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A NOTE ON THE DISTRIBUTION OF HITTING TIMES[1]

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We show that "hitting times" associated with stationary processes have an absolutely continuous distribution, except, possibly, for an atom at the origin. The density is then identified in several special cases.

0. We show that "hitting times" associated with stationary processes have an absolutely continuous distribution except, possibly, for an atom at the origin. Several examples are given.

1. Let $R(R_+)$ denote the real line (half-line $[0, \infty)$) with Borel sets $\mathscr{B}(\mathscr{B}_+)$. We use m for Lebesgue measure. Let θ_t, $t \in R$, be a group of measurable, measure-preserving transformations of a probability space $(\Omega, \mathscr{F}, P)$. In addition, assume $(t, \omega) \to \theta_t(\omega)$ is $\mathscr{B} \times \mathscr{F}$ measurable. We will consider Borel sets $M(\omega)$ indexed by Ω and satisfying

(i) $M(\theta_t \omega) = M(\omega) - t$ for all $t \in R$, $\omega \in \Omega$ and
(ii) $\sup M(\omega) = +\infty$ a.s.

The hitting time of $M(\omega)$ is $\tau(\omega) = \inf\{t > 0 : t \in M(\omega)\}$. Note that $\tau(\omega) > t$ implies $\tau(\omega) = t + \tau(\theta_t \omega)$, the so-called "terminal" property of τ. Assume τ is measurable.

THEOREM. *The distribution of τ is absolutely continuous except, possibly, for an atom at* 0.

This theorem, as well as Section 2, is a direct consequence of the following equation:

$$(*) \qquad \int_0^T I_\Gamma(\tau(\theta_t \omega))\, dt = \int_\Gamma \nu_T(x, \omega)\, dx + \delta_0(\Gamma) \int_0^T I_{\{0\}}(\tau(\theta_t \omega))\, dt\,, \qquad \Gamma \in \mathscr{B}_+$$

where $\nu_T(x, \omega)$ is the number of solutions of $\tau(\theta_t \omega) = x$ for $0 < t \leqq T$ and δ_0 is unit mass at 0.

The proof of (*) is elementary. For each $\omega \in \Omega$, the paths $\tau(\theta_t \omega)$ are right-continuous with left-hand limits on R_+, and have "saw-tooth" appearance, with possibly infinitely many "teeth" in a finite interval. With $\omega \in \Omega$, $b > 0$ and $T > 0$ fixed, it is fairly apparent that $\{t : \tau(\theta_t \omega) \geqq b\}$ meets $[0, T]$ in at most finitely many disjoint (possibly degenerate) closed intervals, say $J_1, J_2, \cdots, J_n$, such that

$$(1) \qquad m(J_k) = \int_b^\infty \nu(k, x, \omega)\, dx\,, \qquad k = 1, 2, \cdots, n$$

where $\nu(k, x, \omega)$ is the number of solutions (necessarily finite) of $\tau(\theta_t \omega) = x$ for

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$t \in J_k$. Summing (1) over $k = 1, 2, \cdots, n$ we find that

$$\int_0^T I_{[b,\infty)}(\tau(\theta_t \omega))\, dt = \int_b^\infty \nu_T(x, \omega)\, dx .$$

Incorporating the case $b = 0$ we obtain (*) for any $\Gamma \in \mathscr{B}_+$ of the form $\Gamma = [b, \infty)$, hence for any $\Gamma \in \mathscr{B}_+$. Taking expectations with $T = 1$ in (*) gives the theorem:

$$P(\tau \in \Gamma) = \int_\Gamma E\nu_1(x)\, dx + \delta_0(\Gamma)P(\tau = 0)\,, \qquad \Gamma \in \mathscr{B}_+ .$$

(Notice that $P(\tau = 0) = Em(M \cap [0, 1])$ if $M(\omega)$ is right-closed a.s.)

2. By (*), the family of additive functionals

$$\begin{aligned} \beta_t(x, \omega) &= (P(\tau = 0))^{-1} \int_0^t I_{\{0\}}(\tau(\theta_s \omega))\, ds\,, && x = 0 \\ &= (E\nu_1(x))^{-1}\nu_t(x, \omega)\,, && x > 0 \end{aligned}$$

is an "occupation-time density" for the process $\tau \circ \theta_t$. (By an *additive functional* we mean a right-continuous, non-decreasing process $\alpha_t(\omega)$ for which $\alpha_0 \equiv 0$ and $\alpha_{t+s} - \alpha_t = \alpha_s \circ \theta_t$ a.s. for each s, t.) Specifically

$$\int_\Gamma \beta_t(x, \omega) P(\tau \in dx) = \int_0^t I_\Gamma(\tau(\theta_s \omega))\, ds$$

for all $t \in R_+$, $\omega \in \Omega$, $\Gamma \in \mathscr{B}_+$. A straightforward calculation (which we will omit) shows that the family of Palm measures (of $\beta_t(x)$)

$$P_{\beta(x)}(A) = E \int_0^1 I_A(\theta_t \omega)\, d\beta_t(x, \omega)\,, \qquad x \in R_+$$

is a regular version of the family of conditional probabilities $P^x(\cdot) = P(\cdot \mid \tau = x)$, $x \in R_+$. Indeed, if $\mathscr{F}$ is separable, $\hat{P}_{\beta(x)} = P^x$ a.e. (m).

