_n) \downarrow 0$ for each t. Thus B_t is a measure for each $t \geqq 0$.

Another straightforward induction argument starting from (6.10) shows that

$$\frac{d^n}{d\alpha^n}[\alpha c^\alpha(1)] = (-1)^{n+1} n!\, c^\alpha[(U^\alpha)^n(1 - \alpha U^\alpha 1)]\,.$$

Since $\alpha U^\alpha 1 \leqq 1$ this implies that $\alpha c^\alpha(1)$ has a completely monotonic derivative. (In the terminology of [3], $\alpha \to \alpha c^\alpha(1)$ is a completely monotonic mapping.) As a result

$$\alpha c^\alpha(1) = b_1 + b_2\alpha + \int_0^\infty (1 - e^{-\alpha t})\nu(dt)$$

where b_1 and b_2 are nonnegative constants and ν is a measure on $(0, \infty)$ satisfying $\int_0^\infty (t \wedge 1)\nu(dt) < \infty$. Since $\alpha c^\alpha(1) \to 0$ as $\alpha \to 0$, $b_1 = 0$, and then (6.7) implies that $b_2 = 0$. Thus if $a_t = \nu((t, \infty))$ for $t > 0$, an integration by parts yields

$$c^\alpha(1) = \int_0^\infty e^{-\alpha t} a_t\, dt\,.$$

Comparing this with (6.12) we see that

$$B_t(1) = \int_0^t a_s\, ds$$

where a is right continuous, decreasing, and $\lim_{t\to\infty} a_t = 0$. If $0 \leqq f \leqq 1$, then the relationship $dB_t(f) + dB_t(1 - f) = dB_t(1) = a_t\, dt$ implies that $dB_t(f)$ is absolutely continuous. Because $B_t(||f||) = ||f||B_t(1)$, for each $f \in \mathbf{B}$, $dB_t(f)$ is absolutely continuous and we may choose a fixed version $b_t(f)$ of its derivative such that for all $t > 0$, $b_t(f) = b_t(f^+) - b_t(f^-)$ and such that $0 \leqq b_t(f) \leqq ||f||a_t$ if $f \in \mathbf{B}^+$. Thus for $f \in \mathbf{B}$, $|b_t(f)| \leqq (\max(||f^+||, ||f^-||))a_t = ||f||a_t$.

We next "regularize" the densities $b_t(f)$. To this end let $(f_n)_{n\geqq 1}$ be a countable *linearly independent* set of elements of norm one in $\mathbf{C}_0$ such that the vector space $\mathbf{H}$ generated by (f_n) is dense in $\mathbf{C}_0$. For each n let $b_t(f_n)$ be the density for $dB_t(f_n)$ chosen at the end of the last paragraph. For each $t > 0$ extend b_t by linearity to all of $\mathbf{H}$. We denote this extension by e_t in order to distinguish it from b_t chosen above. If $f \in \mathbf{H}$, then by the linearity of c^α and e_t

$$c^\alpha(f) = \int_0^\infty e^{-\alpha t} e_t(f)\, dt = \int_0^\infty e^{-\alpha t} b_t(f)\, dt ,$$

and so $e_t(f) = b_t(f)$ almost everywhere (Lebesgue) in t. In particular, $|e_t(f)| \leqq \|f\| a_t$ almost everywhere. Hence there exists $N_0 \subset (0, \infty)$ of Lebesgue measure zero such that if $\mathbf{H}_r$ denotes the vector space over the rationals generated by (f_n), then $|e_t(f)| \leqq \|f\| a_t$ for all $t \notin N_0$ and all $f \in \mathbf{H}_r$. If $f \in \mathbf{H}$, say $f = \lambda_1 f_1 + \cdots + \lambda_n f_n$, choose rationals (r_k^m) such that $r_k^m \to \lambda_k$ as $m \to \infty$ for $1 \leqq k \leqq n$. Then if $f^m = \sum r_k^m f_k$, $\|f^m - f\| \to 0$ and

$$\begin{aligned} e_t(f) &= \sum \lambda_k e_t(f_k) = \lim_m \sum r_k^m e_t(f_k) \\ &= \lim_m e_t(f^m) . \end{aligned}$$

