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PROPERTIES OF OPTIMAL EXTENDED-VALUED STOPPING RULES FOR S_n/n [1]

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Let $X_1, X_2, \cdots$ be i.i.d. nondegenerate mean zero random variables with common distribution function F such that $EX_1^+ \log^+ X_1 < \infty$. Let $S_n = X_1 + \cdots + X_n$ and T_∞ be the collection of randomized extended-valued stopping rules. For $t \in T_\infty$, define $E(S_t/t) = E(S_t/t)1_{\{t<\infty\}}$. There exists $\tau \in T_\infty$ such that $E(S_\tau/\tau) = \sup_{t \in T_\infty} E(S_t/t) < \infty$. We wish to determine whether or not $P(\tau = \infty) > 0$. Let $b^{-1}(x) = 1/(\int_x^\infty (y/x) \log (y/x)\, dF(y))$. We find that $P(X_n \geqq b(n)$ i.o.$) = 1 \Rightarrow P(\tau < \infty) = 1$. As a corollary, if $E(X_1^+)^\alpha < \infty$ for some $\alpha > 1$, $P(\tau < \infty) = 1$. Conversely, if $P(|X_n| \geqq b(n)$ i.o.$) = 0$, then $P(\tau = \infty) > 0$. Moreover, if $g(n) \nearrow$ and $\sum_{n=1}^\infty g(2^n)^{-1} < \infty$ then $P(|X_n| \geqq b(n)g(n)$ i.o.$) = 1 \Rightarrow P(\tau < \infty) = 1$. Examples satisfying these latter conditions are given. An outgrowth of this work is that for *any* i.i.d. sequence $\{X_n\}$ of mean zero random variables and $c < \frac{1}{4}$, $P(S_n \geqq cE|S_n|$ i.o.$) = 1$. The importance of this result stems from the fact that we may also have $S_n(\log n)^\varepsilon/n \to -\infty$ in probability (see [15]).

In order to be completely rigorous, a section was included which provides a useful characterization of the general form of the class of optimal rules for sequential decision problems.

0. Introduction. Imagine that one is playing a game in which he is allowed to view sequentially random variables $X_1, X_2, \cdots$ which are independent and identically distributed with common distribution function F and mean zero. He is allowed to stop viewing at any stage n, based only upon $X_1, \cdots, X_n$, in which case he is rewarded S_n/n, where $S_n = X_1 + \cdots + X_n$. Should he continue forever, (he should live so long), he receives zero.

Combining the works of D. L. Burkholder [1], B. J. McCabe and L. A. Shepp [17], and D. O. Siegmund [20] Theorem 4, it is found that a necessary and sufficient condition which guarantees that there exist an optimal strategy with finite expected payoff is that $EX_1^+ \log^+ X_1 < \infty$. Burgess Davis [8], and R. F. Gundy [12] have obtained similar results. Somewhat prior to these papers, Y. S. Chow and H. Robbins [3] proved the existence of a unique minimal optimal rule for the case in which F concentrates on two points and found that its form was: stop at the first n such that $S_n \geqq a_n$, where $\{a_n\}$ is a strictly increasing sequence of positive constants. A. Dvoretzky [11] and H. Teicher and J. Wolfowitz [24] derived the same results for random variables with finite variance. A by-product of papers [3], [11], and [24] is that in the cases treated, the minimal

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optimal rule stops with probability one. A recent paper, combining the dissertations of M. E. Thompson, A. K. Basu, and W. L. Owen [23] establishes that for a certain class of random variables with infinite variance (those with a finite absolute moment of order $\beta > 1$ whose truncated variance is of dominated variation of order $\nu > 1$), the minimal optimal rule stops with probability one. Most recently, B. Davis [7] proved the same result under the hypothesis that the random variables have a finite absolute moment of order $\beta > 1$. Heretofore, the question of whether there are nondegenerate distribution functions F for which an optimal rule has positive probability of never stopping was open. The intrigue of this problem stems from the fact that $S_n > 0$ infinitely often, so that if one desires he can always obtain a positive return; so how can the optimal rule have positive probability of never stopping? In this work, results are obtained which provide a very general criterion for determining whether or not a given df F gives rise to an optimal rule which stops with probabiiity one. Additionally, a broad class of random variables is found which yields an optimal rule with positive probability of not stopping.

The primary results obtained in this paper are predicated upon knowledge of the form of the optimal rule for sequences

$$\frac{a+S_1}{n+1}, \frac{a+S_2}{n+2}, \cdots, \frac{a+S_k}{n+k}, \cdots,$$

where a is a fixed constant, n a nonnegative integer, $S_k = X_1 + \cdots + X_k$. Hence, Section 1 has been devoted to a treatment of the existence and nature of optimal rules for problems of a most general type. The results of Section 1 relating to proving the existence of an optimal rule under assumptions A_1 and A_2 are drawn from the class notes of T. S. Ferguson. It is believed that Theorem 6 provides a practical means of identifying the form of the class of optimal rules. Indeed, it proved to be the means of characterizing optimal rules for the problem at hand. The fact that the minimal *semi-optimal* rule (see Definition 4) is optimal should clarify what the author felt was an obscurity in Dvoretzky's derivation of the form of the minimal optimal rule on page 488 of [11], beginning with "The analysis of $\cdots$".

Assume that F is nondegenerate and that $EX_1^+ \log^+ X_1 < \infty$. As previously indicated and verified in Section 2, there is a strictly increasing sequence of positive numbers $\{a_n\}$ which characterizes the collection of extended-valued optimal stopping variables in the sense that any extended-valued stopping variable τ such that both

(a) $\tau = n \Rightarrow S_n \geqq a_n$ a.s., and
(b) $P(\tau \leqq \tau_1) = 1$

hold, is optimal, where

$$\tau_1 = \text{1st} \quad k : S_k > a_k \quad \text{if such } k \text{ exists}$$
$$= \infty \quad \text{otherwise.}$$

For a given F as above, either all optimal rules stop with probability one or they all have positive probability of never stopping, according as $P(S_n \geqq a_n \text{ i.o.})$ is one or zero. For $x > 0$, and provided $P(X > t) > 0$ for all t, the function

$$1\Big/\left(\int_x^\infty \frac{y}{x} \log \frac{y}{x}\, dF(y)\right)$$

is a continuous strictly increasing positive function and hence is the functional inverse of a continuous strictly increasing function $b(x)$. We find in Section 3 that if $P(S_n \geqq a_n \text{ i.o.}) = 0$, then $P(X_n \geqq b(n) \text{ i.o.}) = 0$. A corollary establishes that whenever $E(X_1^+)^{1+\varepsilon} < \infty$ for some $\varepsilon > 0$, all optimal rules stop with probability one. The condition $P(X_n \geqq b(n) \text{ i.o.}) = 0$ is not sufficient to guarantee that $P(S_n \geqq a_n \text{ i.o.}) = 0$. Some assumptions must be made regarding the negative part of the X_1 distribution. We find in Section 4 that $P(|X_n| \geqq b(n) \text{ i.o.}) = 0 \Rightarrow P(|S_n| \geqq a_n \text{ i.o.}) = 0$, in which case $P(\tau = \infty) > 0$. In Section 5 it is shown that if $g(n)$ is an increasing positive function such that $\sum_{n=1}^\infty 1/g(2^n) < \infty$, then

$$P(|X_n| \geqq b(n)g(n) \text{ i.o.}) = 1 \Rightarrow P(S_n > a_n \text{ i.o.}) = 1 .$$

Relations between a_n, $b(n)$ and $E|S_n|$ can be found in Sections 4 and 7. For instance, $a_n > \frac{1}{4}E|S_n|$ and if $P(S_n \geqq a_n \text{ i.o.}) = 0$ then $1 \leqq \liminf_{n\to\infty} a_n/(b(n)) \leqq \limsup_{n\to\infty} a_n/(b(n)) \leqq 8$ so that $\limsup_{n\to\infty} E|S_n|/(b(n)) \leqq 32$. H. M. Taylor [22], L. H. Walker [25], and L. A. Shepp [19] have demonstrated that $a_n/n^{\frac{1}{2}}$ tends to a finite nonzero limit whenever $0 < EX_1^2 < \infty$, a result which has been extended by W. L. Owen [18] in case X_1 is in the domain of normal attraction of a stable law of order $1 < \alpha < 2$ to the statement that $\lim_{n\to\infty} a_n/n^{1/\alpha}$ exists and is finite and nonzero. The author conjectures that $\lim_{n\to\infty} a_n/E|S_n|$ exists and is a finite nonzero number provided only that $0 < E|X_1|^{1+\varepsilon} < \infty$ for some $\varepsilon > 0$. In Section 7 it is shown that under the latter hypothesis, $\frac{1}{4} \leqq \liminf_{n\to\infty} a_n/E|S_n| < \infty$. One can also show that if $P(|X_n| \geqq b(n) \text{ i.o.}) = 0$, then $\limsup_{n\to\infty} a_n/E|S_n| = \infty$.

In Section 6, a multitude of distributions F is exhibited that have the property that for $0 < c < 1$,

$$1 = P(S_n > cb(n) \text{ i.o.}) > P(X_n > cb(n) \text{ i.o.}) = 0 .$$

However, in [13] page 259, Feller erroneously claimed that whenever (a) $EX_1 = 0$ (b) $E|X_1|^\alpha = \infty$ for some $\alpha < 2$, and (c) $\{c_n\}$ is a sequence of positive constants such that for some

$$\varepsilon > 0, \quad \frac{c_n}{n} \searrow \quad \text{and} \quad \frac{c_n^{2-\varepsilon}}{n} \nearrow \qquad \text{then} \quad P(S_n > c_n \text{ i.o.}) = P(X_n > c_n \text{ i.o.}) .$$

In Sections 8 and 9 it is demonstrated that for $\alpha > \frac{1}{2}$, $\alpha \neq 1$, all optimal rules for S_n/n^α having finite expected return stop with probability one. In Section 9 it is also proved that for $c < \frac{1}{4}$,

$$EX_1 = 0 \Rightarrow P(S_n \geqq cE|S_n| \text{ i.o.}) = 1 .$$

This improves the result of C. Stone [21] that $P(\limsup_{n\to\infty} S_n/n^{\frac{1}{2}} = \infty) = 1$.

1. Existence and nature of the class of optimal extended-valued stopping rules. Let $X_1, X_2, \cdots$ be random variables defined on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$. Let $\mathscr{B}_n$ be the Borel field generated by $X_1, \cdots, X_n$ and an arbitrary collection of random variables which are independent of $X_{n+1}, X_{n+2}, \cdots$. We assume that $\mathscr{B}_n \subset \mathscr{B}_{n+1}$. Let $\mathscr{B}_\infty = \bigvee_n \mathscr{B}_n$.

DEFINITION 1. Let t be a random variable defined on $(\Omega, \mathscr{F}, \mathscr{P})$ which takes values in $\{1, 2, \cdots, \infty\}$. If $\{t = n\} \in \mathscr{B}_n$ for every n (and consequently $\{t = \infty\} \in \mathscr{B}_\infty$), then t is said to be an *extended-valued stopping rule* with respect to the Borel fields $\mathscr{B}_1, \cdots, \mathscr{B}_\infty$. Let T_∞ be the collection of extended-valued stopping rules with respect to $\mathscr{B}_1, \cdots, \mathscr{B}_\infty$.

DEFINITION 2. Given a stopping rule problem with observables $X_1, X_2, \cdots$ and returns $Y_1, Y_2, \cdots, Y_\infty$, where $Y_n(\omega) = Y_n(X_1(\omega), \cdots, X_n(\omega))$ for $\omega \in \Omega$, an extended valued stopping rule τ is said to be *regular* if for each $n \geqq 1$,

$$E(Y_\tau \mid \mathscr{B}_n) \geqq Y_n \quad \text{a.s.} \qquad \text{on} \quad \{\tau > n\}. \tag{1}$$

Equivalently, τ is regular if for any $n \geqq 1$ and any $\mathscr{B}_n$-measurable set $\Lambda \subseteqq \{\tau > n\}$,

$$EY_\tau 1_\Lambda \geqq EY_n 1_\Lambda. \tag{2}$$

Thus if τ is regular, one does not suffer by continuing whenever τ instructs one to do so. The term *admissible* as used by Chow, Robbins, and Siegmund [4] page 64, is synonymous with our term *regular*, which is to be found both in T. S. Ferguson's notes and de Groot [9].

DEFINITION 3. τ is *strictly regular* if for each $n \geqq 1$ the inequality in (1) is strict a.s. on $\{\tau > n\}$. Equivalently, τ is *strictly regular* if for each $n \geqq 1$ and any $\mathscr{B}_n$-measurable set $\Lambda \subseteqq \{\tau > n\}$ such that $P(\Lambda) > 0$, the inequality in (2) is strict.

In what follows, let A_1 be the assumption that

$$E \sup_{n \leqq \infty} Y_n < \infty \tag{3}$$

and let A_2 be the assumption

$$Y_\infty \geqq \limsup_{n \to \infty} Y_n \quad \text{a.s.} \tag{4}$$

A_1 will allow us to interchange expectation and summation and when used in conjunction with A_2 will guarantee the existence of an optimal extended-valued stopping rule. A_1 also guarantees that our integrals are well defined.

To avoid trivialities, we always assume

$$A_3\text{: for some} \quad t \in T_\infty, \quad EY_t > -\infty. \tag{5}$$

LEMMA 1. *Assume A_1. Let $t \in T_\infty$ be given such that $EY_t > -\infty$. Then $\exists$ strictly regular $t' \in T_\infty \ni$*

$$EY_{t'} \geqq EY_t. \tag{6}$$

Moreover, if t is not regular, we have strict inequality in (6). *Note also that $t' \leqq t$.*

PROOF. We define extended-valued stopping rules t_1 and t' inductively. For $n \geqq 1$,

$$\begin{aligned}(7)\qquad &\{t_1 = n\} = \{t_1 \geqq n, t = n\} \cup \{t_1 \geqq n, t > n, E(Y_t \mid \mathscr{B}_n) < Y_n\}\\ &\{t' = n\} = \{t' \geqq n, t_1 = n\} \cup \{t' \geqq n, t_1 > n, E(Y_{t_1} \mid \mathscr{B}_n) \leqq Y_n\}.\end{aligned}$$

Clearly, t_1 and $t' \in T_\infty$ and $t' \leqq t_1 \leqq t$. Using the identity $\{\tau > n\} = \{\tau \geqq n\}\{\tau = n\}^c$ for $\tau \in T_\infty$ one easily derives equivalent definitions of t_1 and t'; namely, for $n \geqq 1$,

$$\begin{aligned}(7\text{a})\qquad &\{t_1 > n) = \{t_1 \geqq n, t > n, E(Y_t \mid \mathscr{B}_n) \geqq Y_n\}\\ &\{t' > n\} = \{t' \geqq n, t_1 > n, E(Y_{t_1} \mid \mathscr{B}_n) > Y_n\}.\end{aligned}$$

Put $\mathscr{B}_0 = \Omega$. Let Λ be a $\mathscr{B}_k$-measurable set for some $k \geqq 0$. Assume $P(\Lambda) > 0$ and $\Lambda \subseteqq \{t' > k\}$.

$$\begin{aligned}E1_\Lambda Y_{t'} &= \textstyle\sum_{k<n\leqq\infty} E1_{\{t'=n\}\Lambda} Y_n\\ &\geqq \textstyle\sum_{k<n\leqq\infty} E1_{\{t'=n\}\Lambda} Y_{t_1}\\ &= E1_\Lambda Y_{t_1}.\end{aligned}$$

If $k \geqq 1$, then from (7a), $E1_\Lambda Y_{t_1} > E1_\Lambda Y_k$. Hence t' is strictly regular.

Similarly, $E1_\Lambda Y_{t_1} \geqq E1_\Lambda Y_t$. Let $k = 0$ and $\Lambda = \Omega$. Then $EY_{t'} \geqq EY_{t_1} \geqq EY_t$. Suppose t is not regular. Then $\exists k \geqq 1 \ni P\{t > k,\ t_1 = k\} > 0$. By (7),

$$EY_{t_1} 1_{\{t_1=n\}} \geqq EY_t 1_{\{t_1=n\}}.$$

For $n = k$, this inequality is strict. Therefore

$$EY_{t'} \geqq EY_{t_1} > EY_t.$$

COROLLARY 1. *Assume A_1 and A_3. If $\tau \in T_\infty$ and $EY_\tau = \sup_{t\in T_\infty} EY_t$, then τ is regular.*

LEMMA 2. *Assume A_1. Let $t \in T_\infty$ be regular and let $t' \in T_\infty$ be any rule such that $P(t' \leqq t) = 1$. Then $EY_t \geqq EY_{t'}$.*

PROOF.

$$\begin{aligned}EY_t &= \textstyle\sum_{1\leqq n\leqq\infty} EY_t 1_{\{t'=n\}}\\ &\geqq \textstyle\sum_{1\leqq n\leqq\infty} EY_n 1_{\{t'=n\}}\\ &= EY_{t'}.\end{aligned}$$

LEMMA 3. *Assume A_1. Let t' and t'' be (strictly) regular extended-valued stopping rules. Then $t = \max(t', t'')$ is also a (strictly) regular extended-valued stopping rule and*

$$(8)\qquad EY_t \geqq \max(EY_{t'}, EY_{t''}).$$

PROOF.

$$\{t = n\} = (\{t' = n\} \cap \{t'' \leqq n\}) \cup (\{t' \leqq n\} \cap \{t'' = n\}) \in \mathscr{B}_n$$

therefore $t \in T_\infty$.

