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THE NONEXISTENCE OF THE YAGLOM LIMIT FOR AN AGE DEPENDENT SUBCRITICAL BRANCHING PROCESS[1]

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An age dependent subcritical branching process is constructed for which the conditioned limits

$$\lim_{t\to\infty} P\{Z(t) = k \mid Z(t) > 0\} \qquad k \geqq 1,$$

do not exist.

1. Introduction and notation. We consider subcritical branching processes with population size at time t denoted by $Z(t)$. (See [1] or [5] for standard background material.) For a Galton–Watson process, a classical theorem of Yaglom ([4] page 18) asserts

$$\lim_{t\to\infty} P(Z(t) = k/Z(t) > 0) = b(k) \quad \text{exists.} \tag{1}$$

Moreover if $f(s) = \sum_{k=0}^{\infty} p_k s^k$, where p_k is the probability of having k offspring, is not a linear function of s, the limit in (1) is nondegenerate i.e. $b(k) > 0$ for some $k \neq 1$.

Recently a number of authors [2], [4] and [6] have considered the nature of the Yaglom limit in the age dependent case i.e. assuming that the particles have random lifetimes with distribution function denoted by $G(t)$. In particular, it has been shown that for some lifetime distributions, G, in nontrivial cases i.e. $f(s)$ nonlinear, the Yaglom limit may be degenerate. The point of this paper is to construct an example showing that the Yaglom limit need not exist at all.

We shall set

$$F(s, t) = \sum_{k=0}^{\infty} P(Z(t) = k)s^k ,$$

$$X(s, t) = 1 - F(s, t) ,$$

and $Y(t) = E(Z(t))$.

It is known that $F(s, t)$ satisfies the integral equation

$$F(s, t) = s(1 - G(t)) + \int_0^t f\{F(s, t - u)\}\, dG(u) , \tag{2}$$

and hence

$$X(s, t) = (1 - s)(1 - G(t)) + \int_0^t \{1 - f[1 - x(s, t - u)]\}\, dG(u) , \tag{3}$$

and

$$Y(t) = (1 - G(t)) + m \int_0^t Y(t - u)\, dG(u) \tag{4}$$

where $m = \sum_{k=1}^{\infty} kp_k < 1$ (since the processes we consider are subcritical).

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2. Construction of the example. We shall take

$$f(s) = \left(1 - \frac{m}{2}\right) + \frac{m}{2} s^2$$

with $m < 1$.

In this section we shall construct a $G(t)$ and a sequence of times t_k; $k = 1, 2, \cdots$ such that (5), (6) and (7) hold:

$$P(Z(t_k) > 1 \mid Z(t_k) > 0) \geqq \delta \tag{5}$$

for some positive δ, so that the Yaglom limit, if it exists cannot possibly be degenerate.

For every $A > 0$,

$$\frac{1 - G(t_k - A - 1)}{1 - G(t_k - 1)} \to 1 \qquad \text{as } k \to \infty , \tag{6}$$

and

$$\frac{Y(t)}{1 - G(t)} \quad \text{is bounded.} \tag{7}$$

In Section 3 we shall show that (6) and (7) imply the Yaglom limit is degenerate if it exists; this, of course will conclude the proof.

In our construction $G(t)$ will be constant on intervals $t_k \leqq t < t_{k+1}$, so that condition (6) will be satisfied if $t_{k+1} - t_k \to \infty$.

We shall also have $1 - G(t_k) = \rho^k$ for any fixed ρ, $0 < \rho < 1$. (7) will then imply (5) by the following argument: $P(Z(t_k) > 1) \geqq P$ (first particle dies at time t_k and has two offspring). So

$$P(Z(t_k) > 1) \geqq (1 - \rho)\rho^{k-1} \frac{m}{2} . \tag{8}$$

Now an easy comparison argument with the integral equations (3) and (4) shows that

$$P(Z(t) > 0) = X(0, t) \leqq Y(t) .$$

Thus by (8) and (7)

$$\begin{aligned} P(Z(t_k) > 1 \mid Z(t_k) > 0) &\geqq \frac{P(Z(t_k) > 1)}{Y(t_k)}, \\ &\geqq \frac{(1 - \rho)\rho^{k-1}(m/2)}{1 - G(t_k)} \frac{(1 - G(t_k))}{Y(t_k)} \\ &\geqq \frac{\eta(1 - \rho)m}{2\rho} \end{aligned}$$

where $\eta^{-1} = \sup Y(t)/(1 - G(t)$, proving (5).

Hence it remains to choose the times t_k so that (7) holds (with $(t_{k+1} - t_k) \to \infty$).

We take t_1 arbitrary. Given $t_1, \cdots, t_k$ we construct t_{k+1} inductively as follows:

Let $U_k(t)$ be the solution of the integral equation

$$U_k(t) = 1 - G(t_k \wedge t) + m \int_0^{t_k \wedge t} U_k(t-u)\, dG(u) ,$$

$\{G(u)$ is assumed to be defined for $u \leqq t_k\}$. An argument used in [3] shows $U_k(t) \downarrow$; hence $U_k(t)$ tends to a limit, r_k, as $t \to \infty$.

One easily sees that r_k satisfies

$$r_k = (1 - G(t_k)) + m r_k G(t_k)$$

so

$$r_k < \frac{1 - G(t_k)}{1 - m} = \frac{\rho^k}{1 - m} .$$

Choose t_{k+1} so that

$$U_k(t_{k+1}^-) < \frac{\rho^k}{1 - m} ,$$

and $t_{k+1} - t_k > k(t_k - t_{k-1})$. Note that

$$Y(t) = U_k(t) \qquad \text{for } t < t_k ,$$

and $Y(t)$ is monotone decreasing by an argument of [3]. So

$$\frac{Y(t_{k+1})}{1 - G(t_{k+1})} \leqq \frac{Y(t_{k+1}^-)}{1 - G(t_{k+1})} = \frac{U_k(t_{k+1}^-)}{1 - G(t_{k+1})} \leqq \frac{1}{\rho(1 - m)}$$

and if $t_{k+1} \leqq t < t_{k+2}$

$$\frac{Y(t)}{1 - G(t)} \leqq \frac{Y(t_{k+1})}{1 - G(t_{k+1})} \leqq \frac{1}{\rho(1 - m)} .$$

This shows then that (7) holds.

