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ON SYMMETRIC MOMENTS AND LOCAL BEHAVIOR OF THE CHARACTERISTIC FUNCTION

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Let $F(x)$ be a probability distribution function and $\phi(t)$ its characteristic function. Define $\psi(t) = \text{Im}\,\phi(t) = -\int_0^\infty \sin tx\, dG(x)$, where $G(x) = 1 - F(x) - F(-x)$ is the tail difference of F. In general G is known only to be of bounded variation on $[0, \infty]$. The purpose of this paper is to relate differentiability of ψ at 0 and Lipschitz behavior of ψ to asymptotic behavior of G. Let C_α be the class of real valued functions g such that $\int_x^\infty |dg(u)| = o(x^{-\alpha})$ $(x \to +\infty)$, and let μ_α^* be the symmetric moment of order α. Then it is shown that $G \in C_1$ implies $\psi'(0) = \mu_1^*$ if either exists, while $G \in C_\alpha$ implies $\psi \in \text{Lip}\,\alpha$ for $0 < \alpha < 1$. Extensions of these results to higher derivatives and values of $\alpha > 1$ are indicated. Finally it is shown that, for $1 < \alpha < 3$, $G(x) = o(x^{-\alpha})$ $(x \to +\infty)$ and $t^{-\alpha-1}(\psi(t) - t\mu_1^*) \in L(0, 1)$ imply that μ_α^* exists.

1. Introduction. Let $F(x)$ be a distribution function and $\phi(t)$ its characteristic function. Results of Boas [2] relate local behavior of $\text{Re}\,\phi$ to asymptotic behavior of $1 - F(x) + F(-x)$, while Pitman [4] and Zygmund [7] give necessary and sufficient conditions for the existence of $\phi'(0)$. It is natural now to investigate differentiability of $\text{Im}\,\phi$ at 0 and Lipschitz behavior of $\text{Im}\,\phi$. Conditions on F ensuring such behavior are the subject of this paper. Since $\text{Im}\,\phi(t) = -\int_0^\infty \sin tx\, dG(x)$, where G is a function of bounded variation, the results derived here apply in general to the theory of Fourier–Stieltjes transforms of functions of bounded variation.

2. Notation. Let $H(x) = 1 - F(x) + F(-x)$ and $G(x) = 1 - F(x) - F(-x)$ be the tail sum and tail difference of F.

Let $\mu_\alpha^* = \lim_{T\to\infty} \int_{-T}^T \text{sign}\, x|x|^\alpha\, dF(x) = -\int_0^\infty x^\alpha\, dG(x)$ be the symmetric moment of order α, and define $\psi(t) = \text{Im}\,\phi(t) = -\int_0^\infty \sin tx\, dG(x)$. Differentiating this equation formally gives $\psi'(0) = \mu_1^*$, and conditions on G guaranteeing this equality are sought. In this paper the result has not been achieved in its greatest generality, and a subsidiary condition on G has been introduced to make the proofs work. This condition will be shown not to be necessary. Define now the class C_α: $g \in C_\alpha$ if g is a real valued function defined on $[0, \infty]$, having a representation $g = g_1 - g_2$, where g_1 and g_2 are bounded and non-increasing, with $g_1(x)$ and $g_2(x)$ being $o(x^{-\alpha})$ $(x \to +\infty)$. It is useful, and not hard to show that, for $\alpha \geqq 0$, $g \in C_\alpha$ if and only if $\int_x^\infty |dg(u)| < +\infty$ for $x \geqq 0$ and is $o(x^{-\alpha})$ $(x \to +\infty)$.

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Now if $H(x) = o(x^{-\alpha})$ $(x \to +\infty)$ then $G \in C_\alpha$, while if G is ultimately monotonic and $o(x^{-\alpha})$ $(x \to +\infty)$, then $G \in C_\alpha$. Relevant to these points see Pitman [5], who gives results on local behavior of ψ under regularity conditions on G.

3. Derivatives of ψ at 0.

THEOREM 1. *If* $G \in C_1$ *then* $\psi'(0) = \mu_1{}^*$ *if either exists.*

PROOF. Write

$$t\psi(t^{-1}) + \int_0^t x\,dG(x) = \int_0^t (x - t\sin xt^{-1})\,dG(x) + t\int_t^\infty \sin xt^{-1}\,dG(x)\,.$$

Since ψ is an odd function it suffices to show that the right hand side vanishes as $t \to +\infty$. Fix $\varepsilon > 0$ and choose T so large that $|G(x)| < \varepsilon x^{-1}$ whenever $x \geqq T$. Taking $t > T$ and integrating by parts,

$$\int_0^t (x - t\sin xt^{-1})\,dG(x) = o(1) - \int_0^t (1 - \cos xt^{-1})G(x)\,dx \qquad (t \to +\infty)\,.$$

Further,

$$\begin{aligned}\left|\int_0^t (1 - \cos xt^{-1})G(x)\,dx\right| &= \left|\left\{\int_0^T + \int_T^t\right\}(1 - \cos xt^{-1})G(x)\,dx\right|\,,\\ &\leqq o(1) + t\left|\int_{T/t}^1 (1 - \cos x)G(xt)\,dx\right| \qquad (t \to +\infty)\,,\\ &\leqq o(1) + \varepsilon\int_0^1 x^{-1}(1 - \cos x)\,dx \qquad (t \to +\infty)\,,\end{aligned}$$

hence is arbitrarily small. Finally,

$$\left|t\int_t^\infty \sin xt^{-1}\,dG(x)\right| \leqq t\int_t^\infty |dG(x)| = o(1) \qquad (t \to +\infty)\,,$$

since $G \in C_1$.