Let Λ be a $\mathscr{B}_n$-measurable set such that $P(\Lambda) > 0$ and $\Lambda \subseteqq \{t > n\} \cap \{t' \leqq n\}$.

$$EY_t 1_\Lambda = EY_{t''} 1_\Lambda \,(>)\!\geqq EY_n 1_\Lambda \,.$$

Similarly, if $\Lambda \subseteqq \{t > n\} \cap \{t'' \leqq n\}$, $\Lambda \in \mathscr{B}_n$ and $P(\Lambda) > 0$, then

$$EY_t 1_\Lambda \,(>)\!\geqq EY_n 1_\Lambda .$$

Let $\Lambda \in \mathscr{B}_n$, $\Lambda \subseteqq \{t' > n\} \cap \{t'' > n\}$, $P(\Lambda) > 0$.

$$\begin{aligned} EY_t 1_\Lambda 1_{\{t''>t'\}} &= \textstyle\sum_{n<k\leqq\infty} EY_{t''} 1_{\{t''>k,t'=k\}\Lambda} \\ &\geqq \textstyle\sum_{n<k\leqq\infty} EY_k 1_{\{t''>k,t'=k\}\Lambda} \\ &= EY_{t'} 1_\Lambda 1_{\{t''>t'\}} \\ EY_t 1_\Lambda &= EY_t 1_\Lambda 1_{\{t'=t\}} + EY_t 1_\Lambda 1_{\{t''>t'\}} \\ &\geqq EY_{t'} 1_\Lambda 1_{\{t'=t\}} + EY_{t'} 1_\Lambda 1_{\{t''>t'\}} \\ &= EY_{t'} 1_\Lambda \\ &(>)\!\geqq EY_n 1_\Lambda . \end{aligned}$$

Since $\{t > n\} = (\{t > n\} \cap \{t' \leqq n\}) \cup (\{t > n\} \cap (\{t'' \leqq n\}) \cup (\{t' > n\} \cap \{t'' > n\})$, we have shown that t is (strictly) regular. Clearly $P(t' \leqq t) = 1$ and $P(t'' \leqq t) = 1$. Apply Lemma 2 to obtain (8).

THEOREM 1. *Assume A_1 and A_2 hold. Then there exists $\tau \in T_\infty$ such that $EY_\tau = \sup_{t\in T_\infty} EY_t$.*

PROOF. Let $M = \sup_{t\in T_\infty} EY_t$. $A_1 \Rightarrow M < \infty$. If $M = -\infty$, set $\tau \equiv 1$. Thus suppose M is finite. Recalling Lemma 1, $\exists$ strictly regular $t_n \in T_\infty \ni EY_{t_n} > M - n^{-1}$. Put $\tau_1 = t_1$, $\tau_n = \max(\tau_{n-1}, t_n) = \max(t_1, \cdots, t_n)$ and $\tau = \sup_n \tau_n$.

By Lemma 3, τ_n is strictly regular and $EY_{\tau_n} \geqq EY_{t_n} > M - n^{-1}$. $\tau_n \nearrow \tau$.

$$\{\tau = k\} = \textstyle\bigcup_{m=1}^\infty \bigcap_{n=m}^\infty \{\tau_n = k\} \subset \mathscr{B}_k$$

therefore $\tau \in T_\infty \Rightarrow EY_\tau \leqq M$.

$$\begin{aligned} M &= \lim_{n\to\infty} EY_{\tau_n} \leqq E \limsup_{n\to\infty} Y_{\tau_n} \\ &= E \limsup_{n\to\infty} Y_{\tau_n} 1_{\{\tau<\infty\}} + E \limsup_{n\to\infty} Y_{\tau_n} 1_{\{\tau=\infty\}} \\ &= EY_\tau 1_{\{\tau<\infty\}} + E \limsup_{n\to\infty} Y_{\tau_n} 1_{\{\tau=\infty\}} \\ &\leqq EY_\tau 1_{\{\tau<\infty\}} + EY_\infty 1_{\{\tau=\infty\}} && \text{(by } A_2\text{)} \\ &= EY_\tau \leqq M . && \text{therefore } EY_\tau = M . \end{aligned}$$

DEFINITION 4. $t' \in T_\infty$ is *semi-optimal* iff given any $t \in T_\infty$, $n \geqq 1$, and B_n-measurable set $A \subseteqq \{t' = t,\ t > n\} \ni P(A) > 0$,

$$EY_n 1_A \geqq EY_t 1_A \,. \tag{9}$$

t' is *strictly semi-optimal* iff the inequality in (9) is always strict.

LEMMA 4. *Assume A_3. Suppose $\exists \tau \in T_\infty$ which is optimal in the sense that $EY_\tau = \sup_{t\in T_\infty} EY_t < \infty$. Then τ is semi-optimal.*

PROOF. If τ is not semi-optimal, then $\exists\, t \in T_\infty$, $0 < n < \infty$, and a $\mathscr{B}_n$-measurable set $A \subseteqq \{\tau = n,\ t > n\} \ni$

$$EY_t 1_A > EY_\tau 1_A .$$

Put $t' = \tau 1_{A^c} + t 1_A$. $t' \in T_\infty$.

$$\begin{aligned} EY_{t'} &= EY_\tau 1_{A^c} + EY_t 1_A \\ &> EY_\tau 1_{A^c} + EY_\tau 1_A \qquad \text{(since } |EY_\tau 1_{A^c}| < \infty) \\ &= EY_\tau \Rightarrow \Leftarrow \text{contradiction.} \end{aligned}$$

Thus τ is semi-optimal.

LEMMA 5. *If $t' \in T_\infty$ is semi-optimal and $t \in T_\infty$ is strictly regular, then $P(t \leqq t') = 1$.*

PROOF. Let $A_n = \{t' = n,\ t > n\}$. Suppose for some $n \geqq 1$, $P(A_n) > 0$. Since t' is semi-optimal, (8) gives

$$EY_n 1_{A_n} \geqq EY_t 1_{A_n} .$$

t being strictly regular, we have

$$EY_t 1_{A_n} > EY_n 1_{A_n} \Rightarrow \Leftarrow \text{contradiction.}$$

therefore $P(A_n) = 0\ \forall n \geqq 1$.
Hence

$$\begin{aligned} P(t > t') &= P(\textstyle\bigcup_n A_n) = 0 \\ &\Rightarrow P(t \leqq t') = 1 . \end{aligned}$$

THEOREM 2. *Assume A_1 and A_3. Then $\tau \in T_\infty$ is an optimal extended-valued stopping rule* iff *τ is a regular semi-optimal extended-valued stopping rule.*

PROOF. Suppose τ is optimal. Corollary 1 gives the fact that τ is regular, while Lemma 4 implies that τ is semi-optimal. Now assume that $\tau \in T_\infty$ is both regular and semi-optimal. If τ is not optimal, $\exists\, t' \in T_\infty$ such that $EY_{t'} > EY_\tau$. $\exists$ strictly regular $t \in T_\infty \ni EY_t \geqq EY_{t'} > EY_\tau$. By Lemma 5, $P(t \leqq \tau) = 1$.
We now invoke Lemma 2 to obtain $EY_\tau \geqq EY_t > EY_\tau$. $\Rightarrow \Leftarrow$ contradiction. Therefore τ is optimal.

The next two definitions and subsequent results (Theorems 3 to 6) are of interest in that they enable one to identify certain rules as being optimal without requiring one to carry out the verification of the fact that they are *both* regular and semi-optimal.

DEFINITION 5. A (strictly) semi-optimal rule $\tau \in T_\infty$ is *minimal (strictly) semi-optimal* iff for every (strictly) semi-optimal $t \in T_\infty$, $P(\tau \lneqq t) = 1$; or equivalently, iff given $t \in T_\infty$ such that $P(t < \tau) > 0$ it follows that t is not (strictly) semi-optimal.

COMMENT 1. Assuming existence, uniqueness of minimal (strictly) semi-optimal rules is a direct consequence of Definition 5.

THEOREM 3. *Assume A_1, A_2 and A_3. There exists a unique (up to equivalence) minimal semi-optimal rule $\tau \in T_\infty$. Furthermore, $\tau \in T_\infty$ is minimal semi-optimal* iff *τ is optimal and strictly regular.*

PROOF. Combining Theorem 1 and Lemma 1, there is a strictly regular $\tau \in T_\infty$ such that τ is optimal. By Lemma 5, for every semi-optimal rule $t \in T_\infty$ we have $P(t \geqq \tau) = 1$. Thus τ is a minimal semi-optimal rule. This establishes existence of minimal semi-optimal rules and the implication τ optimal and strictly regular $\Rightarrow$ τ minimal semi-optimal. Uniqueness of minimal semi-optimal rules and existence of an optimal strictly regular rule enables us to prove the reverse implication.

DEFINITION 6. A regular rule $\tau \in T_\infty$ is said to be *maximal regular* iff for every regular $t \in T_\infty$, $P(\tau \geqq t) = 1$; or equivalently, iff for every $t \in T_\infty$ such that $P(\tau < t) > 0$, t is not regular.

COMMENT 2. Uniqueness (up to equivalence) of maximal regular rules follows directly from Definition 6.

THEOREM 4. *Assume A_1, A_2 and A_3 hold. Then there exists a unique (up to equivalence) maximal regular $\tau \in T_\infty$ and τ is optimal.*

PROOF. By Theorem 1 $\exists \tau_0 \in T_\infty \ni \tau_0$ is optimal. From Corollary 1, τ_0 is regular. If there exists a maximal regular $\tau \in T_\infty$, $P(\tau_0 \leqq \tau) = 1$; so Lemma 2 $\Rightarrow EY_\tau \geqq EY_{\tau_0} = \sup_{t \in T_\infty} EY_t \geqq EY_\tau \Rightarrow \tau$ is optimal.

To complete the proof of this theorem it will suffice to construct a maximal regular rule. Moreover, in this attempt we may restrict our attention to the collection $\mathscr{O}$ of optimal rules. We must take care to insure that the rule we construct is in T_∞ and hence measurable.

Let $\mathscr{O}_{n,r} = \{t \in \mathscr{O}\colon P(t > n) \geqq r\}$.

Let $r_n = \sup\{r : \mathscr{O}_{n,r} \neq \varnothing\}$.

Fix $k \geqq 1$. We assert that $\forall n \geqq 1$, $(\bigcap_{j=1}^n \mathscr{O}_{j,r_j-1/k}) \neq \varnothing$. $\mathscr{O}_{1,r_1-1/k} \neq \varnothing$. If $\bigcap_{j=1}^{n-1} \mathscr{O}_{j,r_j-1/k} \neq \varnothing$ $\exists t' \in \bigcap_{j=1}^{n-1} \mathscr{O}_{j,r_j-1/k}$. $\exists t'' \in \mathscr{O}_{n,r_n-1/k}$. Let $t = \max(t', t'')$. Being optimal, both t' and t'' are regular so (Lemma 3) t is regular and optimal. $\Rightarrow t \in \mathscr{O}$.

Let $t_0 \in T_\infty \ni P(t \geqq t_0) = 1$. Then $P(t > l) \geqq P(t_0 > l)$ for each l.

For $0 < l < n$, $P(t > l) \geqq P(t' > l) \geqq r_l - 1/k$. For $l = n$, $P(t > n) \geqq P(t'' > n) \geqq r_n - 1/k \Rightarrow t \in \bigcap_{j=1}^n \mathscr{O}_{j,r_j-1/k}$. Hence $\forall n \exists t_n \in \bigcap_{j=1}^n \mathscr{O}_{j,r_j-1/n}$. Let $\tau_1 = t_1$, $\tau_n = \max(\tau_{n-1}, t_n) = \max(t_1, \cdots, t_n)$. Let $\tau = \sup_n \tau_n$.

$$\{\tau = n\} = \bigcup_{m=1}^\infty \bigcap_{k=m}^\infty \{\tau_k = n\} \subset \mathscr{B}_n \Rightarrow \tau \in T_\infty .$$

$$\begin{aligned} EY_\tau &= EY_\tau 1_{\{\tau<\infty\}} + EY_\infty 1_{\{\tau=\infty\}} \\ &= E \limsup_n Y_{\tau_n} 1_{\{\tau<\infty\}} + EY_\infty 1_{\{\tau=\infty\}} \\ &\geqq \limsup_n EY_{\tau_n} 1_{\{\tau<\infty\}} + E \limsup_n Y_{\tau_n} 1_{\{\tau=\infty\}} \\ &\geqq \limsup_n EY_{\tau_n} . \end{aligned}$$

By Lemma 3, $EY_{\tau_n} \geqq EY_{\tau_{n-1}} \geqq EY_{\tau_1} \geqq EY_{t_1}$, $EY_{t_1} = \sup_{t \in T_\infty} EY_t$ since $t_1 \in \mathscr{O}$.

Therefore,

$$\sup_{t \in T_\infty} EY_t \geqq EY_\tau \geqq EY_{t_1} = \sup_{t \in T_\infty} EY_t \Rightarrow \tau \in \mathscr{O}.$$

Fix $k \geqq 1$. Choose $n > k$.

$$P(\tau > k) \geqq P(\tau_n > k) \geqq P(t_n > k) \geqq r_k - 1/n \Rightarrow P(\tau > k) \geqq r_k.$$

By construction of r_k, $t \in \mathscr{O} \Rightarrow P(t > k) \leqq r_k$. Therefore $P(\tau > k) = r_k$.

$\tau \in \mathscr{O} \Rightarrow \tau$ is regular. If τ is not maximal regular $\exists t' \in T_\infty \ni P(t' > \tau) > 0$ where t' is regular. Let $t = \max(\tau, t')$. $EY_t \geqq EY_\tau \Rightarrow t \in \mathscr{O}$. $P(t \geqq \tau) = 1$ and $\exists 0 < k < \infty \ni P(t > \tau = k) > 0$.

$$\begin{aligned} P(t > k) &= P(\tau > k) + P(\tau = k, t > k) \\ &> P(\tau > k) = r_k. \end{aligned}$$

But $P(t > k) > r_k \Rightarrow t \notin \mathscr{O}$ by construction of $r_k \Rightarrow \Leftarrow$ contradiction. Therefore τ is maximal regular.

THEOREM 5. *Assume A_1, A_2 and A_3. $\tau_1 \in T_\infty$ is minimal strictly semi-optimal* iff *τ_1 is maximal regular. Such a rule τ_1 exists, is unique (up to equivalence), and is optimal.*

PROOF. $\exists$ maximal regular $\tau \in T_\infty$ (Theorem 4). If τ is not strictly semi-optimal, $\exists t \in T_\infty$, $1 \leqq k < \infty$, and a $\mathscr{B}_k$-measurable set $A \subseteqq \{\tau = k, t > k\} \ni P(A) > 0$ and

$$EY_\tau 1_A \leqq EY_t 1_A.$$

Let $t' = \tau 1_{A^c} + t 1_A$. $t' \in T_\infty$ and $P(t' > \tau) > 0$. Evaluating $EY_{t'}$,

$$\begin{aligned} EY_{t'} &= EY_\tau 1_{A^c} + EY_t 1_A \\ &\geqq EY_\tau 1_{A^c} + EY_\tau 1_A \\ &= EY_\tau. \end{aligned}$$

The optimality of τ now forces t' to be optimal and hence regular, but then τ is not maximal regular. $\Rightarrow \Leftarrow$ contradiction. Therefore τ is strictly semi-optimal.

Now let $t \in T_\infty$ be any strictly semi-optimal rule. Let $A_k = \{t = k, \tau > k\}$. If $P(A_k) > 0$ for some $k \geqq 1$, then

$$EY_k 1_{A_k} > EY_\tau 1_{A_k} \Rightarrow \tau \quad \text{is not regular} \quad \Rightarrow \Leftarrow \text{contradiction.}$$

Therefore $P(\tau > t) = P(\bigcup_k A_k) = 0$ or

$$P(\tau \leqq t) = 1.$$

$\Rightarrow \tau$ is minimal strictly semi-optimal.

Since minimal strictly semi-optimal rules and maximal regular rules are unique up to equivalence whenever they exist, the entire theorem follows.

THEOREM 6. *Let A_1, A_2, and A_3 hold. Let τ_0 and τ_1 be the minimal semi-optimal and minimal strictly semi-optimal rules, respectively. Let $\mathscr{O}$ be the collection of optimal extended-valued stopping rules.*

Then

$$\mathcal{O} = \{t \in T_\infty : t \text{ is regular and } P(t \geqq \tau_0) = 1\}$$

and

$$\mathcal{O} = \{t \in T_\infty : t \text{ is semi-optimal and } P(t \leqq \tau_1) = 1\}\,.$$

PROOF. Let $t \in T_\infty$ be optimal. Then t is semi-optimal. τ_0 is minimal semi-optimal so $P(t \geqq \tau_0) = 1$. Also, $t \in \mathcal{O} \Rightarrow t$ is regular. $\Rightarrow \mathcal{O} \subseteqq \{t \in T_\infty : t$ is regular and $P(t \geqq \tau_0) = 1\}$. Lemma 2 is sufficient to guarantee the reverse inclusion.

Now let $t \in T_\infty$ be semi-optimal and assume $P(t \leqq \tau_1) = 1$.

$$\begin{aligned} EY_{\tau_1} &= \textstyle\sum_{1 \leqq n \leqq \infty} EY_{\tau_1} 1_{\{t=n\}} \leqq \sum_{1 \leqq n \leqq \infty} EY_n 1_{\{t=n\}} \\ &= EY_t \Rightarrow t \in \mathcal{O}\,. \end{aligned}$$

Therefore $\{t \in T_\infty : t$ is semi-optimal and $P(t \leqq \tau_1) = 1\} \subseteqq \mathcal{O}$.