3. In general, the density $E\nu_1(x)$ of the positive part of τ does not admit a smooth version. For instance, let $h(x) \geqq 0$, $x > 0$, be any non-increasing, integrable function and let $X_t(\omega)$ be the semilinear Markov process with characteristic $\{\alpha, h(x)\}$, $\alpha \geqq 0$. When $h(0+) = \infty$, X_t has a unique, stationary initial distribution (see Horowitz [1]), under which the distribution of the first passage to 0 is

$$(\alpha + \int_0^\infty h(x)\, dx)^{-1}(\alpha\delta_0(dx) + h(x)\, dx) .$$

4. When $M(\omega)$ is discrete, say countable and clustering only at $\pm\infty$, the additive functional $n_t(\omega) \equiv \lim_{x \downarrow 0} \nu_t(x, \omega)$ counts the number of points in $M(\omega) \cap (0, t]$ and is a stationary point process. When $En_1 = \alpha < \infty$, the (normalized) Palm measure $\hat{P}$ of $n_t(\omega)$ is a probability and $\hat{E}\tau = 1/\alpha$. A now well-known inversion formula (see e.g. Ryll-Nardzewski [2]) is

$$(2) \qquad E\xi = \alpha\hat{E} \int_0^{\tau(\omega)} \xi(\theta_s \omega)\, ds\,, \qquad \text{for any random variable } \xi .$$

Denote by $\phi(\lambda)$ and $\hat{\phi}(\lambda)$ the Laplace transforms of τ under P and $\hat{P}$ respectively. From (2), $\phi(\lambda) = \alpha(1 - \hat{\phi}(\lambda))/\lambda$ and consequently $\alpha(1 - \hat{P}(\tau \leqq t)$ is the density of τ under P.

5. As a final example, let N_t be a Poisson process with unit intensity and

$X_t = N_{t+1} - N_t$. This is a strictly stationary (non-Markov) process; we will compute the distribution of $\tau(\omega) = \inf\{t > 0 : X_t(\omega) = 0\}$.

Let $T_1, T_2, \cdots$ be the times between the jumps of N_t. Clearly $P(\tau = 0) = P(T_1 > 1) = e^{-1}$ and for $t > 0$,

$$\begin{aligned} P(\tau \leqq t) &= P(T_1 > 1) + \textstyle\sum_{n=1}^{\infty} P(T_1 \leqq 1, \cdots, T_n \leqq 1, T_{n+1} > 1, \sum_1^n T_i \leqq t) \\ &= e^{-1}(1 + \textstyle\sum_{n=1}^{\infty} u_n(t)) \end{aligned}$$

where $u_n(t) = \int_{D_n(t)} \cdots \int (\exp - \sum_1^n x_i) \prod_1^n dx_i$ and $D_n(t)$ is the region $\{(x_1, x_2, \cdots, x_n) : 0 \leqq x_i \leqq 1, \sum_1^n x_i \leqq t\}$. Observe that $u_1(t) = 1 - \exp[-\min(1, t)]$ and that

$$u_{n+1}(t) = \int_0^{\min(1,t)} e^{-x} u_n(t - x)\, dx\,, \qquad n = 1, 2, \cdots.$$

Summing over $n = 1, 2, \cdots$ results in the renewal equation

$$(3) \qquad u(t) = u_1(t) + \int_0^t u(t - x)\, du_1(x)\,, \qquad u(t) = \textstyle\sum_{n=1}^{\infty} u_n(t)\,.$$

Evidently, $u(t)$ is continuously differentiable for $t \neq 1$; thus $P(\tau \leqq t)$ is absolutely continuous on $(0, \infty)$. In fact, an easy computation with (3) leads to

$$Ee^{-\lambda\tau} = \frac{e^{-1}(1 + \lambda)}{\lambda + e^{-1-\lambda}}\,, \qquad \lambda \geqq 0\,.$$

(For intensity θ we get $e^{-\theta}(\theta + \lambda)/(\lambda + \theta e^{-\theta-\lambda})$.)

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