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LIMIT THEOREMS FOR REVERSIBLE MARKOV PROCESSES

BY MICHAEL L. LEVITAN AND LAWRENCE H. SMOLOWITZ

Villanova University and Hobart College

Consider a reversible (self-adjoint) Markov process with a discrete time parameter and stationary transition probability functions satisfying the Harris recurrence condition. $P^{(n)}(x, S)$ denotes the n-step transition probability function from x to the measurable set S and π is the sigma-finite stationary measure induced by the above hypotheses. Using both a functional analytic representation for reversible probabilities and probabilistic identities, various limits are considered for both general and discrete spaces. The principle result gives necessary and sufficient conditions for sets A and B so that a reversible, aperiodic Markov process satisfies the strong ratio limit property $\lim_{m\to\infty} P^{(n+k)}(\mu, A)/P^{(n)}(\nu, B)=\pi(A)/\pi(B)$ where μ and ν are arbitrary probability distributions defined on the space and k is any integer.

1. Introduction. Let us consider a discrete time parameter Markov process $\{X_k, k \geqq 0\}$ with stationary transition probability functions defined on a general measurable state space $(X, \mathscr{B})$ where $\mathscr{B}$ is a separable (countably generated) Borel field of subsets of X containing single point sets. Furthermore, assume the Harris recurrence condition

CONDITION (C). There exists a sigma-finite measure μ defined on X with $\mu(X) > 0$ such that for every $S \in \mathscr{B}$ with $\mu(S) > 0$, we have $P[X_k \in S$ infinitely often $| X_0 = x] = 1$ for all $x \in X$.

This then implies via [1] the existence of a sigma-finite measure π defined on $(X, \mathscr{B})$, unique up to a constant factor, such that μ is absolutely continuous with respect to π. Furthermore,

(i) $\pi(S) > 0$ implies that $P[X_k \in S$ infinitely often $| X_0 = x] = 1$ for all $x \in X$
(ii) $\pi(S) = \int_X P^{(n)}(x, S)\pi(dx)$ for all $S \in \mathscr{B}$, $n \geqq 0$ (Invariance).

$P^{(n)}(x, S)$ denotes the n-step transition probability from $x \in X$ to $S \in \mathscr{B}$ for $n \geqq 1$, and $P^{(0)}(x, S)$ is the indicator function of S, denoted $\chi_S(x)$.

The above conditions will be assumed to hold throughout this paper and will not be mentioned explicitly. In addition, we require in Sections 2 and 3 that the process be reversible (self-adjoint), i.e., for all $A, B \in \mathscr{B}$,

$$\int_B P^{(1)}(x, A)\pi(dx) = \int_A P^{(1)}(x, B)\pi(dx) .$$

This implies (see [9]) that

$$\int_B P^{(n)}(x, A)\pi(dx) = \int_A P^{(n)}(x, B)\pi(dx) , \qquad n = 1, 2, \cdots .$$

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Reversibility also implies that the number d of "cyclic sets" (see [7]) is either one or two. After Theorem 2.1, we shall always require the additional hypothesis of aperiodicity ($d = 1$).

Our principal result (Section 2) is to obtain a necessary and sufficient condition on sets A and B so that a reversible, aperiodic Markov process satisfies the strong ratio limit property (SRLP)

$$\lim_{n\to\infty} \frac{P^{(n+k)}(\mu, A)}{P^{(n)}(\nu, B)} = \frac{\pi(A)}{\pi(B)}$$

for all probability distributions μ and ν, all integers k and sets A and B satisfying $0 < \pi(A) < \infty$, $0 < \pi(B) < \infty$. In the method of obtaining this result, we also show that

$$\lim_{n\to\infty} \frac{\int_A P^{(2n+2k)}(x, A)\pi(dx)}{\int_A P^{(2n)}(x, A)\pi(dx)} = 1$$

for a reversible (not necessarily aperiodic) Markov process for all $A \in \mathscr{B}$ satisfying $0 < \pi(A) < \infty$.

In Section 3 we consider processes for which the state space X has at least one atom (see [5]) with respect to π. Certain weaker forms of the SRLP are shown to hold for reversible, aperiodic processes without assuming the stronger condition of Section 2. In Section 4 we drop the reversibility hypothesis.

We conclude this section by presenting additional notation and remarks. For $n \geqq 1$,

$${}_BP^{(n)}(x, A) = P[X_k \notin B \text{ for } 1 \leqq k < n, X_n \in A \mid X_0 = x], \quad A, B \in \mathscr{B}.$$

Let $\mathscr{S} = \{C \mid C \in \mathscr{B}, \pi(C) > 0 \text{ and } \inf_{x\in C, y \in C} p^{(n)}(x, y) > 0 \text{ for some } n\}$ where $p^{(n)}(x, \cdot)$ is the component of $P^{(n)}(x, \cdot)$ which is absolutely continuous with respect to π. It has been shown by Orey [7] that $\mathscr{S} \neq \phi$. Levitan [5] has proven that there exist sets $S_i \in \mathscr{S}$ such that $\pi(X - \bigcup_{i=1}^{\infty} S_i) = 0$. Furthermore, he has shown that if the Markov process is aperiodic, then for any $T_i \in \mathscr{S}$, $\bigcup_{i=1}^{n} T_i \in \mathscr{S}$.