Finally, let $t \in \mathcal{O}$. Then t is regular. By Lemma 3, $\tau = \max(t, \tau_1)$ is regular. τ_1 is maximal regular $\Rightarrow P(\tau = \tau_1) = 1 \Rightarrow P(t \leqq \tau_1) = 1$. t is optimal $\Rightarrow t$ is semi-optimal $\Rightarrow$

$$\mathcal{O} = \{t \in T_\infty : t \text{ is semi-optimal and } P(t \leqq \tau_1) = 1\}\,.$$

2. Form of optimal rules for $E(S_t/t)$. With the tools involving the nature of optimal rules at our disposal, we are ready to attack the problem which prompted this research, that of determining when (optimal) rules for maximizing $E(S_t/t)$ stop with probability one.

For the remainder of this paper, unless explicitly stated to the contrary, we assume that $\{X_n\}$ is a sequence of nondegenerate independent identically distributed random variables with mean zero. Our returns $Y_n = y_n(x_1, \cdots, x_n)$ will have the form $(b + x_1 + \cdots + x_n)/(c + n)$ for some $c > 0$ and b real (c and b are fixed for all n). By the strong law of large numbers, $Y_n \to 0$ a.s. So we define $Y_\infty \equiv 0$. For $t \in T_\infty$, EY_t is unambiguously defined and equals $EY_t 1_{\{t<\infty\}}$. Define $S_n = X_1 + \cdots + X_n$.

LEMMA 6. *If for some* $c > 0$, *b real, and* $t' \in T_\infty$, $E(b + S_{t'})/(c + t') \geqq b/c$ *then* $\exists\, t \in T_\infty \ni$

$$t < \infty \Rightarrow S_t > -b \qquad \textit{and} \tag{10}$$

$$E\frac{b + S_t}{c + t} \geqq \frac{b}{c}\,. \tag{11}$$

PROOF. Set $t = t'$ if $t' < \infty$ and $S_{t'} > -b$; $t = \infty$ otherwise.

LEMMA 7. *Let* $c > 0$, *b real and* $t \in T_\infty$ *satisfy* (10) *and* (11). *Assume also that* $P(t < \infty) > 0$. *Then if* $c_1 \geqq c$ *and* $b_1 \leqq b$ *we have*

$$E\frac{b_1 + S_t}{c_1 + t} \geqq \frac{b_1}{c_1} \tag{12}$$

with strict inequality if $b_1 < b$ *or* $c_1 > c$.

PROOF. We may restrict our attention to the set where $t < \infty$. On this set, $c(b + S_t)/(c + t)$ is a strictly increasing function of $c > 0$. Therefore

$$c_1 \geqq c \Rightarrow E\frac{c_1(b + S_t)}{c_1 + t} \geqq E\frac{c(b + S_t)}{c + t} \geqq b \Rightarrow E\frac{b + S_t}{c_1 + t} \geqq \frac{b}{c_1}$$

with strict inequality for $c_1 > c$.

Now take the derivative of $E(b + S_t)/(c_1 + t) - b/c_1$ with respect to b to obtain $E(1/(c_1 + t)) - 1/c_1 < 0$; therefore

$$b_1 < b \Rightarrow E\frac{b_1 + S_t}{c_1 + t} - \frac{b_1}{c_1} > E\frac{b + S_t}{c_1 + t} - \frac{b}{c_1} \geqq 0 .$$

The result now follows immediately.

REMARK 1. Lemma 7 is a slight modification of the statement of Lemma 5 of Dvoretzky [11] and the proof, while substantially the same, is in a more accurate order.

Henceforth unless specifically stated to the contrary, we make the additional assumption that $EX_1^+ \log^+ X_1 < \infty$, where for a random variable Z,

$$\begin{aligned} Z^+ &= Z \quad \text{if} \quad Z \geqq 0 \qquad \text{and} \qquad & \log^+ Z &= \log Z \quad \text{if} \quad Z \geqq 1 \\ &= 0 \quad \text{if} \quad Z < 0 & &= 0 \qquad \text{otherwise.} \end{aligned}$$

All logarithms are taken with respect to the base e.

The latter assumption was found by D. L. Burkholder [1], to be necessary and sufficient to guarantee that

$$E \sup_n \frac{S_n}{n} < \infty , \tag{13}$$

while B. J. McCabe and L. A. Shepp [17] demonstrated that the condition $EX_1^+ \log^+ X_1 < \infty$ was also equivalent to the statement $\sup_{t \in T_\infty} E(S_t/t) < \infty$. (Note: [1] and [17] also show that $EX_1^+ \log^+ X_1$ is equivalent to (a) $E \sup X_n/n < \infty$ and to (b) $\sup_{t \in T_\infty} E(X_t/t) < \infty$.)

Fix a, n where $n \geqq 0$. Let $Y_k = (a + S_k)/(n + k)$, $k = 1, 2, \cdots$, $Y_\infty \equiv 0$. $Y_\infty = 0 = \lim_{k\to\infty} Y_k$. Therefore

A_2 holds. $E \sup_k Y_k \leqq \frac{|a|}{n} + E \sup_k \frac{S_k}{k} < \infty$ by (13).

Thus A_1 is valid. Define

$$M_n(a) = \sup_{t \in T_\infty} E\frac{a + S_t}{n + t} . \tag{14}$$

Applying Theorem 1, $\exists \tau(a, n) \in T_\infty$ such that

$$E\frac{a + S_\tau}{n + \tau} = M_n(a) . \tag{15}$$

LEMMA 8. *$M_n(a)$ is a positive continuous strictly increasing function of a for each $n \geqq 0$.*

PROOF. Fix a. Let

$$t = \min\{k : S_k > -a\}$$
$$= \infty \quad \text{if} \quad S_k \leqq -a \,\forall k$$

X_1 is nondegenerate, so $P(t < \infty) = 1$.

$$M_n(a) \geqq E\frac{a + S_t}{n + t} > 0\,, \tag{16}$$

so $M_n(a)$ is a positive function. Consider n fixed. For $b < a$, $\exists$, by (15), $\tau_b \in T_\infty$ and $\tau_a \in T_\infty \ni M_n(b) = E(b + S_{\tau_b})/(n + \tau_b)$ and $M_n(a) = E(a + S_{\tau_a})/(n + \tau_a)$.

$$M_n(a) - M_n(b) \leqq E\frac{a + S_{\tau_a}}{n + \tau_a} - E\frac{b + S_{\tau_a}}{n + \tau_a} = E\frac{a - b}{n + \tau_a}$$

$$M_n(a) - M_n(b) \leqq \frac{a - b}{n + 1} \tag{17}$$

$$M_n(a) - M_n(b) \geqq E\frac{a + S_{\tau_b}}{n + \tau_b} - E\frac{b + S_{\tau_b}}{n + \tau_b} = E\frac{a - b}{n + \tau_b}$$

$$M_n(a) - M_n(b) > 0 \qquad \text{due to the fact that} \tag{18}$$

$$M_n(b) > 0 \Rightarrow P(\tau_b < \infty) > 0\,. \tag{19}$$

(18) says that $M_n(a)$ is a strictly increasing function of a, which coupled with (17) establishes the continuity of $M_n(a)$.

LEMMA 9. *If* $\exists 0 < n_1 < n_2 < \cdots$ *and* $\{b_{n_k}\}$ *such that* $M_{n_k}(b_{n_k}) \leqq b_{n_k}/n_n$, *then* $\forall n \geqq 1$, *there exists a unique number* a_n *such that*

$$M_n(a_n) = \frac{a_n}{n}\,. \tag{20}$$

PROOF. Fix $k \geqq 1$. For $0 < j \leqq n_k$, $\exists \tau_j \in T_\infty$ such that

(i) $M_j(b_{n_k}) = (b_{n_k} + S_{\tau_j})/(j + \tau_j)$ (see (15));
(ii) $\tau_j < \infty \Rightarrow b_{n_k} + S_{\tau_j} > 0$ (see (10));
(iii) $P(\tau_j < \infty) > 0$. (see (19)).

If for some $0 < j < n_k$, $M_j(b_{n_k}) > b_{n_k}/j$, then by Lemma 7

$$\frac{b_{n_k}}{n_k} < E\frac{b_{n_k} + S_{\tau_j}}{n_k + \tau_j} \leqq E\frac{b_{n_k} + S_{\tau_{n_k}}}{n_k + \tau_{n_k}} = M_{n_k}(b_{n_k})$$

$\Rightarrow \Leftarrow$ contradiction. Therefore $\forall n \geqq 1\ \exists b_n' \ni M_n(b_n') \leqq (b_n'/n)$. Let $g_n(x) = M_n(x) - (x/n) \cdot g_n$ is continuous. $g_n(0) = M_n(0) > 0$. $g_n(b_n') = M_n(b_n') - (b_n'/n) \leqq 0$. Thus there exists an a_n such that $g_n(a_n) = 0$, or $M_n(a_n) = (a_n/n)$. If $\exists a_n' < a_n \ni M_n(a_n') = (a_n'/n)$, then $M_n(a_n) - M_n(a_n') = (a_n - a_n')/n > (a_n - a_n')/(n + 1)$, contradicting (17). Hence a_n is unique. Moreover, $M_n(a_n) > 0 \Rightarrow a_n > 0$.

REMARK 2. Observe that Lemma 7 yields: $a < a_n \Rightarrow M_n(a) > a/n$ whereas

$a > a_n \Rightarrow M_n(a) < a/n$, wherever a_n exists. Furthermore, if a_n and a_{n+1} are defined, then if $M_n(a_{n+1}) \geqq a_{n+1}/n$ then $M_{n+1}(a_{n+1}) > (a_{n+1})/(n+1)$, contradicting the construction of $a_{n+1} \Rightarrow M_n(a_{n+1}) < (a_{n+1})/n \Rightarrow a_{n+1} > a_n$.

In order to establish the existence of $\{b_{n_k}\}$ as in Lemma 9 we need some elementary estimates.

LEMMA 10. *Let* $\{X_n\}$ *be* i.i.d. *with mean zero. Then* $E|S_n| = o(n)$.

PROOF. Fix $\varepsilon > 0$.

$$\int_{\{S_n/n>\varepsilon/2\}} S_n \, dP = n \int_{\{S_n/n>\varepsilon/2\}} X_1 \, dP = o(n) \quad \text{because} \quad P\left(\frac{S_n}{n} > \frac{\varepsilon}{2}\right) \to 0 \, .$$

Therefore $\int_{\{|S_n|/n>\varepsilon/2\}} (|S_n|/n) \, dP \to 0$ as $n \to \infty$;

$$E\frac{|S_n|}{n} = \int_{\{|S_n|/n\leqq\varepsilon/2\}} \frac{|S_n|}{n} dP + \int_{\{|S_n|/n>\varepsilon/2\}} \frac{|S_n|}{n} dP$$

$$\leqq \frac{\varepsilon}{2} + o(1)$$

$$\Rightarrow \lim_{n\to\infty} E\frac{|S_n|}{n} = 0 \, .$$

LEMMA 11. *Let* $\{Z_n^+\}$ *be a submartingale. Then for any* $u > 0$,

$$P(\max_{k\leqq n} Z_k \geqq u) \leqq \frac{EZ_n^+}{u} \, . \qquad \text{See Doob [10] page 314} \, .$$

PROOF. Let $\Lambda_k = \{\omega \mid Z_k \geqq u, \, Z_i < u \text{ for } i < k\}$

$$\Lambda \equiv \{\omega \mid \max_{k\leqq n} Z_k(\omega) \geqq u\} = \bigcup_{k=1}^n \Lambda_k \, .$$

$$P(\Lambda) = \sum_{k=1}^n P(\Lambda_k) = \sum_{k=1}^n \int_{\Lambda_k} 1 \, dP$$

$$\leqq \sum_{k=1}^n \int_{\Lambda_k} \frac{Z_k}{u} dP \leqq \sum_{k=1}^n \int_{\Lambda_k} \frac{Z_n^+}{u} dP$$

$$= \int_\Lambda \frac{Z_n^+}{u} dP \leqq \frac{EZ_n^+}{u} \, .$$

LEMMA 12. *Let* $\{X_n\}$ *be* i.i.d. *nondegenerate mean zero random variables. Fix* $\alpha > 0$. *Define*

$$t_n = \text{1st} \quad k : S_k > \alpha n \qquad \textit{if such } k \textit{ exists}$$
$$= \infty \qquad \textit{otherwise.}$$

Then $E(n + t_n)^{-1} = o(n^{-1})$.

PROOF. $t_n \in T_\infty$ and $P(t_n < \infty) = 1$.

$$E\frac{1}{n+t_n} = \sum_{k=1}^{Mn} \frac{P(t_n = k)}{n+k} + \sum_{k>Mn} \frac{P(t_n = k)}{n+k}$$

$$\leqq \sum_{k=1}^{Mn} \frac{P(t_n = k)}{n} + \sum_{k>Mn} \frac{P(t_n = k)}{n(M+1)}$$

$$\leqq \frac{P(t_n \leqq Mn)}{n} + \frac{1}{Mn}$$

$$= \frac{P(\max_{j \leqq Mn} S_j > \alpha n)}{n} + \frac{1}{Mn}$$

$$\leqq \frac{ES^+_{Mn}}{\alpha n^2} + \frac{1}{Mn} \qquad \text{by Lemma 11}$$

$$\leqq \frac{M}{\alpha n}\frac{E|S_n|}{n} + \frac{1}{Mn}.$$

Fix $\varepsilon > 0$. Choose M so large that $M^{-1} < \frac{1}{2}\varepsilon$. $\lim_n (M/\alpha)(E|S_n|/n) = 0$ by Lemma 10 $\Rightarrow E(n + t_n)^{-1} < \varepsilon/n$ for n large. □

We are now prepared to establish the existence of a unique root a_n of the function

$$g_n(z) = M_n(z) - z/n. \tag{21}$$

LEMMA 13. *Let $\{X_n\}$ be* i.i.d. *with mean zero and $EX_1^+ \log^+ X_1 < \infty$, then for each $n \geqq 1$ there is a unique number a_n such that $M_n(a_n) = (a_n)/n$.*

PROOF. Referring to (13), $\exists K < \infty \ni 0 < \sup_{t \in T_\infty} E(S_t)/t \leqq K$. Put $b_n = 4Kn$.

$$\exists \tau_n \in T_\infty \ni M_n(b_n) = E\frac{b_n + S_{\tau_n}}{n + \tau_n}.$$

If $M_n(b_n) \leqq b_n/n$ for infinitely many n, we are done (by Lemma 9). Otherwise $\exists N \ni n \geqq N \Rightarrow M_n(b_n) > b_n/n$. Lemma 7 gives

$$M_{n+k}(a) > \frac{a}{n+k} \quad \text{for} \quad a \leqq b_n \qquad \text{and}$$

$$\frac{b_n}{2n} < E\frac{(b_n/2) + S_{\tau_n}}{n + \tau_n} \leqq M_n\left(\frac{b_n}{2}\right). \tag{22}$$

$$\exists \tau_n^* \in T_\infty \ni M_n\left(\frac{b_n}{2}\right) = E\frac{(b_n/2) + S_{\tau_n^*}}{n + \tau_n^*}.$$

Let $A_{n,k} = \{\tau_n^* = k, S_k \leqq b_n/2\}$. We show $P(A_{n,k}) = 0$ for $n \geqq N$, $k > 0$. By (22),

$$\sup_{t \in T_\infty} E\left(\frac{(b_n/2) + S_{\tau_n^*+t}}{n + \tau_n^* + t}\,\middle|\, X_1 = x_1, \cdots, X_k = x_n, A_{n,k}\right) \tag{23}$$

$$> \frac{(b_n/2) + S_k}{n + k} \quad \text{since} \quad \frac{b_n}{2} + S_k \leqq b_n.$$

We must conclude, therefore, that if for some $n \geqq N$ there is a $k > 0$ such that $P(A_{n,k}) > 0$, τ_n^* is not semi-optimal. $\Rightarrow$ (by Lemma 4) τ_n^* is not optimal. $\Rightarrow \Leftarrow$ contradiction. Therefore

$$\tau_n^* < \infty \Rightarrow S_{\tau_n^*} > \frac{b_n}{2} \quad \text{a.s.} \tag{24}$$

Now

$$E\frac{(b_n/2)+S_{\tau_n^*}}{n+\tau_n^*} > \frac{b_n}{2n} \Rightarrow 2KnE\frac{1}{n+\tau_n^*} > 2K - E\frac{S_{\tau_n^*}}{n+\tau_n'^*}$$
$$\Rightarrow 2KnE\frac{1}{n+\tau_n^*} > 2K - K$$
$$\Rightarrow E\frac{1}{n+\tau_n^*} > \frac{1}{2n}.$$

Let

$$\begin{aligned} t_n &= \text{1st} \quad k: S_k > 2Kn \qquad &&\text{if such } k \text{ exists} \\ &= \infty &&\text{otherwise.} \end{aligned}$$

$$P(t_n \leqq \tau_n^*) = 1 \quad \text{from (24)} \Rightarrow E\frac{1}{n+t_n} \geqq E\frac{1}{n+\tau_n^*} > \frac{1}{2n}.$$

We now invoke Lemma 12, obtaining

$$E\frac{1}{n+t_n} = o\left(\frac{1}{n}\right). \Rightarrow \Leftarrow \text{contradiction.}$$

Now apply Lemma 9 to obtain the result asserted.[2]

Intuitively, our considerations illustrate that if we have accumulated amount $a > a_n$ by time n, we should continue. whereas if we have accumulated $a > a_n$ by time n, we should stop. We are indifferent if $a = a_n$ at time n. We state this more explicitly and precisely.