Thus if $t \notin N_0$

$$\begin{aligned} |e_t(f)| &= \lim_m |e_t(f^m)| \\ &\leqq a_t \lim_m \|f^m\| = a_t \|f\| . \end{aligned}$$

Consequently for each $t \notin N_0$, e_t is a bounded, positive, linear functional on $\mathbf{H}$ and hence may be extended to $\mathbf{C}_0$ by continuity preserving these properties. But then e_t is a measure and so may be extended as a measure to $\mathbf{B}$ for all $t \notin N_0$. Since

$$c^\alpha(f) = \int_0^\infty e^{-\alpha t} e_t(f)\, dt \tag{6.13}$$

for all $f \in \mathbf{H}$ and hence $\mathbf{C}_0$, and since both sides are measures in f, if follows that (6.13) holds for all $f \in \mathbf{B}$.

If $f \in \mathbf{C}_0^+$ and $\beta \neq \alpha$, β, $\alpha > 0$, then on the one hand

$$c^\alpha U^\beta f = \int_0^\infty e^{-\beta t} c^\alpha P_t f\, dt ,$$

while from (6.8) and (6.12)

$$\begin{aligned} c^\alpha U^\beta f &= (\beta - \alpha)^{-1}[c^\alpha(f) - c^\beta(f)] \\ &= (\beta - \alpha)^{-1} \int_0^\infty (e^{-\alpha t} - e^{-\beta t})\, dB_t(f) \\ &= \int_0^\infty e^{-\alpha t} \int_0^t e^{-(\beta-\alpha)s}\, ds\, dB_t(f) \\ &= \int_0^\infty e^{-(\beta-\alpha)s} \int_s^\infty e^{-\alpha t}\, dB_t(f)\, ds . \end{aligned}$$

Since $t \to c^\alpha P_t f$ is right continuous if $f \in \mathbf{C}_0^+$, it follows that

$$c^\alpha P_t f = e^{\alpha t} \int_t^\infty e^{-\alpha s}\, dB_s(f)$$

for all $t \geqq 0$ and $f \in \mathbf{C}_0$. Here, of course, $\int_t^\infty = \int_{(t,\infty)}$. From (6.12) and (6.13), $dB_s(f) = e_s(f)\, ds$ and using this in the above formula

$$c^\alpha P_t f = \int_0^\infty e^{-\alpha s} e_{t+s}(f)\, ds .$$

But from (6.13)

$$c^\alpha P_t f = \int_0^\infty e^{-\alpha s} e_s(P_t f)\, ds\,,$$

and so for each $t \geqq 0$ and $f \in \mathbf{C}_0$, $e_{t+s}(f) = e_s P_t f$ almost everywhere in s. By Fubini's theorem we can find a set $M_0 \subset (0, \infty)$ of Lebesgue measure zero containing N_0 such that for all $f \in \mathbf{H}_r$ and $s \notin M_0$, $e_{t+s}(f) = e_s(P_t f)$ almost everywhere in t. Let (s_n) be a sequence decreasing to zero with $s_n \notin M_0$ for each n. Each e_{s_n} is a finite measure and we define $\eta_t^n(f) = e_{s_n}(P_{t-s_n} f)$ for $f \in \mathbf{B}$ and $t \geqq s_n$. Clearly η_t^n is a finite measure and for $s \geqq 0$ and $f \in \mathbf{B}$