2. SRLP for a general state space.

NOTATION. $\gamma_n(A) = \int_A P^{(n)}(x, A)\pi(dx)$, $n = 0, 1, 2, \cdots$.

THEOREM 2.1. *If $\{X_k\}$ is a reversible Markov process, then for $A \in \mathscr{B}$ with $0 < \pi(A) < \infty$*

$$\lim_{n\to\infty} \frac{\gamma_{2(n+k)}(A)}{\gamma_{2n}(A)} = 1$$

for all integers k.

PROOF. Under the above hypotheses, Šidák [9] has the following representation:

$$\int_A P^{(n)}(x, B)\pi(dx) = \int_B P^{(n)}(x, A)\pi(dx) = \int_{-1}^{1} t^n \phi_{AB}(dt)$$

where ϕ is a real function of bounded variation on $[-1, 1]$. We show that $\phi_{AA}(t) = v(t)$ is monotonically increasing using the family of projections $\{E_t\}$

(via the spectral theorem) as follows: $-1 \leqq t_1 < t_2 \leqq 1$ implies that

$$v(t_2) - v(t_1) = ((E_{t_2} - E_{t_1})\chi_A, \chi_A) = ((E_{t_2} - E_{t_1})\chi_A, (E_{t_2} - E_{t_1})\chi_A) \geqq 0 .$$

Thus

$$\gamma_{2n}(A) - \gamma_{2n+2}(A) = \int_{-1}^{1} t^{2n}(1 - t^2)v(dt) \geqq 0$$

and, furthermore,

$$2(\gamma_{2n+2}(A)\gamma_{2n-2}(A) - \gamma_{2n}^2(A)) = \int_{-1}^{1}\int_{-1}^{1} t^{2n-2}s^{2n-2}(s^2 - t^2)^2 v(ds)v(dt) \geqq 0 .$$

Since $\pi(A) > 0$, there exists $C \in \mathscr{S}$ with $C \subset A$ (see Theorem 2.1 of [7]). Therefore $d = 1$ or $d = 2$ implies the existence of an N such that for all $2n \geqq N$

$$\gamma_{2n}(A) \geqq \int_C P^{(2n)}(x, C)\pi(dx) > 0 .$$

Hence $\gamma_{2n+2}(A)/\gamma_{2n}(A)$ exists for all sufficiently large n and by the above inequalities $\lim_{n\to\infty} \gamma_{2n+2}(A)/\gamma_{2n}(A)$ exists. By the Borel-Cantelli Lemma, $d = 1$ or $d = 2$ implies that $\sum_{k=0}^{\infty} \gamma_{2k}(A) = \infty$, and so

$$\lim_{n\to\infty} \frac{\gamma_{2n+2}(A)}{\gamma_{2n}(A)} = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \gamma_{2k+2}(A)}{\sum_{k=0}^{n} \gamma_{2k}(A)} = \lim_{n\to\infty}\left[1 + \frac{\gamma_{2n+2}(A) - \gamma_0(A)}{\sum_{k=0}^{n} \gamma_{2k}(A)}\right] = 1 .$$

$\lim_{n\to\infty} \gamma_{2(n+k)}(A)/\gamma_{2n}(A) = 1$ now follows in the usual manner.

We present the following example to illustrate the fact that $\phi_{AB}(t)$ of the above theorem need not be monotonic if $A \neq B$ with the measure of $A \triangle B$ not zero.

EXAMPLE. We define a family of projections $\{E_t\}$ on $\mathscr{L}_2(\lambda)$ with Lebesgue measure λ where $X = [0, 1]$. We then construct $P^{(n)}(x, A)$ as in Šidák [9]. Consider a fixed sequence $\{t_j\}$ satisfying $1 = t_0 > t_1 > t_2 > \cdots > 0$ and $\sum_{j=1}^{\infty} t_j < \frac{1}{2}$ and also the orthonormal family of functions in $\mathscr{L}_2(\lambda)$ given by

$$f_0(x) = 1 , \qquad f_j(x) = 2^{\frac{1}{2}} \sin 2\pi jx , \qquad j = 1, 2, \cdots .$$

Then for each $f \in \mathscr{L}_2(\lambda)$ define

$$\begin{aligned} E_t f(x) &= 0 & -1 \leqq t \leqq 0 \\ &= f(x) - \textstyle\sum_{j=0}^{m} c_j f_j(x) & t_{m+1} \leqq t < t_m,\ m = 0, 1, 2, \cdots \\ &= f(x) & t = t_0 = 1 \end{aligned}$$

where $\{c_j\}$ are the Fourier coefficients of $f(x)$, i.e.,

$$c_j = \int_0^1 f(x)f_j(x)\,dx , \qquad j = 0, 1, 2, \cdots .$$

Thus we define

$$P^{(n)}(x, A) = \int_{-1}^{1} t^n d(E_t \chi_A(x)) = \sum_{j=0}^{\infty} t_j^n \left(\int_A f_j(y)\,dy\right) f_j(x) .$$

Note that for all $x \in X$, $P^{(n)}(x, X) = 1$. Furthermore for $\lambda(A) > 0$,

$$P(x, A) \geqq \int_A \left[1 - 2^{\frac{1}{2}} \sum_{j=1}^{\infty} t_j f_j(y)\right] dy \geqq \int_A \left[1 - 2\sum_{j=1}^{\infty} t_j\right] dy > 0 .$$