THEOREM 7. *Let* $\{X_n\}$ *be* i.i.d. *nondegenerate mean zero random variables with the property that* $EX_1^+ \log^+ X_1 < \infty$ *and let* $\{a_n\}$ *be the unique sequence of numbers defined in accordance with* (21). *Then up to equivalence the collection* $C_{a,n}$ *of extended-valued stopping rules which maximize* $E(a + S_t)/(n + t)$ *over all* $t \in T_\infty$ *can be defined as follows*: $\tau \in C_{a,n}$ iff $\tau \in T_\infty$ *and*

(i) $a + S_k > a_{n+k} \Rightarrow \tau \leqq k$;

(ii) $a + S_j < a_{n+j}$ *for* $j = 1, 2, \cdots, k \Rightarrow \tau > k$;

(iii) $a + S_j \leqq a_{n+j}$ *for* $j = 1, 2, \cdots, k - 1, a + S_k = a_{n+k}$ *and* $\tau \geqq k \Rightarrow \tau = k$ *with probability* P_k *(determined by the mean of a zero-one valued random variable*

[2] Lemma 13 can be proved without Lemmas 10–12 as follows: There exists $\tau \in T_\infty \ni E(S_\tau/\tau) = \sup_{t \in T_\infty} E(S_t/t)$. $\exists 0 < n_1 < n_2 < \cdots$ such that $P(\tau = n_k) > 0$. (If not, $\exists N < \infty \ni P(\tau \leqq N) + P(\tau = \infty) = 1$. Letting

$$\begin{aligned} \tau' &= \tau &&\text{if } \tau \leqq N \\ &= \text{1st} \quad k > N: S_k > 0 &&\text{if } \tau > N \end{aligned}$$

$\tau' \in T_\infty$ and $E(S_{\tau'}/\tau') > E(S_\tau/\tau)$ unless $P(\tau = \infty) = 0$. But if $P(\tau = \infty) = 0$ then we cannot have $P(\tau \leqq N) = 1$ because $\tau < \infty \Rightarrow S_\tau > 0$ (since τ is optimal) $\Rightarrow P(\bigcup_{j=1}^N \{S_j > 0\}) = 1 \Rightarrow P(\bigcap_{j=1}^N \{X_j \leqq 0\}) = 0$, which is impossible since $EX_1 = 0$). Let $B_k = \{b: P(\tau = n_k, S_\tau \leqq b) > 0\}$. $P(\tau = n_k) > 0$ so B_k is nonempty. Choose $b_k \in B_k$. τ is semi-optimal so $\sup_{t \in T_\infty} E(S_{t+\tau}/(t + \tau) \mid \tau = n_k, S_\tau) \leqq S_{n_k}/n_k$. Therefore (Lemma 7 and (15)) $M_{n_k}(b_k) \leqq b_k/n_k$ from which Lemma 13 follows from Lemma 9.

W_k which is independent of $\{X_{k+1}, X_{k+2}, \cdots\}$ but otherwise W_k is arbitrary) and $\tau > k$ with probability $1 - P_k$.

PROOF. Let τ_1 be the first $k \ni a + S_k > a_{n+k}$, if such k exists; $\tau_1 = \infty$ otherwise. τ_1 is strictly semi-optimal. Given $t \in T_\infty \ni P(t < \tau_1) > 0, \exists k \ni P(\tau_1 > t = k) > 0$. $\tau_1 > k \Rightarrow a + S_k \leqq a_{n+k} \Rightarrow t$ is not strictly semi-optimal; therefore τ_1 is minimal strictly semi-optimal $\Rightarrow \tau_1 \in C_{a,n}$, (see Theorem 6).

Let

$$\begin{aligned} \tau_0 &= k \quad \text{if} \quad a + S_k \geqq a_{n+k}, \quad a + S_j < a_{n+j} \quad \text{for} \quad j < k \\ &= \infty \quad \text{if no such } k \text{ exists} \end{aligned}$$

τ_0 is semi-optimal.

Let $t \in T_\infty \ni P(t < \tau_0) > 0$.

$$\exists 0 < k < \infty \ni P(\tau_0 > t = k) > 0 .$$

$\tau_0 > k \Rightarrow a + S_k < a_{n+k} \Rightarrow t$ is not semi-optimal $\Rightarrow \tau_0$ is minimal semi-optimal $\Rightarrow$ (see Theorem 3 or 6) τ_0 is optimal or $\tau_0 \in C_{a,n}$.

A moment's reflection will convince one that those $t \in T_\infty$ which are semi-optimal and for which $P(t \leqq \tau_1) = 1$, since they must also satisfy $P(t \geqq \tau_0) = 1$, are in $C_{a,n}$. Furthermore, the only $t \in C_{a,n}$ are semi-optimal rules such that $P(t \leqq \tau_1) = 1$. By Theorem 6, all optimal rules are contained in $C_{a,n}$ (up to equivalence) and every rule in $C_{a,n}$ is optimal.

Note. It is necessary that W_k be independent of $\{X_{k+1}, X_{k+2}, \cdots\}$ to insure that τ is a bona-fide member of T_∞.

Except for two auxiliary lemmas and a corollary, we can finally proceed with our analysis of when $P(\tau < \infty) = 1$, where $E(S_\tau/\tau) = \sup_{t \in T_\infty} E(S_t/t)$, and $\tau \in T_\infty$.

3. A necessary condition for optimal rules to assume the value $+\infty$ with positive probability.

LEMMA 14. *Let $\{Z_n\}$ be any sequence of random variables and let $\{b_n\}$ be a sequence of positive reals satisfying only $b_n \leqq b_{n+1} \forall n$ and $\lim_{n\to\infty} b_n = \infty$. Then $P(Z_n \geqq b_n$ i.o.$) = P(\max_{j\leqq n} Z_j \geqq b_n$ i.o.$)$ and $P(Z_n > b_n$ i.o.$) = P(\max_{j\leqq n} Z_j > b_n$ i.o.$)$.*

PROOF. Let

$$\begin{aligned} A &= \{\omega \mid Z_n(\omega) \geqq b_n \ \text{ i.o.}\} \\ B &= \{\omega \mid \max_{j\leqq n} Z_j \geqq b_n \ \text{ i.o.}\} . \end{aligned}$$

Clearly $A \subseteq B$. W log assume $B \neq \emptyset$. Let $\omega \in B$. If $\omega \notin A \ \exists N < \infty \ni n \geqq N \Rightarrow Z_n(\omega) < b_n$.

Let $M = \max\{Z_1(\omega), \cdots, Z_N(\omega)\}$. $M < \infty$. $\exists N' \ni b_{N'} > M$.

$$\omega \in B \Rightarrow \exists n' \geqq N' \ni \max_{j\leqq n'} Z_j(\omega) \geqq b_{n'} \geqq b_{N'} > M .$$

Hence $\exists k \leqq n' \ni Z_k(\omega) \geqq b_{n'} \geqq b_k \Rightarrow k \leqq N \Rightarrow Z_k(\omega) \leqq M$ but $Z_k(\omega) \geqq b_n' \Rightarrow \Leftarrow$ contradiction, therefore

$$\omega \in A \Rightarrow A = B \Rightarrow P(A) = P(B) .$$

The second assertion is proved similarly. We prove this lemma because we will require the use of a corollary.

COROLLARY 2. *Let* $\{Z_n\}$ *be a sequence of random variables and* $\{b_n\}$ *a sequence of positive real numbers such that* $b_n \leqq b_{n+1}\ \forall\, n$ *and* $\lim_{n\to\infty} b_n = \infty$. *Then* $P(Z_n \geqq b_n$ i.o.$) = 0 \Rightarrow \lim_{n\to\infty} P(\max_{j\leqq n} Z_j \geqq b_n) = 0$, *and*

$$P(Z_n > b_n \text{ i.o.}) = 0 \Rightarrow \lim_{n\to\infty} P(\max_{j\leqq n} Z_j > b_n) = 0 .$$

PROOF.

$$\begin{aligned} P(\max_{j\leqq n} Z_j \geqq b_n) &\leqq P(\textstyle\bigcup_{m=n}^{\infty} \{\max_{j\leqq m} Z_j \geqq b_m\}) \\ &\Rightarrow \lim_{n\to\infty} P(\max_{j\leqq n} Z_j \geqq b_n) \leqq \lim_{n\to\infty} P(\textstyle\bigcup_{m=n}^{\infty} \{\max_{j\leqq m} Z_j \geqq b_m\}) \\ &= P(\max_{j\leqq n} Z_j \geqq b_n \text{ i.o.}) \\ &= P(Z_n \geqq b_n \text{ i.o.}) \qquad \text{by Lemma 14} \\ &= 0 . \end{aligned}$$

The other assertion is proved analogously.

We now reemphasize that $\{X_n\}$ will denote a fixed sequence of nondegenerate independent identically distributed mean zero random variables satisfying $EX_1^+ \log^+ X_1 < \infty$. We let $(\Omega, \mathscr{F}, P)$ be the probability space on which $\{X_n)$ is defined and let F be the distribution function of X_1. $\{a_n\}$ will be the unique sequence of roots of the functions

$$g_n(z) = \sup_{t\in T_\infty} E\left(\frac{z+S_t}{n+t}\right) - \frac{z}{n} .$$

Recall $S_n = X_1 + \cdots + X_n$. Remark 2, following Lemma 9, affirms that

$$0 < a_1 < a_2 < \cdots . \tag{25}$$

Our next goal is to determine what $P(S_n \geqq a_n \text{ i.o.}) = 0$ implies about F. Toward this end, define

$$\begin{aligned} t_n^0 &= k \qquad \text{if} \quad S_k \geqq a_{n+k}, \quad S_j < a_{n+j} \quad \text{for} \quad j < k \\ &= \infty \qquad \text{if no such } k \text{ exists.} \end{aligned}$$

Theorem 7 can be utilized to verify that

$$E\,\frac{S_{t_n^0}}{n+t_n^0} = \sup_{t\in T_\infty} E\,\frac{S_t}{n+t} . \tag{26}$$

Define

$$\begin{aligned} t_n' &= \text{1st} \quad k : S_k \geqq a_{n+k} - \tfrac{1}{2}a_n \\ &= \infty \qquad \text{if no such } k \text{ exists.} \end{aligned}$$

Again by Theorem 7,

$$E\,\frac{\frac{1}{2}a_n + S_{t_n'}}{n+t_n'} = \sup_{t\in T_\infty} E\,\frac{\frac{1}{2}a_n + S_t}{n+t} . \tag{27}$$

LEMMA 15.

$$\frac{ES_n^+}{2n} < E\frac{S_{t_n^0}}{n+t_n^0} < \frac{a_n}{n}.$$

PROOF.

$$E\frac{S_{t_n^0}}{n+t_n^0} < \sup_{t\in T_\infty} E\frac{a_n+S_t}{n+t} = \frac{a_n}{n}.$$

Now for the left-hand inequality: Fix $n \geqq 1$. Let

$$\begin{aligned} t' &= \text{1st} \quad k \geqq n: S_k > 0 \qquad \text{if such } k \text{ exists} \\ &= \infty \qquad\qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

X_1 is nondegenerate so $P(t' < \infty) = 1$,

$$\begin{aligned} E\frac{S_{t_n^0}}{n+t_n^0} &= \sup_{t\in T_\infty} E\frac{S_t}{n+t} \geqq E\frac{S_{t'}}{n+t'} \\ &= \sum_{k=n}^\infty \int_{\{t'=k\}} \frac{S_k\,dP}{n+k} > \int_{\{t'=n\}} \frac{S_n}{2n}\,dP = E\frac{S_n^+}{2n}. \end{aligned}$$

LEMMA 16.

$$\frac{a_n}{8n} < E\frac{S_{t_n^0}}{n+t_n^0}.$$

PROOF. *Case* 1. $a_n \leqq 4ES_n^+$.

$$\frac{a_n}{8n} \leqq \frac{ES_n^+}{2n} < E\frac{S_{t_n^0}}{n+t_n^0} \qquad \text{by Lemma 15.}$$

Case 2. $a_n > 4ES_n^+$. Remark 2 following Lemma 9 gives

$$\begin{aligned} \frac{a_n}{2n} &< \sup_{t\in T_\infty} E\frac{(a_n/2)+S_t}{n+t} = E\frac{(a_n/2)+S_{t_n'}}{n+t_n'} \\ &< E\frac{S_{t_n'}}{n+t_n'} + \frac{a_n}{2n}P(t_n' < n) + \frac{a_n}{4n}P(t_n' \geqq n) \\ &= E\frac{S_{t_n'}}{n+t_n'} + \frac{a_n}{4n}(1+P(t_n' < n)) \end{aligned}$$

$$\begin{aligned} P(t_n' < n) &= P\left(\bigcup_{k<n}\left\{S_k \geqq a_{n+k} - \frac{a_n}{2}\right\}\right) \\ &\leqq P\left(\bigcup_{k<n}\left\{S_k > \frac{a_n}{2}\right\}\right) \qquad \text{since } 0 < a_1 < a_2 < \cdots \\ &\leqq \frac{2ES_n^+}{a_n} \qquad \text{by Lemma 11} \\ &< \tfrac{1}{2} \end{aligned}$$

Therefore

$$E\frac{S_{t_n'}}{n+t_n'} > \frac{a_n}{n}\left(\frac{1}{2} - \frac{1}{4}\cdot\frac{3}{2}\right) = \frac{a_n}{8n}$$

$$E\frac{S_{t_n^0}}{n+t_n^0} \geqq E\frac{S_{t_n'}}{n+t_n'} \qquad \text{so the proof is complete.}$$

COROLLARY 3.
$$\frac{8a_n}{n} > \frac{a_{n+k}}{n+k} \qquad \textit{for all } n, k \geqq 1 .$$

PROOF.
$$\frac{a_{n+k}}{8(n+k)} < E\frac{S^0_{t_{n+k}}}{n+k+t^0_{n+k}} < \sup_{t\in T_\infty} E\frac{S_t}{n+t} < \sup_{t\in T_\infty} E\frac{a_n+S_t}{n+t} = \frac{a_n}{n};$$
therefore $8a_n/n > a_{n+k}/(n+k)$.

LEMMA 17. *Assume* $P(S_n \geqq a_n \text{ i.o.}) = 0$. *Then*
$$\int_{a_n}^\infty x \log\frac{a^{-1}(x)}{n}\, dF(x) \geqq \frac{a_n}{8n} - o\left(\frac{a_n}{n}\right).$$

PROOF. a_n is a strictly increasing function. Extend this to a strictly increasing continuous function $a(\cdot)$ defined for $x \geqq 0$. Thus the functional inverse $a^{-1}(\cdot)$ of $a(\cdot)$ is uniquely determined.

$$\begin{aligned}
\frac{a_n}{8n} < E\frac{S_{t_n^0}}{n+t_n^0} &< E\frac{a_{n+t_n^0}+X_{t_n^0}}{n+t_n^0} \\
&= E\frac{a_{n+t_n^0}}{n+t_{n^0}} + E\frac{X_{t_n^0}1_{\{X_{t_n^0}\leqq a_{n+t_n^0}\}}}{n+t_n^0} \\
&\quad + \sum_{k=1}^\infty \frac{\int_{\{t_n^0=k,\, X_k>a_{n+k}\}} X_k\, dP}{n+k} \\
&\leqq 2E\frac{a_{n+t_n^0}}{n+t_n^0} + \sum_{k=1}^\infty \frac{\int_{a_{n+k}}^\infty x\,dF(x)}{n+k} \\
&< 16\frac{a_n}{n}P(t_n^0<\infty) + \sum_{k=n+1}^\infty \sum_{j=k}^\infty \frac{\int_{a_j}^{a_{j+1}} x\,dF(x)}{k} \\
&= 16\frac{a_n}{n}P(t_n^0<\infty) + \sum_{j=n+1}^\infty \left(\sum_{k=n+1}^j \frac{1}{k}\right)\int_{a_j}^{a_{j+1}} x\,dF(x) \\
&\leqq 16\frac{a_n}{n}P(t_n^0<\infty) + \sum_{j=n+1}^\infty \log\frac{j}{n}\int_{a_j}^{a_{j+1}} x\,dF(x) \\
&\leqq 16\frac{a_n}{n}P(t_n^0<\infty) + \sum_{j=n+1}^\infty \int_{a_j}^{a_{j+1}} x\log\frac{a^{-1}(x)}{n}\,dF(x) \\
&\leqq 16\frac{a_n}{n}P(t_n^0<\infty) + \int_{a_n}^\infty x\log\frac{a^{-1}(x)}{n}\,dF(x)
\end{aligned}$$

$$\text{(28)} \qquad \begin{aligned} P(t_n^0<\infty) = P\left(\bigcup_{k=1}^\infty\{S_k \geqq a_{n+k}\}\right) &\leqq P\left(\bigcup_{k=1}^n\{S_k \geqq a_n\}\right) \\ &\quad + P\left(\bigcup_{k=n+1}^\infty\{S_k \geqq a_k\}\right) \to 0+0 \end{aligned}$$

by Corollary 2 and assumption, respectively. Therefore
$$\int_{a_n}^\infty x \log\frac{a^{-1}(x)}{n}\, dF(x) > \frac{a_n}{8n} - o\left(\frac{a_n}{n}\right).$$

We examine the content of Lemma 17 somewhat more closely. Assuming that $P(S_n \geqq a_n \text{ i.o.}) = 0$, then for $\varepsilon > 0$ and y sufficiently large,

(29) $$\int_y^\infty x \log \frac{a^{-1}(x)}{a^{-1}(y)}\, dF(x) \geqq \frac{y}{(8+\varepsilon)a^{-1}(y)};$$

$$(8+\varepsilon)a^{-1}(y) \geqq \frac{y}{\int_y^\infty x \log (a^{-1}(x)/a^{-1}(y))\, dF(x)}.$$

Now if $P(X_n \geqq a_n \text{ i.o.}) = 0$, then

$$\textstyle\sum_n P(X_n \geqq a_n) < \infty \qquad \text{(Borel–Cantelli Lemma)}$$

which implies

$$\textstyle\sum_n P(X_1 \geqq a_n) < \infty \qquad (X_n\text{'s identically distributed})$$

and

$$\textstyle\sum_n P(a^{-1}(X_1) \geqq n) < \infty \qquad (a^{-1}(\cdot) \text{ is an increasing function})$$

and therefore

(30) $$Ea^{-1}(X_1)1_{\{X_1 \geqq 0\}} < \infty .$$

Using (29) it would then follow that

$$\int_0^\infty \frac{dF(y)}{\int_y^\infty (x/y) \log (a^{-1}(x)/a^{-1}(y))\, dF(x)} < \infty .$$

Unfortunately this integral is not independent of $a(x)$.