$$(6.14) \qquad \eta_t^n(P_s f) = e_{s_n}(P_{t+s-s_n} f) = \eta_{t+s}^n(f)\,.$$

Therefore $t \to \eta_t^n(f)$ is right continuous on $[s_n, \infty)$ if $f \in \mathbf{C}_0$. Next if $f \in \mathbf{H}_r$, $\eta_t^n(f) = e_t(f)$ almost everywhere on $[s_n, \infty)$. Consequently for $f \in \mathbf{H}_r$, $\eta_t^{n+1}(f) = \eta_t^n(f)$ almost everywhere, and hence everywhere, on $[s_n, \infty)$. But $\mathbf{H}_r$ is dense in $\mathbf{C}_0$ and each η_t^n is a measure, and therefore $\eta_t^{n+1}(f) = \eta_t^n(f)$ for all $t \geqq s_n$ and $f \in \mathbf{B}$. Hence for each $t > 0$, $\eta_t(t) = \lim_n \eta_t^n(f)$ is a well-defined finite measure, and passing to the limit in (6.14) it is evident that (η_t) is an entrance law. Finally for $f \in \mathbf{H}_r$, $\eta_t(f) = e_t(f)$ almost everywhere and so from (6.13)

$$c^\alpha(f) = \int_0^\infty e^{-\alpha t} \eta_t(f)\, dt\,,$$

and once again, since both sides are measures in f, this then holds for all $f \in \mathbf{B}$. This completes the proof of Theorem 6.9 under the assumption that $\lim_{\alpha \to 0} c^\alpha(1) < \infty$.

For the general case, fix $\beta > 0$ for the moment. Define $c_\beta^\alpha(f) = c^{\beta+\alpha}(f)$ for $\alpha > 0$, $P_t^\beta = e^{-\beta t} P_t$, and $V^\alpha = U^{\beta+\alpha}$. Then (V^α) is the resolvent of the semigroup (P_t^β), and it is immediate that (c_β^α) satisfies (6.8) with respect to (V^α). Obviously (6.7) holds for (c_β^α) and $\lim_{\alpha \to 0} c_\beta^\alpha(1) = c^\beta(1) < \infty$. Therefore by what was proved above there exists an entrance law (ξ_t^β) relative to (P_t^β) such that

$$(6.15) \qquad c^{\alpha+\beta}(f) = c_\beta^\alpha(f) = \int_0^\infty e^{-\alpha t} \xi_t^\beta(f)\, dt\,.$$

Define $\eta_t^\beta(f) = e^{\beta t} \xi_t^\beta(f)$. Clearly η^β is an entrance law for the semigroup (P_t) and by (6.15),

$$(6.16) \qquad c^\gamma(f) = \int_0^\infty e^{-\gamma t} \eta_t^\beta(f)\, dt \qquad \text{for all} \quad \gamma > \beta\,.$$

For $f \in \mathbf{C}_0$, $\eta_t^\beta(f)$ is right continuous in t, and (6.16) shows, by uniqueness of Laplace transforms, that $\eta_t^\beta(f)$ does not depend on β. There exists therefore an entrance law (η_t) for (P_t) such that for all $f \in \mathbf{B}$ and all $\gamma > 0$,

$$c^\gamma(f) = \int_0^\infty e^{-\gamma t} \eta_t(f)\, dt\,.$$

This completes the proof of Theorem 6.9.

The following corollary gives a representation theorem for finite excessive measures.

(6.17) COROLLARY. *Let μ be a finite excessive measure. Then there exists a unique entrance law η_t such that*

$$(6.18) \qquad \mu - \alpha \mu U^\alpha = \int_0^\infty e^{-\alpha t} \eta_t\, dt$$

for $\alpha > 0$. *In particular* $\int_0^\infty \eta_t(1)\,dt \leqq \mu(1) < \infty$. *If, in addition* $\mu P_t \to 0$ *as* $t \to \infty$, *then* $\mu = \int_0^\infty \eta_t\,dt$.

PROOF. Define $c^\alpha = \mu - \alpha\mu U^\alpha$. Then for each $\alpha > 0$, c^α is a finite measure and $c^\alpha(1) \to 0$ as $\alpha \to \infty$. Using the resolvent equation it is easy to check that (6.8) holds. This proves (6.18). If $\mu P_t \to 0$ as $t \to \infty$, then $\alpha\mu U^\alpha \to 0$ as $\alpha \to 0$, and so letting $\alpha \to 0$ in (6.18) we obtain $\mu = \int_0^\infty \eta_t\,dt$.

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