Since $\int_0^1 f_j(y)\,dy = 0$ for $j \geqq 1$ and $\int_A f_j(y)\,dy = -\int_{X-A} f_j(y)\,dy$

$$P(x, A) \leqq \int_A dx + \int_{X-A} 2\sum_{j=1}^{\infty} t_j\,dx \leqq \int_X dx = 1 .$$

Thus $P^{(n)}(x, A)$ defined by this family $\{E_t\}$ is a probability function and it is clearly aperiodic, reversible and recurrent.

$$\int_B P^{(n)}(x, A)\,dx = \int_{-1}^{1} t^n \phi_{AB}(dt) = \sum_{j=0}^{\infty} t_j^n \left(\int_A f_j(x)\,dx\right)\left(\int_B f_j(y)\,dy\right).$$

We see that $\phi_{AB}(dt) = 0$ except at $t = 0$, and $t = t_j$ where

$$\phi_{AB}(dt) = \left(\int_A f_j(x)\,dx\right)\left(\int_B f_j(y)\,dy\right)$$

and clearly this quantity may be negative for some values of j. We now explicitly demonstrate the possible non-monotonic character of $\phi_{AB}(t)$ for $B \subset A$, $\lambda(A - B) > 0$ using the above example with $A = [\frac{1}{4}, 1]$ and $B = [\frac{1}{4}, \frac{1}{2}]$. Then

$$P^{(n)}(x, A) = \frac{3}{4} - \frac{t_1^n}{\pi} \sin 2\pi x - \frac{t_2^n}{\pi} \sin 4\pi x + \cdots$$

and

$$\int_B P^{(n)}(x, A)\,dx = \frac{3}{16} - \frac{t_1^n}{2\pi^2} + \frac{t_2^n}{2\pi^2} + \cdots$$

and thus

$$\begin{aligned} \phi_{AB}(dt) = \phi_{AB}(t + 0) - \phi_{AB}(t - 0) &= \tfrac{3}{16} && t = 1 \\ &= 0 && t_1 < t < 1 \\ &= -\frac{1}{2\pi^2} && t = t_1 \\ &= 0 && t_2 < t < t_1 \\ &= \frac{1}{2\pi^2} && t = t_2 \\ &\ \ \vdots && \ \ \vdots \end{aligned}$$

We now wish to consider how Theorem 2.1 may be employed in order to yield information concerning the existence of $\lim_{n\to\infty} P^{(n+k)}(x, A)/P^{(n)}(y, B)$. We note that the requirement of having $0 < \pi(A) < \infty$ and $0 < \pi(B) < \infty$ is not sufficient by considering Example 3.1 of Krengel [4]. In this example, he showed that $\limsup_{n\to\infty} P^{(n)}(0, A)/P^{(n)}(0, 0) = \infty$ although $\pi(0) = \pi(A) = 1$ and the discrete space is both aperiodic and reversible.

For the remainder of this paper we assume aperiodicity ($d = 1$). We now consider the following condition:

(*) Either (1) there exist constants M and N such that for all $x \in X$ and all $n \geqq N$,

$$\frac{P^{(2n)}(x, A)}{\gamma_{2n}(A)} < M$$

or (2) there exist constants M and N such that for all $x \in X$ and all $n \geqq N$

$$\frac{P^{(2n+1)}(x, A)}{\gamma_{2n}(A)} < M.$$

REMARK 1. We observe that (1) and (2) are actually equivalent, i.e., (1) implies (2) using

$$P^{(2n+1)}(x, A) = \int_X P(x, dw) P^{(2n)}(w, A)$$

and (2) implies (1) when reversibility holds by a similar argument.

REMARK 2. (*) is a slight variation of one of the hypotheses in Jain's Theorem 3 [2], and we use a similar method to show that in the reversible case, the conclusion of the theorem is obtained without the requirement of his condition (F). Furthermore, we somewhat strengthen Jain's result in the reversible case by obtaining it for all sets A and B satisfying $0 < \pi(A) < \infty$, $0 < \pi(B) < \infty$ and (*). Finally, we show that if there exists one set $E \in \mathscr{S}$ satisfying (*) then all $A \in \mathscr{S}$ satisfy it.

REMARK 3. We note that the state 0 of Krengel's Example 3.1 does not satisfy (*). If it did,

$$\limsup_{n\to\infty} \frac{P^{(n)}(0, A)}{P^{(n)}(0, 0)} = \limsup_{n\to\infty} \sum_{x\in A} \frac{P^{(n)}(0, x)}{P^{(n)}(0, 0)}$$
$$= \limsup_{n\to\infty} \sum_{x\in A} \frac{\pi(x)}{\pi(0)} \frac{P^{(n)}(x, 0)}{P^{(n)}(0, 0)} \leqq M \frac{\pi(A)}{\pi(0)} = M .$$

LEMMA 2.1. *If $\pi(A) > 0$, $A \in \mathscr{B}$ then there exists some m such that*

$$\pi(\{x \in A \mid {}_AP^{(2m+1)}(x, A) > 0\}) > 0 .$$

PROOF. Let $T_n = \{x \in A \mid {}_AP^{(n)}(x, A) > 0\}$ and assume that $\pi(T_{2n+1}) = 0$ for all $n \geqq 0$. Since $\pi(\bigcup_{n=0}^{\infty} T_{2n+1}) = 0$, invariance implies that $\pi(C_i) = 0$ where $C_i = \{x \in X \mid P^{(i)}(x, \bigcup_{n=0}^{\infty} T_{2n+1}) > 0\}$, $i \geqq 0$. Thus $\pi(\bigcup_{i=0}^{\infty} C_i) = 0$.