Being interested in what $P(S_n \geqq a_n \text{ i.o.}) = 0$ implies about the distribution of X_1, we seek an estimate of $a^{-1}(x)/a^{-1}(y)$ for $x > y$ in terms of x/y. We find in fact that such an estimate allows us to prove that

$$\int_0^\infty \frac{dF(y)}{\int_y^\infty (x/y) \log (x/y)\, dF(x)} < \infty .$$

LEMMA 18. *If* $P(S_n \geqq a_n \text{ i.o.}) = 0$, *then* $P(X_n \geqq 5a_n \text{ i.o.}) = 0$.

PROOF.

$$P(-S_{n-1} > 2ES_n^-) \leqq \frac{ES_{n-1}^-}{2ES_n^-} < \tfrac{1}{2}$$

$$ES_n^- = ES_n^+ .$$

$$\begin{aligned} P(S_{n-1} \geqq -2ES_n^+) &= 1 - P(S_{n-1} < -2ES_n^+) \\ &= 1 - P(-S_{n-1} > 2ES_n^-) > \tfrac{1}{2} . \end{aligned}$$

Hence if $X_n \geqq 5a_n$ i.o., then $S_n \geqq 5a_n - 2ES_n^+$ i.o. $a_n > ES_n^+/2$ so

$$S_n \geqq a_n \quad \text{i.o.,} \qquad \text{a contradiction.}$$

Therefore $P(X_n \geqq 5a_n \text{ i.o.}) = 0$.

For a more detailed proof: Choose $m \ni P(\bigcup_{n \geqq m} \{S_n \geqq a_n\}) \leqq \frac{1}{4}$ and suppose

$P(X_n \geqq 5a_n \text{ i.o.}) = 1$. Then $\exists m \leqq c_m < \infty \ni P(\bigcup_{n=m}^{c_m} \{X_n \geqq 5a_n\}) > \frac{3}{4}$. Let $M = \max\{n : m \leqq n \leqq c_m \text{ and } X_n \geqq 5a_n\}$. If no such n exists, set $M = \infty$.

$$\begin{aligned}\tfrac{1}{4} &\geqq P(\textstyle\bigcup_{n\geqq m} \{S_n \geqq a_n\}) \geqq P(\bigcup_{n=m}^{c_m} \{S_n \geqq a_n\}) \\ &\geqq \textstyle\sum_{k=m}^{c_m} P(\bigcup_{n=m}^{c_m} \{S_n \geqq a_n\}, M = k) \geqq \sum_{k=m}^{c_m} P(S_k \geqq a_k, M = k) \\ &\geqq \textstyle\sum_{k=m}^{c_m} P(S_{k-1} \geqq -2ES_k^+, M = k) = \sum_{k=m}^{c_m} P(S_{k-1} \geqq -2ES_k^+)P(M = k) \\ &> \tfrac{1}{2} \textstyle\sum_{k=m}^{c_m} P(M = k) = \frac{1}{2}P(\bigcup_{n=m}^{c_m} \{X_n \geqq a_n\}) > \frac{3}{8}, \qquad \text{a contradiction.}\end{aligned}$$

Therefore $P(X_n \geqq 5a_n \text{ i.o.}) = 0$.

Define $f_n(x) = E(S_{t_n^0})/(x + t_n^0)$ for $x \geqq 0$. For $n \leqq x \leqq n + 1$, put $f(x) = \max(f_n(x), f_{n+1}(x))$. We may assume that our extension of a_n satisfies

$$\frac{a(x)}{8x} < f(x) < \frac{a(x)}{x}, \qquad \text{since } xf(x) \nearrow. \tag{31}$$

$t_n^0 < \infty \Rightarrow S_{t_n^0} > 0$. $(S_{t_n^0})/(x + t_n^0)$ is integrable. Thus so is $(S_{t_n^0})/(x + t_n^0)^k$ for $x \geqq 0$, $k \geqq 1$. Moreover, $S_{t_n^0}/(x + t_n^0)^k$ is continuously differentiable in x for $x \geqq 0$. Therefore $f_n(x)$ is infinitely differentiable and $f_n^{(k)}(x) = (-1)^k(k - 1)!\, E(S_{t_n^0})/(x + t_n^0)^{k+1}$. Consequently $f(x)$ is a continuous, piecewise infinitely differentiable function. (In fact, $f(x)$ is convex.)

Lemma 19. *Fix $0 < \varepsilon < 1$. If either*

(i) $P(S_n \geqq a_n \text{ i.o.}) = 0$ *or*

(ii) $a_n \geqq (8/\varepsilon^2)ES_n^+$ *for n sufficiently large then $x^{2\varepsilon}f(x)$ is increasing from some point on.*

Proof. Choose $M = 1/\varepsilon$. Since $x^{2\varepsilon}f(x)$ is continuous, piece-wise differentiable and positive, we need but show that $\log x^{2\varepsilon}f(x)$ has positive derivative for all sufficiently large x at which f is differentiable.

Hence, to verify: $\exists X_0 \ni x > X_0$ and $f'(x)$ exists $\Rightarrow -f'(x) < (2\varepsilon f(x))/(x)$.

Fix x at which f is differentiable. $\exists n \ni f(y) = f_n(y)$ for all y in a neighborhood of x. So $f'(x) = f_n'(x)$.

$$\begin{aligned}-f_n'(x) = E\frac{S_{t_n^0}}{(x + t_n^0)^2} &< \frac{1}{(x+1)} E\frac{S_{t_n^0} 1_{\{t_n^0 \leqq Mn\}}}{x + t_n^0} \\ &+ \frac{1}{x + [Mn] + 1} E\frac{S_{t_n^0} 1_{\{t_n^0 > Mn\}}}{x + t_n^0}.\end{aligned}$$

Set

$$A_n(x) = E\frac{S_{t_n^0}}{x + t_n^0} 1_{\{t_n^0 \leqq Mn\}}.$$

Then

$$-f'(x) < \frac{1}{x+1} A_n(x) + \frac{f(x) - A_n(x)}{x + 1 + [Mn]}.$$

Recalling $n - 1 < x < n + 1$, we have $[x + 1 + [Mn]]^{-1} < (Mx)^{-1}$ for n large. Now since $f(x) - A_n(x) \geqq 0$, $-f'(x) < A_n(x)/(x + 1) + (Mx)^{-1}(f(x) - A_n(x))$

and therefore

$$-f'(x) < \frac{A_n(x)}{x}\left(1 - \frac{1}{M}\right) + \frac{f(x)}{Mx}.$$

We have reached the parting of the ways. Suppose 1 holds.

$$A_n(x) = \sum_{k=1}^{[Mn]} \int_{\{t_n^0=k\}} \frac{S_k}{x+k}\,dP \leqq \sum_{k=1}^{[Mn]} \int_{\{t_n^0=k\}} \frac{a_{n+k}}{x+k}\,dP$$

$$+ \sum_{k=1}^{[Mn]} \frac{\int_{\{t_n^0=k, X_k \leqq 5a_{n+k}\}} X_k\,dP}{x+k} + \sum_{k=1}^{[Mn]} \frac{\int_{\{t_n^0=k, X_k > 5a_{n+k}\}} X_k\,dP}{x+k}$$

$$\leqq \sum_{k=1}^{[Mn]} \int_{\{t_n^0=k\}} \frac{6a_{n+k}\,dP}{x+k} + \sum_{k=1}^{[Mn]} \frac{\int_{5a_{n+k}}^\infty x\,dF(x)}{x+k}$$

$$< 50\frac{a_n}{n} P(t_n^0 < \infty) + \frac{[Mn]}{n} \int_{5a_n}^\infty x\,dF(x).$$

As shown in (28), $P(t_n^0 < \infty) \to 0$ as $n \to \infty$. We now demonstrate that $\int_{5a_n}^\infty x\,dF(x) = o(a_n/n)$. By Lemma 18, $P(X_n \geqq 5a_n \text{ i.o.}) = 0$ and hence as in (30), we have $Ea^{-1}(X_1/5)1_{\{X_1 \geqq 0\}} < \infty$. By change of variables to $U = a^{-1}(X_1/5)$,

$$\int_{5a_n}^\infty x\,dF(x) = \int_n^\infty 5a(u)\,dF_U(u) = 5\int_n^\infty \frac{a(u)}{u}\cdot u\,dF_U(u) \tag{32}$$

$$\leqq 40\,\frac{a(n)}{n} \int_n^\infty u\,dF_U(u).$$

Thus $\int_{5a_n}^\infty x\,dF(x) = o(a_n/n)$

$$\frac{a_n}{n} < \frac{a_{n+1}}{n} < 2\,\frac{a_{n+1}}{n+1} < 16f(n+1) \leqq 16f(x)$$

Therefore $A_n(x) = o(f(x))$.

$$-f'(x) < o\left(\frac{f(x)}{x}\right) + \frac{f(x)}{Mx} < \frac{2\varepsilon f(x)}{x} \qquad \text{for } x \text{ large.}$$

Now assume 2 holds.

$$A_n(x) = \sum_{k=1}^{[Mn]} \int_{\{t_n^0=k\}} \frac{S_k}{x+k} < \frac{1}{x+1} \sum_{k=1}^{]Mn]} \int_{\{t_n^0=k\}} S_k$$

$$= \frac{1}{x+1} \sum_{k=1}^{[Mn]} \int_{\{t_n^0=k\}} S_{[Mn]} \leqq \frac{ES^+_{[Mn]}}{x+1}$$

$$\leqq \frac{MES_n^+}{x+1} < \frac{a_n\varepsilon}{8(x+1)}$$

$$-f'(x) < \frac{a_n\varepsilon}{8(x+1)x}(1-\varepsilon) + \frac{\varepsilon f(x)}{x} < \frac{\varepsilon(1-\varepsilon)f(x+1)}{x} + \frac{\varepsilon f(x)}{x}$$

$$< \frac{2\varepsilon f(x)}{x} \qquad \text{since } f(x+1) < f(x).$$

COROLLARY 4. *Whenever* $P(S_n \geqq a_n \text{ i.o.}) = 0$, $P(X_n \geqq a_n \text{ i.o.}) = 0$.

PROOF. By Lemma 19, $\exists c \ni c < y < x \Rightarrow y^{\frac{1}{2}}f(y) < x^{\frac{1}{2}}f(x)$. By (31),

$$\frac{a(z)}{8(z)^{\frac{1}{2}}} < z^{\frac{1}{2}}f(z) < \frac{a(z)}{z^{\frac{1}{2}}}, \qquad \text{so}$$

$$\frac{a(y)}{8(y)^{\frac{1}{2}}} < y^{\frac{1}{2}}f(y) < x^{\frac{1}{2}}f(x) < \frac{a(x)}{x^{\frac{1}{2}}}.$$

In particular, $a(Ky) > K^{\frac{1}{2}}a(y)/8$ for $K \geqq 1$. According to Lemma 18, $P(X_n \geqq 5a_n \text{ i.o.}) = 0$ whence $\sum_n P(X_1 \geqq 5a_n) < \infty$. Put $K = 1600$. Then $a(Ky) > 5a(y)$ for y large.

$$\Rightarrow \sum_n P(X_1 \geqq a(Kn)) < \infty$$
$$\Rightarrow \sum_n P\left(\frac{a^{-1}(X_1)}{K} \geqq n\right) \leqq \infty$$
$$\Rightarrow Ea^{-1}(X_1)1_{\{X_1 \geqq 0\}} < \infty$$
$$\Rightarrow \sum_n P(a^{-1}(X_1) \geqq n) < \infty$$
$$\Rightarrow \sum P(X_n \geqq a_n) < \infty$$
$$\Rightarrow P(X_n \geqq a_n \quad \text{i.o.}) = 0.$$

THEOREM 8. *If* $P(S_n \geqq a_n \text{ i.o.}) = 0$, *then*

$$\int_0^\infty \frac{dF(y)}{\int_y^\infty (x/y)\log(x/y)\,dF(x)} < \infty.$$

PROOF. As in Corollary 4, one can actually show that for any $\varepsilon > 0\ \exists c \ni c < y < x \Rightarrow x/y < (8a(x)/(a(y))^{1+\varepsilon}$. Put $u = a(x)$ and $v = a(y)$. For $a(c) < v < u$,

$$\frac{a^{-1}(u)}{a^{-1}(v)} < \left(\frac{8u}{v}\right)^{1+\varepsilon}. \tag{33}$$

As noted in (29), for u sufficiently large,

$$\frac{v}{(8+\varepsilon)a^{-1}(v)} \leqq \int_v^\infty u \log\frac{a^{-1}(u)}{a^{-1}(v)}\,dF(u)$$
$$\leqq (1+\varepsilon)\int_v^\infty u \log\frac{u}{v}\,dF(u) + (1+\varepsilon)\log 8 \int_v^\infty u\,dF(u).$$

Recalling the technique used to obtain (32), but this time incorporating $Ea^{-1}(X_1)1_{\{x_1 \geqq 0\}} < \infty$,

$$\int_v^\infty u\,dF(u) = o\left(\frac{v}{a^{-1}(v)}\right).$$

Hence for fixed $\varepsilon > 0\ \exists v_0 \ni v \geqq v_0 \Rightarrow (8+\varepsilon)a^{-1}(v) \geqq v/\int_v^\infty u\log(u/v)\,dF(u)$.

The latter is a nonnegative increasing (in fact continuous) function for $v \geqq 0$ since the numerator increases and the denominator decreases as v increases. Therefore

$$\int_0^\infty \frac{dF(v)}{\int_v^\infty (u/v)\log(u/v)\,dF(u)} \leqq \int_0^{v_0}(8+\varepsilon)a^{-1}(v_0)\,dF(v)$$
$$+ \int_{v_0}^\infty (8+\varepsilon)a^{-1}(v)\,dF(v) < \infty.$$

REMARK 3. Observe that whenever $P(S_n \geqq a_n \text{ i.o.}) = 0$, we obtain an upper bound for $E|S_n|$ in terms of the distribution of X_1^+, since $a_n > \frac{1}{2}ES_n^+ = \frac{1}{4}E|S_n|$ and since given $\varepsilon > 0$,

$$a^{-1}(y) \geqq \frac{1}{(8+\varepsilon)\int_y^\infty (x/y)\log(x/y)\,dF(x)}$$

for large y.

REMARK 4. It is well known that for $0 \leqq b_1 \leqq b_2 \cdots \lim_{n\to\infty} b_n = \infty$, $P(Z_n > b_n \text{ i.o.}) = P(Z_n \geqq b_n \text{ i.o.})$ for $Z_n = X_n$, $Z_n = |X_n|$, $Z_n = S_n$, or $Z_n = |S_n|$.

COROLLARY 5. *If for some $\varepsilon > 0$, $E(X_1^+)^{1+\varepsilon} < \infty$, then $P(S_n > a_n \text{ i.o.}) = 1$ so that every optimal rule stops with probability one.*

PROOF. If not, then by the Hewitt–Savage zero-one law, $P(S_n > a_n \text{ i.o.}) = 0 = P(S_n \geqq a_n \text{ i.o.})$. Thus, according to a result preceding Remark 3,

$$\frac{v}{(8+\varepsilon)a^{-1}(v)} < \int_v^\infty u \log\frac{u}{v}\,dF(u)$$

for large v. Setting $v = a_n$,

$$\frac{a_n}{(8+\varepsilon)n} < \int_{a_n}^\infty u\log\frac{u}{a_n}\,dF(u) < \frac{\int_{a_n}^\infty u^{1+\varepsilon/2}\log(u/a_n)\,dF(u)}{a_n^{\varepsilon/2}}$$

$$\Longrightarrow \frac{a_n^{1+\varepsilon/2}}{n} = o(1) \Longrightarrow \frac{a_n}{n^\alpha} = o(1) \qquad \text{where } \alpha = \frac{1}{1+(\varepsilon/2)}\,.$$

By Lemma 19, given $\delta > 0$, $x^{\delta/2}f(x)$ is eventually increasing $\Longrightarrow \lim_{x\to\infty} x^\delta f(x) = \infty$. Noting that $n^\delta f(n) < a_n/n^{1-\delta}$ from (31) forces us to conclude that for any $\alpha < 1$, $\lim_{n\to\infty} a_n/n^\alpha = \infty$, which then provides the desired contradiction.

Thus if $E(X_1^+)^{1+\varepsilon} < \infty$ for some $\varepsilon > 0$ then all optimal extended-valued stopping rules stop with probability one.