3. It remains to prove the assertion of the previous section, namely that (6) and (7) imply that the Yaglom limit is degenerate if it exists. We prove a slightly stronger version for applications elsewhere.

LEMMA. *Assume there is a sequence $t_k \to \infty$ such that for every $A > 0$,*

$$\frac{1 - G(t_k - A)}{1 - G(t_k)} \to 1 \qquad \textit{as } k \to \infty \tag{9}$$

and

$$\frac{Y(t_k)}{1 - G(t_k)} \leqq M < \infty . \tag{10}$$

Then if the Yaglom limit exists it is degenerate.

PROOF. If the Yaglom limit exists,

$$\lim_{t \to \infty} \frac{X(s, t)}{X(0, t)} = a(s)$$

exists.

To show the limit is degenerate, it suffices to show $a(s) = 1 - s$. See [4].

Fix s. Let $\varepsilon > 0$. Choose A so large that

$$(11) \qquad \left| \frac{X(s, t)}{X(0, t)} - a(s) \right| < \varepsilon$$

for $t > A$,

$$(12) \qquad \left| \frac{1 - f(1 - X(s, t))}{mX(s, t)} - 1 \right| < \varepsilon$$

for $t > A$, and

$$(13) \qquad \left| \frac{1 - f(1 - X(0, t))}{mX(0, t)} - 1 \right| < \varepsilon$$

for $t > A$. (12) and (13) can be arranged since $X(s, t) \to 0$ as $t \to \infty$. See [5].

Now as $n \to \infty$

$$X(s, t_n) = (1 - s)(1 - G(t_n)) + \int_0^{t_n - A} \{1 - f[1 - X(s, t_n - u)]\} \, dG(u) + o(1 - G(t_n))$$

by (9). So

$$\begin{aligned} X(s, t_n) &= (1 - s)(1 - G(t_n)) + \int_0^{t_n - A} \left\{ \frac{\{1 - f[1 - X(s, t_n - u)]\}}{X(s, t_n - u)} \times \frac{X(s, t_n - u)}{X(0, t_n - u)} X(0, t_n - u) \right\} dG(u) + o(1 - G(t_n)) \\ &\leqq (1 - s)(1 - G(t_n)) + m(1 + \varepsilon)(a(s) + \varepsilon) \int_0^{t_n - A} X(0, t_n - u) \, dG(u) + o(1 - G(t_n)) \\ &\leqq (1 - s)(1 - G(t_n)) + m(1 + \varepsilon)(a(s) + \varepsilon) \int_0^{t_n} X(0, t_n - u) \, dG(u) + o(1 - G(t_n)) \\ &\leqq (1 - s)(1 - G(t_n)) + ma(s) \int_0^{t_n} X(0, t_n - u) \, dG(u) + \varepsilon c(s) Y(t_n) , \end{aligned}$$

where $c(s)$ is bounded uniformly in ε. Similarly

$$X(0, t_n) \geqq (1 - G(t_n)) + m \int_0^{t_n} X(0, t_n - u) \, dG(u) - \varepsilon c(s) Y(t_n) .$$

Thus

$$\frac{X(s, t_n)}{X(0, t_n)} \leqq \{(1 - s)(1 - G(t_n)) + ma(s) \int_0^{t_n} X(0, t_n - u) \, dG(u) + \varepsilon c(s) Y(t_n)\}\{1 - G(t_n) + m \int_0^{t_n} X(0, t_n - u) \, dG(u) - \varepsilon c(s) Y(t_n)\}^{-1} .$$

Now take n so large that $|X(s, t_n)/X(0, t_n) - a(s)| < \varepsilon$. Then we have

$$a(s)\{1 - G(t_n) + m \int_0^{t_n} X(0, t_n - u) \, dG(u)\} \leqq (1 - s)(1 - G(t_n)) + a(s)m \int_0^{t_n} X(0, t_n - u) \, dG(u) + \varepsilon c(s) Y(t_n) .$$

Hence

$$a(s) \leqq (1 - s) + \frac{\varepsilon c(s) Y(t_n)}{1 - G(t_n)} .$$

Since $Y(t_n)/(1 - G(t_n))$ is bounded, we conclude $a(s) \leqq 1 - s$. A similar argument shows $a(s) \geqq 1 - s$. Thus $a(s) = 1 - s$, and the proof is complete.

REFERENCES

[1] ATHREYA, K. B. and NEY, P. (1972). *Branching Processes.* Springer Verlag, Heidelberg.

[2] CHISTYAKOV, V. P. (1964). *A theorem on sums of independent positive random variables and its applications to branching random processes. Theor. Probability Appl.* **7** 640–648.

[3] CHOVER, J. and NEY, P. (1968). *The non-linear renewal equation. J. D'Analyse Mathematique* **21** 381–413.

[4] CHOVER, J. NEY, P. and WAINGER, S. (1973) *Degeneracy properties of subcritical branching processes. Ann. Probability* **1** 663–673.

[5] HARRIS, T. (1963). *The Theory of Branching Processes.* Springer Verlag, Heidelberg.

[6] RYAN, T. A. (1968). *On age dependent branching processes.* Ph. D. dissertation, Cornell University.

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