Due to aperiodicity, for all $x \in X$ there exists some odd integer ν_x such that $P^{(\nu_x)}(x, A) > 0$. Let $A_{2n+1} = \{x \in A \mid P^{(2n+1)}(x, A) > 0\}$. $A = \bigcup_{n=0}^{\infty} A_{2n+1}$, hence, $\pi(A) > 0$ implies that $\pi(A_{2n+1}) > 0$ for at least one n. Let n_0 be the smallest value for n such that $\pi(A_{2n_0+1}) > 0$. For every $x \in A_{2n_0+1}$, there must be at least one section of the entire path where x eventually moves from A to A in $2k+1$ steps without visiting A in between, $k = k(x)$. This implies that $A_{2n_0+1} \subset \bigcup_{i=0}^{\infty} C_i$ and thus $\pi(A_{2n_0+1}) = 0$ which is a contradiction, i.e., there is some m such that $\pi(T_{2m+1}) > 0$.

THEOREM 2.2. *A necessary and sufficient condition for an aperiodic, reversible Markov process to satisfy the strong ratio limit property*

$$\lim_{n\to\infty} \frac{P^{(n+k)}(\mu, A)}{P^{(n)}(\nu, B)} = \frac{\pi(A)}{\pi(B)}$$

for all integers k, all sets $A, B \in \mathscr{B}$ with $0 < \pi(A) < \infty$, $0 < \pi(B) < \infty$ and all probability distributions μ and ν on X is that A and B satisfy (*).

PROOF. Necessity follows exactly as in Jain's Theorem 3 [2]. To show sufficiency, we first show that for all probability distributions μ and all integers m

$$\liminf_{n\to\infty} \frac{P^{(2n-m)}(\mu, A)}{\gamma_{2n}(A)} = \frac{1}{\pi(A)} \liminf_{n\to\infty} \frac{\gamma_{2n-m}(A)}{\gamma_{2n}(A)} . \tag{1}$$

Note that both sides of the above equation are finite by (*) and Theorem 2.1. Without loss of generality assume

$$\Delta = \liminf_{n\to\infty} \frac{P^{(2n-m)}(\mu, A)}{\gamma_{2n}(A)} - \frac{1}{\pi(A)} \liminf_{n\to\infty} \frac{\gamma_{2n-m}(A)}{\gamma_{2n}(A)} \geqq 0 .$$

Denoting the probability measure $\sigma(\cdot) = \pi(\cdot \cap A)/\pi(A)$, we have

$$\begin{aligned} 0 \leqq \Delta &\leqq \limsup_{n\to\infty} \left| \frac{P^{(2n-m)}(\mu, A) - P^{(2n-m)}(\sigma, A)}{\gamma_{2n}(A)} \right| \\ &= \limsup_{n\to\infty} \left| \int_X [P^{(n_0)}(\mu, dx) - P^{(n_0)}(\sigma, dx)] \frac{P^{(2n-m-n_0)}(x, A)}{\gamma_{2n}(A)} \right| \\ &\leqq 2M \int_X |P^{(n_0)}(\mu, dx) - P^{(n_0)}(\sigma, dx)| . \end{aligned}$$

The Jamison–Orey Theorem [3] thus implies that $\Delta = 0$.

Next denote $a_j = \liminf_{n\to\infty} \gamma_{2n-j}(A)/\gamma_{2n}(A)$ for $j = 0, \pm 1, \pm 2, \cdots$ and observe that $a_{j+2} = a_j$ for all j by Theorem 2.1. Also

$$\begin{aligned} a_j &= \liminf_{n\to\infty} \int_A \sum_{v=1}^{2n-j} \int_{A\ A}P^{(v)}(x, dy) \frac{P^{(2n-j-v)}(y, A)}{\gamma_{2n}(A)} \pi(dx) \\ &\geqq \int_A \sum_{v=1}^{\infty} \int_{A\ A}P^{(v)}(x, dy) \liminf_{n\to\infty} \frac{P^{(2n-j-v)}(y, A)}{\gamma_{2n}(A)} \pi(dx) \\ &= \int_A \sum_{v=1}^{\infty} \int_{A\ A}P^{(v)}(x, dy) \frac{1}{\pi(A)} \liminf_{n\to\infty} \frac{\gamma_{2n-j-v}(A)}{\gamma_{2n}(A)} \pi(dx) \end{aligned}$$

by (1). Thus

$$a_j \geqq \sum_{v=1}^{\infty} \alpha_v a_{j+v}$$

where

$$\alpha_v = \frac{1}{\pi(A)} \int_{A\ A}P^{(v)}(x, A)\pi(dx) .$$

By Lemma 2.1, there exists m such that $\alpha_{2m+1} > 0$. Let $a_i = \min(a_0, a_1)$; setting $j = i$,

$$a_i \geqq \sum_{v=1}^{\infty} \alpha_v a_i + \alpha_{2m+1}(a_{i+2m+1} - a_i) = a_i + \alpha_{2m+1}(a_{i+2m+1} - a_i)$$

since $\sum_{v=1}^{\infty} \alpha_v = 1$. Therefore

$$a_i \geqq a_{i+2m+1} = a_{i+1}$$

implying that $a_i = a_{i+1}$, so $a_j = a_0 = 1$ for all j, i.e.,

$$\liminf_{n\to\infty} \frac{\gamma_{2n+1}(A)}{\gamma_{2n}(A)} = 1 .$$

From $\gamma_n(A) = \int_{-1}^{1} t^n v(dt)$ we have $\gamma_{2n+1}(A) - \gamma_{2n}(A) \leqq 0$ for all n, so $\lim_{n\to\infty} \gamma_{2n+1}(A)/\gamma_{2n}(A) = 1$. $\lim_{n\to\infty} \gamma_{n+1}(A)/\gamma_n(A) = 1$ now follows from Theorem 2.1.