4. A sufficient condition for optimal rules to assume the value $+\infty$ with positive probability. We will reserve making any further deductions from the hypothesis that $P(S_n > a_n \text{ i.o.}) = 0$ until later. Bearing in mind the essential content of Theorem 8, namely that if an optimal rule has positive probability of not stopping, then a certain integral is finite, one might wonder whether the converse is also true.

We find in the succeeding that the converse is valid in a quite general setting and that our proof enables us to construct random variables whose optimal rules have positive probability of never stopping. In addition, we find that the converse itself is not always valid.

LEMMA 20. *Assume*

$$\int_0^\infty \frac{x\,dF(x)}{\int_x^\infty y\log(y/x)\,dF(y)} < \infty\,.$$

Then

$$\lim_{x\to\infty}\frac{\int_x^\infty y\,dF(y)}{\int_x^\infty y\log(y/x)\,dF(y)} = 0\,.$$

PROOF. Put

$$b^{-1}(y) = \frac{y}{\int_y^\infty z \log(z/y)\, dF(z)}$$

and note that $y/b^{-1}(y)$ is a decreasing positive function.

$$0 \leqq \lim_{x\to\infty} \frac{\int_x^\infty y\, dF(y)}{\int_x^\infty y \log(y/x)\, dF(y)} = \lim_{x\to\infty} \frac{\int_x^\infty y/b^{-1}(y)\cdot b^{-1}(y)\, dF(y)}{\int_x^\infty y \log(y/x)\, dF(y)}$$

$$\leqq \limsup_{x\to\infty} \frac{x}{b^{-1}(x)} \frac{\int_x^\infty b^{-1}(y)\, dF(y)}{(x/b^{-1}(x))} = 0 \qquad \text{since } \int_0^\infty b^{-1}(y)\, dF(y) < \infty .$$

LEMMA 21. *Assume*

$$\int_0^\infty \frac{x\, dF(x)}{\int_x^\infty y \log(y/x)\, dF(y)} < \infty .$$

Let $1 < K < \infty$ *be given. Let*

$$b^{-1}(x) = \frac{Mx}{\int_x^\infty y \log(y/x)\, dF(y)}$$

for some fixed $M > 0$. *Then* $\lim_{x\to\infty} b^{-1}(Kx)/b^{-1}(x) = K$ *and hence also*

$$\lim_{x\to\infty} b(Kx)/b(x) = K .$$

Note. It can easily be shown that regardless of F, $b^{-1}(\cdot)$ is a *continuous* function, so that $b(\cdot)$ is a strictly increasing continuous function defined for *all* $x > 0$.

$$\frac{b^{-1}(Kx)}{b^{-1}(x)} = \frac{K\int_x^\infty y \log(y/x)\, dF(y)}{\int_{Kx}^\infty y(\log(y/x) - \log K)\, dF(y)}$$

$$= \frac{K}{1 - \frac{\int_x^{Kx} y\log(y/x)\, dF(y)}{\int_x^\infty y\log(y/x)\, dF(y)} - \frac{\log K \int_{Kx}^\infty y\, dF(y)}{\int_x^\infty y\log(y/x)\, dF(y)}} .$$

Replacing $\log y/x$ by $\log K$ in the appropriate integrand we have

$$K < \frac{b^{-1}(Kx)}{b^{-1}(x)} < \frac{K}{1 - \frac{\log K \int_x^\infty y\, dF(y)}{\int_x^\infty y \log(y/x)\, dF(y)}} . \tag{34}$$

Lemma 20 now implies that $\lim_{x\to\infty} b^{-1}(Kx)/b^{-1}(x)$ exists, and equals K by (34). To show that $\lim_{x\to\infty} b(Kx)/b(x) = K$, fix $0 < \varepsilon < K$. For x sufficiently large,

$$\frac{b^{-1}((K-\varepsilon)x)}{b^{-1}(x)} < K < \frac{b^{-1}(Kx)}{b^{-1}(x)} . \tag{35}$$

Put $y = b^{-1}(x)$, multiply through by y, and then apply the increasing function $b(\cdot)$ to (35) to obtain $(K-\varepsilon)b(y) < b(Ky) < Kb(y)$, whence $K - \varepsilon < b(Ky)/b(y) < K$ for y sufficiently large $\Rightarrow \lim_{y\to\infty} b(Ky)/b(y) = K$.

COROLLARY 6. *If* $P(S_n \geqq a_n \text{ i.o.}) = 0$, *then* $\limsup_{n\to\infty} a_n/b(n) \leqq 8$ *and* $\limsup_{n\to\infty} E|S_n|/b(n) \leqq 32$.

PROOF. From Remark 3,

$$\limsup_{x\to\infty} \frac{b^{-1}(x)}{a^{-1}(x)} \leqq 8 .$$

As in the proof of Lemma 21, we then obtain $\limsup_{n\to\infty} (a_n)/b(n) \leqq 8$. Now use the fact that

$$a_n > \frac{E|S_n|}{4} .$$

LEMMA 22. *Let* $\{X_n\}$ *be* i.i.d. *random variables with mean-zero and common distribution function F. Assume that* $P(|X_n| > b(n) \text{ i.o.}) = 0$, *where*

$$b^{-1}(x) = \frac{Mx}{\int_x^\infty y \log (y/x)\, dF(y)}$$

for $x > 0$ *and some fixed* $M > 0$. *Then* $P(|S_n| > b(n) \text{ i.o.}) = 0$.

PROOF. By Feller [13] page 259, footnote 3, it suffices to verify that $b(n)/n \searrow$ and $\liminf_{n\to\infty} b(2n)/b(n) > 2^{\frac{1}{2}}$. Lemma 21 gives $\lim_{n\to\infty} b(2n)/b(n) = 2 > 2^{\frac{1}{2}}$. Putting $u = b(n)$, $b(n)/n \searrow$ iff $u/b^{-1}(u) \searrow$ iff $\int_u^\infty y \log (y/u)\, dF(y) \searrow$. Since the latter function obviously decreases as u increases, the desired result follows.

LEMMA 23. *Let* $\{X_n\}$ *be* i.i.d. *random variables with mean zero. Let* $0 < b_1 \leqq b_2 \leqq \cdots$ *be given such that* $P(S_n > b_n \text{ i.o.}) = 0$. *Then* $P(\bigcap_n \{S_n \leqq b_n\}) > 0$.

PROOF. Let $g(N) = P(\bigcap_{n\geqq N} \{S_n \leqq b_n\})\cdot g(1) \leqq g(2) \leqq \cdots \lim_{N\to\infty} g(N) = 1$. Thus $\exists N \geqq 1 \ni g(N) > 0$. Let $A = \{u\colon P(\bigcap_{n\geqq N} \{S_n - S_N \leqq b_n - u\}) > 0\}$. Let $u_0 = \sup A$. Clearly, $(-\infty, u_0) \in A$.

$$\begin{aligned} 0 < g(N) = P(\textstyle\bigcap_{n\geqq N} \{S_n \leqq b_n\}) &= EP(\textstyle\bigcap_{n\geqq N} \{S_n \leqq b_n\} \mid S_N) \\ &= EP(\textstyle\bigcap_{n\geqq N} \{S_n - S_N \leqq b_n - S_N\} \mid S_N) \end{aligned}$$

Therefore $P(S_N \in A) > 0$.

$$\begin{aligned} P(S_1 \leqq b_1, \cdots, S_{N-1} \leqq b_{N-1}, S_N \in A) \\ &\geqq P(X_1 \leqq 0, \cdots, X_{N-1} \leqq 0, S_N \in A) \\ &= [P(X_1 \leqq 0)]^{N-1} P(S_N \in A \mid X_1 \leqq 0, \cdots, X_{N-1} \leqq 0) . \end{aligned}$$

Clearly,

$$P(S_N \in A \mid X_1 \leqq 0, \cdots, X_{N-1} \leqq 0) \geqq P(S_N \in A \mid X_j > 0 \ \text{ for some } \ j < N) ,$$

and at least one of these two numbers must be positive, owing to the fact that $P(S_N \in A) > 0$; hence

$$P(S_N \in A \mid X_1 \leqq 0, \cdots, X_{N-1} \leqq 0) > 0 .$$

$$\Rightarrow P(S_1 \leqq b_1, \cdots, S_{N-1} \leqq b_{N-1}, S_N \in A) > 0$$

$$\Rightarrow P(\textstyle\bigcap_n \{S_n \leqq b_n\}) > 0 .$$

COROLLARY 7. *Under the same hypotheses as in Lemma* 23, $P(\bigcap_n \{S_n < b_n\}) > 0$.

PROOF. Let $0 < \varepsilon < b_1$. Put $b_n' = b_n - \varepsilon$, $P(S_n > b_n' \text{ i.o.}) = 0$ and $0 < b_1' \leqq b_2' \leqq \cdots$; therefore

$$0 < P(\bigcap_n \{S_n \leqq b_n'\}) \leqq P(\bigcap_n \{S_n < b_n\}) .$$

THEOREM 9. *Let* $\{X_n\}$ *be* i.i.d. *nondegenerate mean zero random variables with common distribution function* F, *satisfying*

(36) $$EX_1^+ \log^+ X_1 < \infty$$ *and*

(37) $$\textstyle\int_{-\infty}^{\infty} b^{-1}(|x|)\, dF(x) < \infty$$ *where*

$$b^{-1}(|x|) = \frac{|x|}{\int_{|x|}^{\infty} y \log (y/|x|)\, dF(y)} \qquad \text{if} \quad |x| > 0$$
$$= 0 \qquad \text{if} \quad x = 0 .$$

Then every $\tau \in T_\infty \ni E(S_\tau/\tau) = \sup_{t \in T_\infty} E(S_t/t)$ *has positive probability of assuming the value* $+\infty$. (Note that (37) is equivalent to the assumption

(38) $$P(|X_n| > b_n \text{ i.o.}) = 0 ,$$

where b_n, of course, is defined by the relation $b(b^{-1}(x)) = x$ and in accordance with the note to Lemma 21.)

PROOF. The minimal optimal rule τ_0 has the form

$$\tau_0 = \text{1st} \quad k : S_k \geqq a_k \qquad \text{if such } k \text{ exists}$$
$$= \infty \qquad \text{otherwise}$$

where

$$\frac{a_n}{n} = \sup_{t \in T_\infty} E \frac{a_n + S_t}{n + t} .$$

Note: $0 < a_1 < a_2 < \cdots$

$$P(\tau_0 = \infty) = P(\bigcap_k \{S_k < a_k\}) .$$

If we can find $0 < b_1 < b_2 < \cdots \ni b_k < a_k$ and $P(S_k > b_k \text{ i.o.}) = 0$, then by Lemma 23, $P(\bigcap_k \{S_k \leqq b_k\}) > 0$. $\Rightarrow P(\bigcap_k \{S_k < a_k\}) > 0$, or $P(\tau_0 = \infty) > 0$.

Given any $\tau \in T_\infty \ni E(S_\tau/\tau) = \sup_{t \in T_\infty} E(S_t/t)$, $P(\tau \geqq \tau_0) = 1$ (Theorem 3), therefore $P(\tau = \infty) > 0$.

We construct $b(x)$ by defining its inverse function. Fix $1 < M < \infty$. Let $b^{-1}(x) = Mx/\int_x^\infty y \log (y/x)\, dF(y)$ for $x > 0$. $b^{-1}(x)$ is a positive, finite, strictly increasing continuous function and is therefore the inverse of a strictly increasing positive continuous function $b(x)$.

Let

$$t_n = \text{1st} \quad k : X_k \geqq b(n + k) \qquad \text{if such } k \text{ exists}$$
$$= \infty \qquad \text{otherwise}$$

$t_n \in T_\infty$. At this point we require only the assumption

$$\textstyle\int_0^\infty b^{-1}(x)\, dF(x) < \infty .$$

Estimating $E(X_{t_n}/(n+t_n))$,

$$E\frac{X_{t_n}}{n+t_n} = \sum_{k=1}^{\infty} \frac{\int_{\{t_n \geq k, X_k \geq b(n+k)\}} X_k \, dP}{n+k}$$

$$= \sum_{k=1}^{\infty} P(t_n \geq k) \frac{\int_{b(n+k)}^{\infty} x \, dF(x)}{n+k}$$

$$\geq P(t_n = \infty) \sum_{k=n+1}^{\infty} \int_{b(k)}^{\infty} \frac{x \, dF(x)}{k}$$

$$= P(t_n = \infty) \sum_{k=n+1}^{\infty} \sum_{j=k}^{\infty} \int_{b(j)}^{b(j+1)} \frac{x \, dF(x)}{k}$$

$$= P(t_n = \infty) \sum_{j=n+1}^{\infty} \left(\sum_{k=n+1}^{j} \frac{1}{k}\right) \int_{b(j)}^{b(j+1)} x \, dF(x)$$

$$\geq P(t_n = \infty) \sum_{j=n+1}^{\infty} \log \frac{j+1}{n+1} \int_{b(j)}^{b(j+1)} x \, dF(x)$$

$$\geq P(t_n = \infty) \sum_{j=n+1}^{\infty} \int_{b(j)}^{b(j+1)} x \log \frac{b^{-1}(x)}{n+1} \, dF(x)$$

$$= P(t_n = \infty) \int_{b(n+1)}^{\infty} x \log \frac{b^{-1}(x)}{n+1} \, dF(x)$$

$b^{-1}(x)/x \nearrow$. Therefore $x \geq u \Rightarrow b^{-1}(x)/x \geq b^{-1}(u)/u \Rightarrow b^{-1}(x)/b^{-1}(u) \geq x/u$. Let $u = b(n+1)$. Then

$$E\frac{X_{t_n}}{n+t_n} \geq P(t_n = \infty) \int_{b(n+1)}^{\infty} x \log \frac{x}{b(n+1)} \, dF(x)$$

$$= P(t_n = \infty) \frac{Mb(n+1)}{n+1} > \left(M - \frac{\varepsilon}{2}\right) \frac{b(n)}{n}$$

for n sufficiently large owing to the facts

$$P(t_n = \infty) = P(\textstyle\bigcap_k \{X_k < b(n+k)\})$$
$$= 1 - P(\textstyle\bigcup_{k=n+1} \{X_k \geq b(k)\})$$
$$\geq 1 - \sum_{k=n+1}^{\infty} P(X_k \geq b(k)) \to 1 \qquad \text{as } n \to \infty$$

because

$$P(X_k \geq b(k) \text{ i.o.}) = 0\,; \qquad b(n+1) > b(n)\,; \qquad \text{and} \qquad (n+1)/n \to 1\,.$$

$$E \sum_{i=1}^{t_n - 1} \frac{X_i}{n+t_n} = \sum_{k=2}^{\infty} \sum_{i=1}^{k-1} \frac{P(t_n = k) E(X_i \mid t_n = k)}{n+k}$$

$$= \sum_{k=2}^{\infty} \sum_{i=1}^{k-1} \frac{P(t_n = k) E(X_i \mid X_i < b(n+i))}{n+k}$$

$$\geq \sum_{k=2}^{\infty} \sum_{i=1}^{k-1} \frac{P(t_n = k) E(X_i \mid X_i < b(n+1))}{n+k}$$

$$= E(X_1 \mid X_1 < b(n+1)) \sum_{k=2}^{\infty} \frac{P(t_n = k)(k-1)}{n+k}$$

$$= \frac{-\int_{b(n+1)}^{\infty} x \, dF(x)}{P(X_1 < b(n+1))} \sum_{k=2}^{\infty} \frac{(t_n = k)(k-1)}{n+k}\,.$$

Let $u = b^{-1}(x)$

$$\geqq -\frac{\int_{n+1}^{\infty} b(u)\, dF_u(u)P(t_n < \infty)}{P(X_1 < b(n+1))}$$

$$= -\frac{\int_{n+1}^{\infty} (b(u)/u)\cdot u\, dF_u(u)P(t_n < \infty)}{P(X < b(n+1))}$$

$$\geqq -\frac{(b(n+1)/(n+1)) \int_{n+1}^{\infty} u\, dF_u(u)P(t_n < \infty)}{P(X < b(n+1))}$$

$$\geqq -\frac{b(n)}{n} o(1) \qquad \text{since } \frac{b(n)}{n} > \frac{b(n+1)}{n+1},$$

$P(X < b(n+1)) \to 1$ as $n \to \infty$ and $\int_{n+1}^{\infty} u\, dF_u(u) \to 0$ as $n \to \infty$, also $P(t_n < \infty) \to 0$ as $n \to \infty$).

Fix $\varepsilon > 0$ and $M > 1 + \varepsilon$. $\exists N \ni n \geqq N$

$$\Rightarrow E\frac{b(n) + S_{t_n}}{n + t_n} > E\frac{S_{t_n}}{n + t_n} = E \sum_{i=1}^{t_n - 1} \frac{X_i}{n + t_n} + E\frac{X_{t_n}}{n + t_n}$$

$$\geqq -\frac{\varepsilon}{2}\frac{b(n)}{n} + \left(M - \frac{\varepsilon}{2}\right)\frac{b(n)}{n} > \frac{b(n)}{n} \qquad \text{for} \quad n \geqq N\,.$$ [3]

$$\Rightarrow a_n > b(n) \qquad \text{for} \quad n \geqq N\,.$$

$$\Rightarrow P(X_n \geqq a_n \quad \text{i.o.}) = 0\,.$$

Critical use of (37) is now made to obtain the conclusion of Lemma 22, namely that

$$P(|S_n| > b(n) \ \text{ i.o.}) = 0\,.$$

$$\Rightarrow \text{(by Lemma 23)} \quad P(\textstyle\bigcap_n \{S_n \leqq b(n)\}) > 0$$

$$\Rightarrow P(\textstyle\bigcap_n \{S_n < a_n\}) > 0\,.$$

Given any optimal rule $\tau \in T_\infty$, $\tau = n \Rightarrow S_n \geqq a_n$ a.s. by Theorem 7. Therefore $P(\tau = \infty) \geqq P(\bigcap_n \{S_n < a_n\}) > 0$.

COROLLARY 8. *If* $P(X_n \geqq b(n)\,\text{i.o.}) = 0$, *then* $\liminf_{n\to\infty} a_n/b(n) \geqq 1$.

PROOF. Let

$$b_M^{-1}(x) = \frac{M}{\int_x^{\infty} (y/x) \log (y/x)\, dF(y)} \cdot b_1^{-1}(x) = b^{-1}(x)\,.$$

Let $M = 1 + 1/m$. $b_M^{-1}(n) < b^{-1}(Mn) < (1 + 2/m)b^{-1}(n)$ for n large (Lemma 21).
By Theorem 9, $a_n > b_M(n)$ for n large.

$$a^{-1}(n) < b_M^{-1}(n) < (1 + 2/m)b^{-1}(n)$$

and so $\limsup_{n\to\infty} a^{-1}(n)/b^{-1}(n) \leqq 1$, from which it follows that

$$\liminf_{n\to\infty} a_n/b(n) \geqq 1\,.$$

[3] Compare proof of result given by McCabe and Shepp in [17].

REMARK 5. Observe that whenever $P(S_n \geqq a_n \text{ i.o.}) = 0$, combining Corollaries 4, 6, and 8 gives

$$1 \leqq \liminf_{n\to\infty} a_n/b(n) \leqq \limsup_{n\to\infty} a_n/b(n) \leqq 8 \, .$$

Thus for fixed $\varepsilon > 0$, if $P(S_n \geqq (8+\varepsilon)b(n) \text{ i.o.}) = 1$, then $P(S_n > a_n \text{ i.o.}) = 1$, and therefore all optimal rules stop with probability one.

We now derive a condition which ensures that $P(S_n > a_n \text{ i.o.}) = 1$ even though $Eb^{-1}(X_1^+) < \infty$. This condition will be expressed directly in terms of the distribution of X.

5. Certain stopping: Another sufficient condition.[4]

LEMMA 24. *Let* $\{X_n\}$ *be* i.i.d. *nondegenerate mean zero random variables. Define* $\{n_k\}$ *inductively as follows: Set* $n_1 = 1$. *Having defined* n_{k-1}, *let* $n_k = \min\{j > n_{k-1}\colon E|S_j| > 2E|S_{n_{k-1}}|\}$. *Let* $\{c_n\}$ *be any sequence of eventually positive, non-decreasing numbers such that for some* k_0

$$\sum_{k=k_0}^{\infty} \frac{1}{c_{n_k}} < \infty \, .$$

Then $P(|S_n| > c_n E|S_n| \text{ i.o.}) = 0$.

PROOF.

$$P(\textstyle\bigcup_{j\geqq n_k} \{|S_j| \geqq c_j E|S_j|\}) \leqq \sum_{l=k}^{\infty} P(\bigcup_{n_l \leqq j < n_{l+1}} \{|S_j| \geqq c_j E|S_j|\})$$
$$\leqq \textstyle\sum_{l=k}^{\infty} P\{\max_{j<n_{l+1}} |S_j| \geqq c_{n_l} E|S_{n_l}|\}$$

(since $c_j E|S_j|$ is non-decreasing).

$$\leqq \sum_{l=k}^{\infty} \frac{E|S_{n_{l+1}-1}|}{c_{n_l} E|S_{n_l}|} \qquad \text{(by Lemma 11)}$$

$$\leqq 2 \sum_{l=k}^{\infty} \frac{1}{c_{n_l}} \to 0 \qquad \text{as } k \to \infty \, ;$$

therefore $P(|S_n| \geqq c_n E|S_n| \text{ i.o.}) = 0$.

COROLLARY 9. *Let* $\{X_n\}$ *be as in Lemma* 24. *Let* $g(x)$ *be any eventually positive non-decreasing function of* x *such that*

$$\sum_{k=k_0}^{\infty} \frac{1}{g(2^k)} < \infty$$

for k_0 *sufficiently large, then*

$$P(|S_n| \geqq g(E|S_n|)E|S_n| \ \text{ i.o.}) = 0 \, .$$

PROOF. With n_k as in Lemma 24, we need only verify that

$$\sum_{k=k_0}^{\infty} \frac{1}{g(E|S_{n_k}|)} < \infty \, .$$

W log we may assume $k_0 = 1$.

$$E|S_{n_k}| > 2E|S_{n_{k-1}}| \Rightarrow E|S_{n_k}| > 2^{k-1} E|S_{n_1}|$$
$$= 2^{k-1} E|X_1| \, ;$$

[4] Compare with the contrapositive of section title 3.

$\exists$ integer b such that $2^b E|X_1| \geqq 2$.

$$\sum_{k=b+1}^{\infty} \frac{1}{g(E|S_{n_k}|)} \leqq \sum_{k=b+1}^{\infty} \frac{1}{g(2^{k-b-1}2^b E|X_1|)}$$
$$\leqq \sum_{k=1}^{\infty} \frac{1}{g(2^k)} < \infty .$$

In particular, for any $\alpha > 1$, $P(|S_n| \geqq (\log(E|S_n|))^\alpha E|S_n| \text{ i.o.}) = 0$.

THEOREM 10. *Let $g(x)$ be any function satisfying the conditions of Corollary 9. If $P(|X_n| \geqq g(n)b(n) \text{ i.o.}) = 1$, then $P(S_n > a_n \text{ i.o.}) = 1$.*

PROOF. Suppose $P(S_n > a_n \text{ i.o.}) = 0$. Corollary 6 yields $E|S_n| < 50b(n)$ for n large. Therefore $P(|X_n| \geqq (g(n)/50)E|S_n| \text{ i.o.}) = 1$, from which one concludes that $P(|S_n| \geqq (g(n)/100)E|S_n| \text{ i.o.}) = 1$, a contradiction of Corollary 9.

Theorems 9 and 10 involve conditions on the negative part of the distribution of X_1 when $Eb^{-1}(X_1^+) < \infty$. Examples of random variables which obey such conditions are provided by Corollary 10.

COROLLARY 10. *Let X_1 be a random variable having density*

$$(39) \qquad \begin{aligned} f(x) &= \frac{C_1}{x^2(\log x)^\alpha} & x &\geqq e \\ &= 0 & |x| &< e \\ &= \frac{C_2}{x^2(\log|x|)^\beta} & x &\leqq -e , \end{aligned}$$

where $\alpha > 2$, $\beta > 1$ and C_1 and C_2 are positive constants chosen so that $\int_{-\infty}^{\infty} f(x)\,dx = 1$ and $EX_1 = 0$. Then

$$(40) \qquad \beta > \alpha - 1 \Rightarrow P(S_n \geqq a_n \quad \text{i.o.}) = 0 ;$$

$$(41) \qquad 1 < \beta < \alpha - 2 \Rightarrow P(S_n > a_n \quad \text{i.o.}) = 1 .$$

PROOF. First consider $\lim_{x\to\infty} (\log x)^\varepsilon \int_x^\infty [y(\log y)^{1+\varepsilon}]^{-1}\,dy$ for some fixed $\varepsilon > 0$. Using L'Hospital's Rule

$$\lim_{x\to\infty} \frac{\int_x^\infty [y(\log y)^{1+\varepsilon}]^{-1}\,dy}{(\log x)^{-\varepsilon}} = \lim_{x\to\infty} \frac{-[x(\log x)^{1+\varepsilon}]^{-1}}{-\varepsilon((\log x)^{-\varepsilon-1}/x)} = \frac{1}{\varepsilon} .$$

Consequently, $\int_x^\infty [(y/x)\log(y/x)/y^2(\log y)^\alpha]\,dy$ is asymptotic to

$$\frac{1}{x}\left(\frac{1}{(\alpha-2)(\log x)^{\alpha-2}} - \frac{1}{(\alpha-1)(\log x)^{\alpha-2}}\right) = \frac{(\log x)^{-\alpha+2}}{x(\alpha-1)(\alpha-2)} ,$$

which implies that $b^{-1}(x)$ is asymptotic to $[(\alpha-1)(\alpha-2)x(\log x)^{\alpha-2}]/C_1$.

$$\lim_{x\to\infty} \int_x^\infty b^{-1}(y)\,dF(y) = \lim_{x\to\infty} \int_x^\infty \frac{(\alpha-1)(\alpha-2)y(\log y)^{\alpha-2}}{y^2(\log y)^\alpha}\,dy$$
$$= \lim_{y\to\infty} (\alpha-1)(\alpha-2) \int_x^\infty \frac{1}{y\log^2 y}\,dy = 0 ;$$

therefore $Eb^{-1}(X_1^+) < \infty$.

If $\beta > \alpha - 1$, then

$$\lim_{y\to\infty} \int_{-\infty}^{-y} b^{-1}(|x|)\, dF(x)$$
$$= \frac{C_2(\alpha-1)(\alpha-2)}{C_1} \lim_{y\to\infty} \int_{-\infty}^{-y} \frac{dx}{|x|(\log|x|)^{1+(\beta-\alpha+1)}} = 0 .$$

Hence $Eb^{-1}(|x_1|) < \infty$. Applying Theorem 9, $P(S_n \geqq a_n \text{ i.o.}) = 0$.

Suppose $1 < \beta < \alpha - 2$. Let $g(x) = (\log x)^{1+\delta}$ where $0 < \delta \leqq \alpha - \beta - 2$, $0 < \sum_{n=1}^{\infty} (g(2^n))^{-1} > \infty$ $g(x) \nearrow$ for $x > 1$. We show $P(|X_n| \geqq g(n)b(n) \text{ i.o.}) = 1$.

$$\int_{-\infty}^{-e} \frac{|x|(\log|x|)^{\alpha-\delta-3}}{x^2(\log|x|)^{\beta}}\, dx \geqq \int_{-\infty}^{-e} \frac{dx}{|x|\log|x|} = \infty ; \qquad \text{therefore}$$

$$P\left(\frac{|X_n|(\log|X_n|)^{\alpha-\delta-3}}{K} \geqq n \ \text{ i.o.}\right) = 1 \quad \forall\, 0 < K < \infty .$$

$\exists K < \infty \ni b(x) < Kx/2(\log x)^{\alpha-2}$ for x large.
Let

$$h(x) = g(x)b(x) < \frac{K}{2}\, \frac{x}{(\log x)^{\alpha-\delta-3}} \qquad \text{for } x \text{ large; therefore}$$

$$h^{-1}(x) > \frac{x(\log x)^{\alpha-\delta-3}}{K} \qquad \text{for } x \text{ large}$$

$$1 = P\left(\frac{|X_n|(\log|X_n|)^{\alpha-\delta-3}}{K} \geqq n \ \text{ i.o.}\right) \leqq P(h^{-1}(|X_n|) \geqq n \ \text{ i.o.}) .$$

Therefore $P(|X_n| \geqq h(n) \text{ i.o.}) = 1$. We now invoke Theorem 10, obtaining $P(S_n > a_n \text{ i.o.}) = 1$.

6. $P(S_n > c_n \text{ i.o.}) \neq P(X_n > c_n \text{ i.o.})$. Let $\{c_n\}$ be a sequence of positive numbers such that $c_n/n \searrow$ and $c_n^{2-\varepsilon}/n \nearrow \infty$ for some $\varepsilon > 0$. Let $\{X_n\}$ be i.i.d. mean zero random variables such that $E|X_1|^{1+\delta} = \infty$ for some $0 < \delta < 1$. Letting $S_n = X_1 + \cdots + X_n$, Feller [13] proved that

$$P(|S_n| > c_n \ \text{ i.o.}) = P(|X_n| > c_n \ \text{ i.o.}) . \tag{42}$$

According to [13] page 259, footnote 3, the condition $c_n^{2-\varepsilon}/n \nearrow \infty$ can be replaced by the requirement that $\liminf_{n\to\infty} c_{2n}/c_n > 2^{\frac{1}{2}}$. Feller further stated (page 259) that the absolute values could be removed from both sides of (42) without affecting the validity of the equation. As an outgrowth of our results, we can produce examples to contradict Feller's latter assertion.

COROLLARY 11. *Let X_1 have a density as in Corollary* 10, *satisfying* (39) *and* (41). *For* $0 < \varepsilon < 1$,

$$1 = P(S_n > (1-\varepsilon)b(n) \ \text{ i.o.}) > P(X_n > (1-\varepsilon)b(n) \ \text{ i.o.}) = 0$$

where

$$b^{-1}(x) = \frac{1}{\int_x^{\infty} (y/x)\log(y/x)\, dF(y)} .$$

REMARK 6. Corollary 11 will provide the desired counterexample since $b(n)/n \searrow 0$ and $\lim_{n\to\infty} b(2n)/b(n) = 2 > 2^{\frac{1}{2}}$ (see Lemma 22).

PROOF OF COROLLARY 11. As verified in Corollary 10, $Eb^{-1}(X_1^+) < \infty$. Therefore $P(2/(1-\varepsilon)b^{-1}(X_n) > n \text{ i.o.}) = 0$. Equivalently, $P(X_n > b(n(1-\varepsilon)/2) \text{ i.o.}) = 0$. Owing to Lemma 21, $b(n(1-\varepsilon)/2) < (1-\varepsilon)b(n)$ for n large. Therefore $P(X_n > (1-\varepsilon)b(n) \text{ i.o.}) = 0$. Corollary 8 gives $a_n > (1-\varepsilon)b(n)$ for n large, which combined with Corollary 10 yields $P(S_n > (1-\varepsilon)b(n) \text{ i.o.}) = 1$.

Curiously enough, Feller himself has a counterexample in his book ([14] page 247).

7. $\liminf_{n\to\infty} a_n/E|S_n| < \infty$ **whenever** $E|X_1|^{1+\varepsilon} < \infty$. I suspect that whenever $E|X_1|^{1+\varepsilon} < \infty$ for some $\varepsilon > 0$ that $\lim_{n\to\infty} a_n/E|S_n|$ exists and is finite. This has effectively been shown (though not stated in this form) when X_1 has finite variance or is in the domain of normal attraction of a stable law of order $\alpha > 1$. (see [25], [19] and [18]). Recently, it has come to my attention that Thompson, Basu, and Owen [23] and B. Davis [7] have virtually proven that $\liminf_{n\to\infty} a_n/E|S_n| < \infty$ if $E|X_1|^{1+\varepsilon} < \infty$ for some $\varepsilon > 0$. Another proof of this fact is presented here. We dispense with the implicit convention regarding the special meaning of $\{b_n\}$.

THEOREM 11. *Let $\{X_n\}$ be a sequence of pairwise independent identically distributed mean zero random variables. Given $\{b_n\}$ such that*

$$b_n > 0 \quad \text{and} \quad \frac{b_n}{n} \searrow \tag{43}$$

$$\frac{b_n^2}{n} \nearrow \infty \tag{44}$$

$$P(|X_n| \geqq b_n \ \text{ i.o.}) = 0 \tag{45}$$

then $E|S_n| = o(b_n)$.

PROOF. Let

$$\begin{aligned} Y_{n,j} &= X_j \qquad \text{if} \quad |X_j| \leqq b_n \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Let $Z_{n,j} = X_j - Y_{n,j}$. Let $T_n = \sum_{j=1}^n Y_{n,j}$, $V_n = \sum_{j=1}^n Z_{n,j}$, $S_n = T_n + V_n$, $ES_n = 0 \Rightarrow (ET_n)^2 = (EV_n)^2$.

$$\begin{aligned} E|S_n| &\leqq E|T_n| + E|V_n| \leqq (ET_n^2)^{\frac12} + E|V_n| \\ &= (\operatorname{Var} T_n + (ET_n)^2)^{\frac12} + E|V_n| \\ &= (\textstyle\sum_{j=1}^n \operatorname{Var} Y_{n,j} + (EV_n)^2)^{\frac12} + E|V_n| \\ &\leqq (\textstyle\sum_{j=1}^n EY_{n,j}^2 + (E|V_n|)^2)^{\frac12} + E|V_n| \\ &\leqq (nEY_{n,1}^2 + (\textstyle\sum_{j=1}^n E|Z_{n,j}|)^2)^{\frac12} + \textstyle\sum_{j=1}^n E|Z_{n,j}| \\ &= (nEY_{n,1}^2 + (nE|Z_{n,1}|)^2)^{\frac12} + nE|Z_{n,1}|\,; \end{aligned}$$

(44) gives $b_n < b_{n+1}$.
Extend b_n to a continuous function $b(x)$ for $x > 0$ such that $b(x)/x \searrow$ and $b^2(x)/x \nearrow \infty$.

$$\begin{aligned} 0 = P(|X_n| \geqq b_n \ \text{ i.o.}) &\Leftrightarrow Eb^{-1}(|X_1|) < \infty \\ &\Leftrightarrow \lim_{n\to\infty} Eb^{-1}(|X_1|)1_{\{b^{-1}|X_1| \geqq (n)\}} = 0\,. \end{aligned}$$

Let

$$\begin{aligned} u &= b^{-1}(|X_1|) \\ E|Z_{n,1}| &= E|X_1|1_{\{|X_1|\geqq b(n)\}} = Eb(u)1_{\{u\geqq n\}} \\ &= E\frac{b(u)}{u}\cdot u1_{\{u\geqq n\}} \leqq \frac{b(n)}{n}Eu1_{\{u\geqq n\}} \\ &= o\left(\frac{b(n)}{n}\right). \end{aligned}$$

$b^2(n)/n \nearrow \infty$ so $\exists$ integers $c_n \nearrow \infty \ni b^2(c_n)/c_n = o(b^2(n)/n)$.
Put $u = b^{-1}(|X_1|)$

$$\begin{aligned} EY_{n,1}^2 &= EX_1^2 1_{\{|X_1|\leqq b_n\}} \\ &= Eb^2(u)1_{\{u\leqq n\}} \\ &= E\frac{b^2(u)}{u}\cdot u1_{\{u\leqq c_n\}} + E\frac{b^2(u)}{u}\cdot u1_{\{c_n<u\leqq n\}} \\ &\leqq \frac{b^2(c_n)}{c_n}Eu + \frac{b^2(n)}{n}Eu1_{\{c_n<u\}} \\ &= o\left(\frac{b^2(n)}{n}\right) + o\left(\frac{b^2(n)}{n}\right) = o\left(\frac{b^2(n)}{n}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} E|S_n| &\leqq (o(b^2(n)) + (o(b(n))^2)^{\frac{1}{2}} + o(b(n)) = o(b(n)) \\ E|S_n| &= o(b(n)) . \end{aligned}$$

COROLLARY 12. *Let $\{X_n\}$ be a sequence of pairwise independent identically distributed random variables with mean zero. Assume $\exists 1 \leqq \alpha < 2$ such that $E|X_1|^\alpha < \infty$. Then $E|S_n| = o(n^{1/\alpha})$.*

PROOF.

$$\begin{aligned} E|X_1|^\alpha < \infty &\Rightarrow P(|X_n|^\alpha \geqq n \text{ i.o.}) = 0 \\ &\Rightarrow P(|X_n| \geqq n^{1/\alpha} \text{ i.o.}) = 0 \\ &\Rightarrow E|S_n| = o(n^{1/\alpha}) \end{aligned}$$

by Theorem 11 since $n^{1/\alpha}/n \searrow$ and $n^{2/\alpha}/n \nearrow \infty$.