We now observe that the proof that $\Delta = 0$ works equally well to show that

$$\limsup_{n\to\infty} \frac{P^{(2n-m)}(\mu, A)}{\gamma_{2n}(A)} = \frac{1}{\pi(A)} \limsup_{n\to\infty} \frac{\gamma_{2n-m}(A)}{\gamma_{2n}(A)},$$

so

$$\lim_{n\to\infty} \frac{P^{(n+k)}(\mu, A)}{\gamma_n(A)} = \frac{1}{\pi(A)}.$$

Thus

$$\lim_{n\to\infty} \frac{P^{(n+k)}(\mu, A)}{P^{(n)}(\nu, B)} = \lim_{n\to\infty} \frac{P^{(n+k)}(\mu, A)/\gamma_n(A)}{P^{(n)}(\nu, B)/\gamma_n(B)} \lim_{n\to\infty} \frac{\gamma_n(A)}{\gamma_n(B)}.$$

Writing

$$\frac{\gamma_n(A)}{\gamma_n(B)} = \frac{\gamma_n(A)}{\int_B P^{(n)}(x, A)\pi(dx)} \frac{\int_A P^{(n)}(x, B)\pi(dx)}{\gamma_n(B)}$$

by reversibility and denoting $\tau(\cdot) = \pi(\cdot \cap B)/\pi(B)$,

$$\lim_{n\to\infty} \frac{\gamma_n(A)}{\gamma_n(B)} = \lim_{n\to\infty} \frac{\pi(A)P^{(n)}(\sigma, B)/\gamma_n(B)}{\pi(B)P^{(n)}(\tau, A)/\gamma_n(A)} = \frac{\pi(A)/\pi(B)}{\pi(B)/\pi(A)}.$$

Therefore

$$\lim_{n\to\infty} \frac{P^{(n+k)}(\mu, A)}{P^{(n)}(\nu, B)} = \frac{\pi(A)}{\pi(B)}.$$

COROLLARY. *A necessary and sufficient condition for an aperiodic, reversible Markov process to satisfy the* SRLP *for all sets* $A, B \in \mathscr{S}$ *is that there exists a set* $E \in \mathscr{S}$ *satisfying* (*).

PROOF. By Jain's Lemma 2.1 [2], there exist constants j and M_1 such that $P^{(n)}(x, E) \leqq M_1 P^{(n+j)}(x, A)$ for all $x \in X$ and all n. Thus

$$\gamma_n(E) \leqq M_1 \int_E P^{(n+j)}(x, A)\pi(dx) = M_1 \int_A P^{(n+j)}(x, E)\pi(dx) \leqq M_1^2 \gamma_{n+2j}(A).$$

Since E satisfies (*), $\lim_{n\to\infty} \gamma_{n+1}(E)/\gamma_n(E) = 1$, so it follows that for some M and all sufficiently large n, $P^{(n)}(x, E)/\gamma_n(E) < M$ for all $x \in X$. Then $P^{(n)}(x, A) \leqq M_2 P^{(n+j)}(x, E) < MM_2\gamma_{n+j}(E)$, for some constant M_2. $\lim_{n\to\infty} \gamma_{n+1}(E)/\gamma_n(E) = 1$ implies that for all sufficiently large n

$$P^{(n)}(x, A) < 2MM_2\gamma_{n-2j}(E) < 2MM_1^2M_2\gamma_n(A).$$

Thus all sets in $\mathscr{S}$ satisfy (*) and the SRLP follows.

The following results are obtained without assuming (*).

LEMMA 2.2. *Let* $A \in \mathscr{B}$ *with* $0 < \pi(A) < \infty$ *and* τ *be some finite measure defined on* X. *Then if there exist a set* $B \in \mathscr{B}$ *with* $\pi(B) > 0$ *and integers* r, s *and* t *such that*

$$\liminf_{n\to\infty} \frac{P^{(2n-2r)}(y, A)}{\gamma_{2n}(A)} \geqq \frac{1}{\pi(A)} \quad \text{and} \quad \liminf_{n\to\infty} \frac{P^{(2n-2s+1)}(y, A)}{\gamma_{2n}(A)} \geqq \frac{1}{\pi(A)}$$

for all $y \in B$ *and*

$$\limsup_{n\to\infty} \frac{P^{(2n-t)}(\tau, A)}{\gamma_{2n}(A)} \leqq \frac{\tau(X)}{\pi(A)}$$

then for all integers k

$$\lim_{n\to\infty} \frac{P^{(2n-t-2k)}(\tau, A)}{\gamma_{2n}(A)} = \frac{\tau(X)}{\pi(A)}.$$