THEOREM 12. *Let $\{X_n\}$ be* i.i.d. *mean zero nondegenerate random variables such that $E|X_1|^\alpha < \infty$ for some $\alpha > 1$. Then*

$$\liminf_{n\to\infty} \frac{a_n}{E|S_n|} < \infty .$$

PROOF. Assume $\lim_{n\to\infty} a_n/E|S_n| = \infty$. Recalling Lemma 19, $\forall \delta>0$, $x^\delta f(x)\to\infty$. $f(n) < a_n/n$ so $a_n/(n^{1-\delta}) \to \infty$. Incorporating Corollary 5, $\forall \delta > 0$, $P(S_n > n^{1-\delta}$ i.o.$) = 1$. However, combining Lemma 24 with Corollary 12 we find that

$$P(|S_n| > n^{1/\alpha+\delta_1} \text{ i.o.}) = 0 \quad \forall \delta_1 > 0 .$$

Choose δ_1 so that $1/\alpha + \delta_1 < 1$ to obtain the desired contradiction.

8. For $\frac{1}{2} < \alpha < 1$, optimal rules for $E(S_t/t^\alpha)$ are finite a.s.

THEOREM 13. *Let $\{X_n\}$ be identically nondegenerate mean zero random variables. Fix $\frac{1}{2} < \alpha < 1$. Assume that* $\limsup_n S_n/n^\alpha = 0$ a.s. *and that* $E\sup_n' S_n/n^\alpha < \infty$. *Then* $\exists \tau \in T_\infty \ni E(S_\tau/\tau^\alpha) = \sup_{t \in T_\infty} E(S_t/t^\alpha)$ *and for each such* τ, $P(\tau < \infty) = 1$. *Again we define*

$$E\frac{S_t}{t^\alpha} \equiv E\frac{S_t 1_{(t<\infty)}}{t^\alpha} \qquad \textit{for } t \in T_\infty .$$

PROOF. Theorem 1 guarantees the existence of an optimal $\tau \in T_\infty$ and of $\tau_n(a) \in T_\infty \ni$

$$E\frac{a + S_{\tau_n(a)}}{(n + \tau_n(a))^\alpha} = \sup_{t \in T_\infty} E\frac{a + S_t}{(n + t)^\alpha} .$$

Lemmas 6 and 7 can be modified to show that if

$$E\frac{a + S_{\tau_n(a)}}{(n + \tau_n(a))^\alpha} \geqq \frac{a}{n^\alpha}$$

then $m \geqq n$ and $a' \leqq a$ with $m - n + a - a' > 0$.

$$\Rightarrow \frac{a' + S_{\tau_n(a)}}{(m + \tau_n(a))^\alpha} > \frac{a'}{m^\alpha} .$$

For optimal $\tau \in T_\infty$, let $A_n = \{a\colon P(S_n \leqq a, \tau = n) > 0\}$. A_n is nonempty whenever $P(\tau = n) > 0$. We may suppose, to obtain a contradiction, that $P(\tau = \infty) > 0$. It follows that for every N, $P(\tau \leqq N) + P(\tau = \infty) < 1$, for if not, let

$$\begin{aligned} \tau' &= \tau && \text{if } \tau \leqq N \\ &= \text{1st } k > N\colon S_k > 0 && \text{if } \tau = \infty . \end{aligned}$$

Then $\tau' \in T_\infty$ and since $P(S_n > 0 \text{ i.o.}) = 1$,

$$E\frac{S_{\tau'}}{(\tau')^\alpha} > E\frac{S_\tau}{\tau^\alpha} \Rightarrow \Leftarrow \text{contradiction.}$$

Hence $\exists$ infinitely many numbers $n_1 < n_2 < \cdots$ such that $P(\tau = n_k) > 0 \Rightarrow A_{n_k}$ is nonempty. For $b_k \in A_{n_k}$,

$$\sup_{t \in T_\infty} E\frac{b_k + S_t}{(n_k + t)^\alpha} \leqq \frac{b_k}{(n_k)^\alpha} .$$

The analogues of Lemma 8 and Lemma 9 verify the existence of unique numbers a_n such that

$$\sup_{t \in T_\infty} E\frac{a_n + S_t}{(n + t)^\alpha} = \frac{a_n}{n^\alpha} \qquad \text{and} \quad 0 < a_1 < a_2 < \cdots .$$

The class of rules which maximize $E((a + S_t)/(n + t)^\alpha)$ over $t \in T_\infty$ has the same form as that of Theorem 7. Thus $P(S_n \geqq a_n \text{ i.o.}) = 0$ and, letting

$$\begin{aligned} t_n &= \text{1st } k\colon S_k \geqq a_{n+k} && \text{if such } k \text{ exists} \\ &= \infty && \text{otherwise,} \end{aligned}$$

$$E\frac{S_{t_n}}{(n + t_n)^\alpha} = \sup_{t \in T_\infty} E\frac{S_t}{(n + t)^\alpha} .$$

As in Lemma 15, one can obtain

$$\frac{ES_n{}^+}{2n^\alpha} < \frac{ES_n{}^+}{(2n)^\alpha} > E\frac{S_{t_n}}{(n+t_n)^\alpha}$$

and then the technique of Lemma 16 can be utilized to give

$$\frac{1}{8}\frac{a_n}{n^\alpha} < E\frac{S_{t_n}}{(n+t_n)^\alpha} < \frac{a_n}{n^\alpha}$$

so that in particular we have $a_{n+m}/(n+m)^\alpha < 8a_n/n^\alpha$. As in Lemma 17 and Corollary 4,

$$\begin{aligned}P(S_n \geqq a_n \text{ i.o.}) = 0 &\Rightarrow P(X_n \geqq a_n \text{ i.o.}) = 0\\ &\Rightarrow \textstyle\sum_n P(X_n \geqq a_n) < \infty \Rightarrow \sum P(X_1 \geqq a_n) < \infty\\ &\Rightarrow \textstyle\sum_n P(a^{-1}(X_1) \geqq n) < \infty \Rightarrow Ea^{-1}(X_1)1_{\{X_1>0\}} < \infty\end{aligned}$$

$$\begin{aligned}\frac{a_n}{8n^\alpha} &< \textstyle\sum_{k=1}^\infty \int_{\{t_n=k\}} \frac{S_k\,dP}{(n+k)^\alpha} < 2\sum_{k=1}^\infty \int_{\{t_n=k\}} \frac{a_{n+k}\,dP}{(n+k)^\alpha} + \sum_{k=1} \frac{\int_{a_{n+k}}^\infty x\,dF(x)}{(x+k)^\alpha}\\ &< \frac{16a_n}{n^\alpha}P(t_n<\infty) + \textstyle\sum_{k=n+1}^\infty \sum_{j=k}^\infty \int_{a_j}^{a_{j+1}} \frac{x\,dF(x)}{k^\alpha}\\ &= o\left(\frac{a_n}{n^\alpha}\right) + \textstyle\sum_{j=n+1}^\infty \left(\sum_{k=n+1}^j \frac{1}{k^\alpha}\right) \int_{a_j}^{a_{j+1}} x\,dF(x)\\ &< o\left(\frac{a_n}{n^\alpha}\right) + \textstyle\sum_{j=n+1}^\infty \left(\int_n^j \frac{1}{x^\alpha}\,dx\right)\left(\int_{a_j}^{a_{j+1}} x\,dF(x)\right)\\ &= o\left(\frac{a_n}{n^\alpha}\right) + \frac{1}{1-\alpha}\textstyle\sum_{j=n+1}^\infty (j^{1-\alpha} - n^{1-\alpha}) \int_{a_j}^{a_{j+1}} x\,dF(x)\\ &< o\left(\frac{a_n}{n^\alpha}\right) + \frac{1}{1-\alpha}\textstyle\sum_{j=n+1}^\infty \int_{a_j}^{a_{j+1}} x(a^{-1}(x))^{1-\alpha}\,dF(x)\\ &= o\left(\frac{a_n}{n^\alpha}\right) + \frac{1}{1-\alpha}\textstyle\int_{a_{n+1}}^\infty x(a^{-1}(x))^{1-\alpha}\,dF(x)\,.\end{aligned}$$

For n large,

$$\int_{a_n}^\infty \frac{xa^{-1}(x)}{(a^{-1}(x))^\alpha}\,dF(x) > \frac{(1-\alpha)a_n}{20n^\alpha}$$

$$\frac{a_{n+m}}{8(n+m)^\alpha} < \frac{a_n}{n^\alpha} \Rightarrow \frac{z}{(a^{-1}(z))^\alpha} < \frac{8y}{(a^{-1}(y))^\alpha}$$

for $z > y$. Therefore $\int_{a_n}^\infty [xa^{-1}(x)/(a^{-1}(x))^\alpha]\,dF(x) < 8a_n/n^\alpha \int_{a_n}^\infty a^{-1}(x)\,dF(x)$; n large $\Rightarrow$ $\int_{a_n}^\infty a^{-1}(x)\,dF(x) > (1-\alpha)/160 > 0$; $\Rightarrow Ea^{-1}(X_1)1_{\{X_1>0\}} = \infty \Rightarrow \Leftarrow$ contradiction. Therefore $P(\tau < \infty) = 1$.

9. $EX_1 = 0 \Rightarrow P(S_n \geqq (4+\varepsilon)^{-1}E|S_n| \text{ i.o.}) = 1$. An optimal rule takes advantage of large fluctuations of a sequence of random variables. Thus, in order to discover something about the frequency of certain large fluctuations of a particular sequence of random variables, one might employ optimal extended-valued stopping rules τ, examining $P(\tau = \infty)$. Following this prescription, we discover a result much stronger than the well-known fact that $P(S_n > 0 \text{ i.o.}) = 1$ for a nondegenerate random walk generated by a mean-zero random variable.

Let $\{X_n\}$ be a sequence of nondegenerate i.i.d. mean-zero random variables. Fix $\varepsilon > 0$. Let $Y_n = S_n/(n^{1+\varepsilon})$ and $Y_\infty = 0$. We will first show that $\exists \tau \in T_\infty \ni EY_\tau = \sup_{t \in T_\infty} EY_t < \infty$. By the strong law of large numbers, $Y_n \to Y_\infty$ a.s.

According to Theorem 1, we need only prove

LEMMA 25.

$$E \sup_n \frac{S_n}{n^{1+\varepsilon}} < \infty .$$

PROOF.

$$E \sup_n \frac{S_n}{n^{1+\varepsilon}} \leqq E \sum_{n=1}^{\infty} \frac{|X_n|}{n^{1+\varepsilon}} = E|X_1| \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} < \infty .$$

Analogues of Lemmas 6–9 remain valid in this setting. Using a proof similar to that given in Lemma 13 or that presented in Theorem 13 one can show the existence of unique numbers a_n such that

$$\sup_{t \in T_\infty} E \frac{a_n + S_t}{(n + t)^{1+\varepsilon}} = \frac{a_n}{n^{1+\varepsilon}} .$$

We want to show that $P(S_n > a_n \text{ i.o.}) = 1$. As in Lemma 15,

$$\begin{aligned} \frac{ES_n^+}{(2n)^{1+\varepsilon}} &< \sup_{t \in T_\infty} E \frac{S_t}{(n + t)^{1+\varepsilon}} \\ &< \sup_{t \in T_\infty} E \frac{a_n + S_t}{(n + t)^{1+\varepsilon}} = \frac{a_n}{n^{1+\varepsilon}} . \end{aligned}$$

Therefore

$$a_n > \frac{ES_n^+}{2^{1+\varepsilon}} = \frac{E|S_n|}{2^{2+\varepsilon}} .$$

Let

$$\begin{aligned} t_n^0 &= \text{1st} \quad k : S_k \geqq a_{n+k} \qquad \text{if such } k \text{ exists,} \\ &= \infty \qquad\qquad\qquad\qquad \text{otherwise;} \end{aligned}$$

$$E \frac{S_{t_n^0}}{(n + t_n^0)^{1+\varepsilon}} = \sup_{t \in T_\infty} E \frac{S_t}{(n + t)^{1+\varepsilon}} .$$

Continuing further, we can parrot Lemma 16 and Corollary 3, obtaining for sufficiently small $\varepsilon > 0$, $a_n/(10n^{1+\varepsilon}) < ES_{t_n^0}/(n + t_n^0)^{1+\varepsilon}$ and thus $10a_n/(n^{1+\varepsilon}) > (a_{n+k})/(n + k)^{1+\varepsilon}$ for all n, $k \geqq 1$.

LEMMA 26.

$$P(S_n > a_n \text{ i.o.}) = 1 .$$

PROOF.

$$\frac{a_n}{10n^{1+\varepsilon}} < E\frac{S_{t_n^0}}{(n+t_n^0)^{1+\varepsilon}} \leqq \sum_{k<\infty} \int_{\{t_n^0=k\}} \frac{2a_{n+k}\,dP}{(n+k)^{1+\varepsilon}} + \sum_{k<\infty} \int_{a_{n+k}}^{\infty} \frac{x\,dF(x)}{(n+k)^{1+\varepsilon}}$$
$$< \frac{20a_n}{n^{1+\varepsilon}} P(t_n^0 < \infty) + \sum_{k=n+1}^{\infty} \sum_{j=k}^{\infty} \int_{a_j}^{a_{j+1}} \frac{x\,dF(x)}{k^{1+\varepsilon}}$$
$$= \frac{20a_n}{n^{1+\varepsilon}} P(t_n^0 < \infty) + \sum_{j=n+1}^{\infty} \left(\sum_{k=n+1}^{j} \frac{1}{k^{1+\varepsilon}}\right) \int_{a_j}^{a_{j+1}} x\,dF(x)$$
$$\leqq \frac{20a_n}{n^{1+\varepsilon}} P(t_n^0 < \infty) + \frac{1}{\varepsilon n^{\varepsilon}} \int_{a_n}^{\infty} x\,dF(x)\,.$$

Now assume $P(S_n > a_n \text{ i.o.}) = 0$. Then $P(t_n^0 < \infty) \to 0$ as $n \to \infty$, and so

$$\frac{\varepsilon a_n}{20n} < \int_{a_n}^{\infty} x\,dF(x)$$

for n large and

$$a^{-1}(u) > \frac{\varepsilon}{20} \frac{u}{\int_u^{\infty} x\,dF(x)} \qquad \text{for } u \text{ large.}$$

Since $a_n > ES_n^+/(2^{1+\varepsilon})$ one can prove that $P(X_n > a_n \text{ i.o.}) = 0$. Equivalently, $P(S_n > a_n \text{ i.o.}) = 0 \Rightarrow Ea^{-1}(X^+) < \infty$. Hence

$$\lim_{z\to\infty} \int_z^{\infty} \left(\frac{u}{\int_u^{\infty} x\,dF(x)}\right) dF(u) = 0\,.$$

However,

$$\int_z^{\infty} \frac{u}{\int_u^{\infty} x\,dF(x)}\,dF(u) \geqq \frac{\int_z^{\infty} u\,dF(u)}{\int_z^{\infty} x\,dF(x)} = 1\,.$$

This contradiction establishes that $P(S_n > a_n \text{ i.o.}) = 1$.

THEOREM 14. *Let $\{X_n\}$ be i.i.d. mean zero random variables. Let $S_n = X_1 + \cdots + X_n$. Then $\forall c < \frac{1}{4}$, $P(S_n \geqq cE|S_n| \text{ i.o.}) = 1$.*

PROOF. This follows immediately from Lemma 26 because $a_n > cE|S_n|$.

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