PROOF. Let $a_j(x) = \liminf_{n\to\infty} P^{(2n-j)}(x, A)/\gamma_{2n}(A)$ for $j = 0, \pm 1, \pm 2, \cdots$ and observe that $a_{j+2}(x) = a_j(x)$ for all j by Theorem 2.1.

$$\begin{aligned} a_j(x) &= \liminf_{n\to\infty} \left[\sum_{v=1}^{2n-j-1} \int_B \frac{{}_B P^{(v)}(x, dy) P^{(2n-j-v)}(y, A)}{\gamma_{2n}(A)} + \frac{{}_B P^{(2n-j)}(x, A)}{\gamma_{2n}(A)} \right] \\ &\geqq \sum_{v=1}^{\infty} \int_B {}_B P^{(v)}(x, dy) a_{j+v}(y) \\ &\geqq \frac{1}{\pi(A)} \sum_{v=1}^{\infty} {}_B P^{(v)}(x, B) = \frac{1}{\pi(A)} \end{aligned}$$

for all $x \in X$ and all j. We get the desired result by applying Theorem 2.1 and Fatou's Lemma to $P^{(2n-t-2k)}(\tau, A)/\gamma_{2n}(A)$.

COROLLARY. *If A satisfies the hypothesis of Lemma 2.2 along with τ and σ, two finite measures on X with respective integers t and t', then for all integers k*

$$\lim_{n\to\infty} \frac{P^{(2n-t-2k)}(\tau, A)}{P^{(2n-t')}(\sigma, A)} = \frac{\tau(X)}{\sigma(X)}.$$

THEOREM 2.3. *Let $A \in \mathscr{B}$ with $0 < \pi(A) < \infty$. If there exist a set $B \in \mathscr{B}$ with $\pi(B) > 0$ and integers r and s such that*

$$\liminf_{n\to\infty} \frac{P^{(2n-2r)}(y, A)}{\gamma_{2n}(A)} \geqq \frac{1}{\pi(A)} \qquad \text{and} \qquad \liminf_{n\to\infty} \frac{P^{(2n-2s+1)}(y, A)}{\gamma_{2n}(A)} \geqq \frac{1}{\pi(A)}$$

for all $y \in B$, then for all integers k

$$\lim_{n\to\infty} \frac{\gamma_{n+k}(A)}{\gamma_n(A)} = 1.$$

PROOF. As in Theorem 2.2, for n sufficiently large $\gamma_{2n+1}(A)/\gamma_{2n}(A) \leqq 1$. Letting $\tau(\cdot) = \pi(\cdot \cap A)$ in Lemma 2.2, we have $\lim_{n\to\infty} \gamma_{2n+1}(A)/\gamma_{2n}(A) = 1$. This, together with Theorem 2.1 yields $\lim_{n\to\infty} \gamma_{n+k}(A)/\gamma_n(A) = 1$.

3. A space with one or more atoms.

THEOREM 3.1. *Let $\{q\}$ be an atom (see ([5]) of X. Then for all $A, B \in \mathscr{S}$ and all integers k*

(i) $\lim_{n\to\infty} P^{(n+k)}(x, A)/P^{(n)}(y, B) = \pi(A)/\pi(B)$ *in measure $\pi_1 \times \pi_2$ on $(X \times X, \mathscr{B} \times \mathscr{B})$, where π_1 and π_2 are any finite measures on X with $\pi_1 \ll \pi$, $\pi_2 \ll \pi$.*

(ii) $\lim_{n\to\infty} P^{(n+k)}(\mu, A)/P^{(n)}(\nu, B) = \pi(A)/\pi(B)$ *for any probability measures μ and ν on X with $\mu \ll \pi$, $\nu \ll \pi$, $d\mu/d\pi$ and $d\nu/d\pi$ bounded with supports in $\mathscr{S}$.*

PROOF. We first observe that $\lim_{n\to\infty} P^{(n+k)}(q, q)/P^{(n)}(q, q) = 1$ (denoting $\{q\}$ by q). This follows by using the method of Theorem 2.2 and letting q play the role of A. (That is, $a_j = \liminf_{n\to\infty} P^{(2n-j)}(q, q)/P^{(2n)}(q, q) \geqq \sum_{v=1}^{\infty} {}_q P^{(v)}(q, q) a_{j+v}$ and with Lemma 2.1 we have $a_j = 1$ for all j). (i) now follows from Corollary 3.4 of Levitan [5] and (ii) from Corollary 3.3 of Lin [6].

COROLLARY 1. *Let $\{q\}$ be an atom of X. Then for all $A, B, C, D \in \mathscr{S}$ and for all integers k*

$$\lim_{n\to\infty} \frac{\int_A P^{(n+k)}(x, B)\pi(dx)}{\int_C P^{(n)}(x, D)\pi(dx)} = \frac{\pi(A)\pi(B)}{\pi(C)\pi(D)} .$$

PROOF. Define the probability measures $\mu(\cdot) = \pi(\cdot \cap A)/\pi(A)$ and $\nu(\cdot) = \pi(\cdot \cap C)/\pi(C)$ and apply (ii) of Theorem 3.1.

REMARK. Krengel's set A of Example 3.1 [4] is not in $\mathscr{S}$. This is shown by noting that for any fixed m, there exist points (r, a_r), $(s, a_s) \in A$ with $\pi((s, a_s)) > 0$ such that $P^{(m)}((r, a_r), (s, a_s)) = 0$ implying that $p^{(m)}((r, a_r), (s, a_s)) = 0$.

From [5], we know that there exist sets $A_i \in \mathscr{S}$ such that $\lim_{i\to\infty} \pi(A - A_i) = 0$. Thus, although $0 \in \mathscr{S}$ implying that $\lim_{n\to\infty} P^{(n+k)}(0, A_i)/P^{(n)}(0, 0) = \pi(A_i)/\pi(0)$ for all i and for all k, $\lim_{n\to\infty} P^{(n)}(0, A)/P^{(n)}(0, 0)$ does not exist.

COROLLARY 2. *Let $\{q\}$ be an atom of X. Then for all $A \in \mathscr{S}$ and for all integers k*

$$\lim_{n\to\infty} \frac{\gamma_{n+k}(A)}{\gamma_n(A)} = 1 .$$

COROLLARY 3. *If X is a discrete space then for all $A, B \in \mathscr{S}$ and for all integers k*

$$\lim_{n\to\infty} \frac{P^{(n+k)}(x, A)}{P^{(n)}(y, B)} = \frac{\pi(A)}{\pi(B)} .$$

REMARK. This result has been obtained in Corollary 3.5 of Levitan [5] under weaker hypotheses.

In the discrete case, we may strengthen Lemma 2.2.

LEMMA 3.1. *Let $A \in \mathscr{B}$ with $0 < \pi(A) < \infty$ and τ be some finite measure defined on the discrete space X. Then if there exist $x \in X$ and integers r and s such that*

$$\liminf_{n\to\infty} \frac{P^{(2n-r)}(x, A)}{\gamma_{2n}(A)} \geqq \frac{1}{\pi(A)} \quad \textit{and} \quad \limsup_{n\to\infty} \frac{P^{(2n-s)}(\tau, A)}{\gamma_{2n}(A)} \leqq \frac{\tau(X)}{\pi(A)}$$

then for all integers k

$$\lim_{n\to\infty} \frac{P^{(2n-s-2k)}(\tau, A)}{\gamma_{2n}(A)} = \frac{\tau(X)}{\pi(A)} .$$

PROOF. Let $a_j(x) = \liminf_{n\to\infty} P^{(2n-j)}(x, A)/\gamma_{2n}(A)$. Following the proof of Lemma 2.2 with $B = \{y\}$ we have

$$a_j(x) \geqq \sum_{v=1}^{\infty} {}_yP^{(v)}(x, y)a_{j+v}(y) \tag{2}$$

for any $x, y \in X$.

Letting $y = x$, for some odd v, ${}_xP^{(v)}(x, x) > 0$ by Lemma 2.1. For $a_m(x) = \min(a_0(x), a_1(x))$ we have

$$a_m(x) \geqq a_m(x) + [a_{m+1}(x) - a_m(x)]\,{}_xP^{(v)}(x, x)$$

implying that $a_{m+1}(x) = a_m(x) = a_0(x)$ for all $x \in X$. From (2) we therefore have

$a_0(x) \geqq a_0(y)$. Interchanging the roles of x and y yields $a_0(x) = a_0(y)$ for all $x, y \in X$. Hence for all $x \in X$, $a_0(x) \geqq 1/\pi(A)$. The result now follows as in Lemma 2.2.

COROLLARY. *If A satisfies the hypothesis of Lemma 3.1 along with τ and σ, two finite measures on X with respective integers t and t', then for all integers k*

$$\lim_{n\to\infty} \frac{P^{(2n-t-2k)}(\tau, A)}{P^{(2n-t')}(\sigma, A)} = \frac{\tau(X)}{\sigma(X)} .$$

THEOREM 3.2. *Let $A \in \mathscr{B}$ with $0 < \pi(A) < \infty$ and the space X be discrete. Then if there exist $x \in X$ and an integer t such that*

$$\liminf_{n\to\infty} \frac{P^{(2n-t)}(x, A)}{\gamma_{2n}(A)} \geqq \frac{1}{\pi(A)}$$

then for all integers k

$$\lim_{n\to\infty} \frac{\gamma_{n+k}(A)}{\gamma_n(A)} = 1 .$$

PROOF. This follows the proof of Theorem 2.3.

REMARK. A sufficient conditon for $\lim_{n\to\infty} \gamma_{n+k}(A)/\gamma_n(A) = 1$ in Theorem 3.2 is that either

(i) there exist $x \in X$ and an integer t such that for infinitely many n,

$$\sum_{i=1}^{\infty} P^{(2n)}(q_i, A) \frac{\pi(q_i)}{\pi(A)} \leqq P^{(2n-t)}(x, A)$$

where $A = \{q_i, i = 1, 2, \cdots\}$, or

(ii) there exist $x \in X$ and an integer t such that for infinitely many n,

$$P^{(2n)}(q_i, A) \leqq P^{(2n-t)}(x, A) \qquad \text{for all } q_i \in A .$$

Note that for each fixed n, there obviously exists a point $x(n) \in X$ satisfying the inequality in (i) with $t = 0$.

4. Space without the reversibility hypothesis.

NOTATION. $\delta_n(A, B) = \int_A P^{(n)}(x, B)\pi(dx)$, $n = 0, 1, 2, \cdots$.

THEOREM 4.1. *The existence of $G \in \mathscr{S}$ satisfying both (*) and*

$$\lim_{n\to\infty} \frac{\gamma_{n+1}(G)}{\gamma_n(G)} = 1$$

is a necessary and sufficient condition for

$$\lim_{n\to\infty} \frac{P^{(n+k)}(\mu, A)}{P^{(n)}(\nu, A)} = 1$$

to hold for all probability distributions μ and ν on X, all $A \in \mathscr{S}$ and all integers k.

PROOF. Necessity follows as in Jain's Theorem 3 [2]. Writing

$$\frac{P^{(n+k)}(x, A)}{\delta_n(G, A)} = \frac{P^{(n+k)}(x, A)}{P^{(n+j+k)}(x, G)} \frac{P^{(n+j+k)}(x, G)}{\gamma_{n+j+k}(G)} \frac{\gamma_{n+j+k}(G)}{\gamma_{n-j}(G)} \frac{\gamma_{n-j}(G)}{\delta_n(G, A)}$$

for any integer k and fixed integer j to be determined later, we are able to show that there exists M such that for sufficiently large n, $P^{(n+k)}(x, A)/\delta_n(G, A) < M$. The first and fourth terms on the right side of the above equation are uniformly bounded for all $x \in X$ via Jain's Lemma 2.1 [2] (thus determining j). The second term is uniformly bounded by (*) and the third term is bounded since $\lim_{n\to\infty} \gamma_{n+1}(G)/\gamma_n(G) = 1$. Proceeding as in Theorem 2.2, we may write without loss of generality

$$0 \leqq \Delta = \liminf_{n\to\infty} \frac{P^{(n+1)}(\mu, A)}{\delta_n(G, A)} - \frac{1}{\pi(G)} \liminf_{n\to\infty} \frac{\delta_n(G, A)}{\delta_n(G, A)}$$
$$\leqq \limsup_{n\to\infty} \int_X |P^{(n_0+1)}(\mu, dx) - P^{(n_0)}(\sigma, dx)| \frac{P^{(n-n_0)}(x, A)}{\delta_n(G, A)}$$

where $\sigma(\cdot) = \pi(\cdot \cap G)/\pi(G)$. Since $P^{(n-n_0)}(x, A)/\delta_n(G, A)$ is uniformly bounded on X, we may apply the Jamison-Orey Theorem to show that $\Delta = 0$. Corresponding results hold for $\limsup_{n\to\infty} P^{(n+1)}(\mu, A)/\delta_n(G, A)$ and we have

$$\lim_{n\to\infty} \frac{P^{(n+1)}(\mu, A)}{\delta_n(G, A)} = \frac{1}{\pi(G)} .$$

Similarly,

$$\lim_{n\to\infty} \frac{P^{(n)}(\mu, A)}{\delta_n(G, A)} = \frac{1}{\pi(G)} \qquad \text{and so} \qquad \lim_{n\to\infty} \frac{P^{(n+k)}(\mu, A)}{P^{(n)}(\nu, A)} = 1 .$$

In order to extend Theorem 4.1 to obtain the SRLP, we have the following:

DEFINITION. A Markov process will be called weakly reversible if there exists a set $C \in \mathscr{S}$ such that for all $A \in \mathscr{S}$

$$\lim_{n\to\infty} \frac{\delta_n(C, A)}{\delta_n(A, C)} = 1 .$$

THEOREM 4.2. *If there exists a set $G \in \mathscr{S}$ satisfying both (*) and*

$$\lim_{n\to\infty} \frac{\gamma_{n+1}(G)}{\gamma_n(G)} = 1$$

and in addition the Markov process is weakly reversible, then for all probability distributions μ and ν on X, all $A, B \in \mathscr{S}$, and all integers k

$$\lim_{n\to\infty} \frac{P^{(n+k)}(\mu, A)}{P^{(n)}(\nu, B)} = \frac{\pi(A)}{\pi(B)} .$$

PROOF. For any $A \in \mathscr{S}$, let $\sigma_A = \pi(\cdot \cap A)/\pi(A)$. Then $\delta_n(A, C) = \pi(A)P^{(n)}(\sigma_A, C)$.

$$\frac{P^{(n+k)}(\mu, A)}{P^{(n)}(\nu, B)} = \frac{\dfrac{P^{(n+k)}(\mu, A)}{\delta_{n+k}(C, A)} \dfrac{\delta_{n+k}(C, A)}{\delta_{n+k}(A, C)} \dfrac{\delta_{n+k}(A, C)}{\delta_{n+k}(G, C)} \dfrac{\delta_{n+k}(G, C)}{\delta_{n+k}(C, G)} \dfrac{\delta_{n+k}(C, G)}{\gamma_{n+k}(G)} \gamma_{n+k}(G)}{\dfrac{P^{(n)}(\nu, B)}{\delta_n(C, B)} \dfrac{\delta_n(C, B)}{\delta_n(B, C)} \dfrac{\delta_n(B, C)}{\delta_n(G, C)} \dfrac{\delta_n(G, C)}{\delta_n(C, G)} \dfrac{\delta_n(C, G)}{\gamma_n(G)} \gamma_n(G)} .$$

The result now follows using Theorem 4.1, the weak reversibility condition and the hypothesis on G.

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MICHAEL L. LEVITAN
DEPARTMENT OF MATHEMATICS
VILLANOVA UNIVERSITY
VILLANOVA, PENNSYLVANIA 19085

LAWRENCE H. SMOLOWITZ
HOBART COLLEGE
GENEVA, NEW YORK 14456