REMARKS. (i) The condition $G \in C_1$ can be replaced by $G(x) = o(x^{-\alpha})$ $(x \to +\infty)$ $(\alpha > 1)$. Then $\mu_1{}^*$ exists.

(ii) The condition $G \in C_1$ is not necessary in Theorem 1. Define

$$\begin{aligned}G(x) &= 0 & 0 \leqq x < 1\\ &= ax^{-2}\sin x & x \geqq 1\,.\end{aligned}$$

G is of bounded variation hence for some choice of $a \neq 0$ it is the tail difference of a distribution. Clearly, by remark (1), $\psi'(0) = \mu_1{}^* = a\int_1^\infty x^{-2}\sin x\,dx$. However $G \notin C_1$. To see this, for $x > 1$,

$$\begin{aligned}a^{-1}x\int_x^\infty |dG(u)| &= x\int_x^\infty |u^{-2}\cos u - 2u^{-3}\sin u|\,du\\ &\geqq \left|x\int_x^\infty u^{-2}|\cos u|\,du - 2x\int_x^\infty u^{-3}|\sin u|\,du\right|\\ &= \int_1^\infty u^{-2}|\cos ux|\,dx + o(1) \qquad (x \to +\infty)\\ &\to \int_1^\infty u^{-2}\,du = 1 \qquad (x \to +\infty)\,.\end{aligned}$$

Hence $\int_x^\infty |dG(u)| \neq o(x^{-1})$ $(x \to +\infty)$ and so $G \notin C_1$.

(iii) Let $n \geqq 0$ be an integer. Then $G \in C_{2n+1}$ implies $\psi^{(2n+1)}(0) = (-1)^n\mu_{2n+1}^*$ if either exists, while $G \in C_{2n}$ implies $\psi^{(2n)}(0) = 0$. Proof of this extension is similar to Theorem 1 and will be omitted.

4. Lipschitz behavior of ψ.

THEOREM 2. (I) *For* $0 < \alpha < 1$ *let* $G \in C_\alpha$. *Then* $\psi(t+h) - \psi(t) = o(|h|^\alpha)$ *uniformly in* t *as* $h \to 0$.

(II) *Let* $n \geqq 0$ *be an integer and* $2n+1 < \alpha < 2n+3$. *Then* $G \in C_\alpha$ *implies*

$$\psi(t+h) - \psi(t) - h\psi'(t) - \frac{h^2}{2!}\psi''(t) \cdots - \frac{h^{2n+1}}{(2n+1)!}\psi^{(2n+1)}(t) = o(|h|^\alpha)$$

uniformly in t *as* $h \to 0$.

PROOF. Proof of (II) is similar to the proof of (I) and will be omitted. To simplify the notation write h for $|h|^{-1}$. Then

$$\begin{aligned} h^\alpha(\psi(t+2h^{-1}) - \psi(t)) &= h^\alpha \int_0^\infty (\sin tx - \sin(t+2h^{-1})x)\, dG(x) \\ &= -2h^\alpha\{\textstyle\int_0^h + \int_h^\infty\} \cos(t+h^{-1})x \sin xh^{-1}\, dG(x)\,. \end{aligned}$$

The second integral is $o(1)$ uniformly in t as $h \to +\infty$, since $G \in C_\alpha$. To deal with the first integral, define $A(x) = \int_x^\infty \cos(t+h^{-1})u\, dG(u)$, which is in C_α, hence in particular $A(x) = o(x^{-\alpha})$ $(x \to +\infty)$. Then, integrating by parts

$$\begin{aligned} &h^\alpha \textstyle\int_0^h \cos(t+h^{-1})x \sin xh^{-1}\, dG(x) \\ &\qquad = o(1) + h^{\alpha-1} \textstyle\int_0^h \cos xh^{-1} A(x)\, dx \qquad (h \to +\infty) \\ &\qquad = o(1) + h^{\alpha-1} \textstyle\int_0^h o(x^{-\alpha})\, dx \qquad (h \to +\infty) \\ &\qquad = o(1) \qquad (h \to +\infty)\,, \end{aligned}$$

uniformly in t, which proves the theorem.

REMARKS. (i) Theorem 2 is most useful at $t = 0$. In this case (II) can be strengthened. In fact, if $n \geqq 0$ is an integer and $2n+1 < \alpha < 2n+3$, then $G(x) = o(x^{-\alpha})$ $(x \to +\infty)$ implies

$$\psi(t) - t\mu_1{}^* + \frac{t^3}{3!}\mu_3{}^* + \cdots + \frac{(-1)^{n+1}}{(2n+1)!} t^{2n+1}\mu_{2n+1}^* = o(t^\alpha) \qquad (t \to 0+)\,.$$

(ii) As an application of Theorem 2, Lemma 3 of Koopman [3] and Lemma 1 of Binmore and Stratton [1] can be deduced immediately.

5. Sufficient condition for existence of $\mu_\alpha{}^*$. Von Bahr [6] has given an expression for the absolute moment of non-integer order of F in terms of an integral of the real part of the characteristic function. The next theorem gives an analogous result for the symmetric moments.

THEOREM 3. *For* $1 < \alpha < 3$ *let* $G(x) = o(x^{-\alpha})$ $(x \to +\infty)$ *and let* $t^{-\alpha-1}(\psi(t) - t\mu_1{}^*) \in L(0, 1)$. *Then* $\mu_\alpha{}^*$ *exists and*

$$\mu_\alpha{}^* = \frac{2}{\pi}\Gamma(\alpha+1)\cos\frac{\alpha\pi}{2} \int_0^\infty t^{-\alpha-1}(\psi(t) - t\mu_1{}^*)\, dt\,.$$

PROOF. Take $1 < \alpha \leqq 2$. Now $\mu_1{}^*$ exists and equals $\int_0^\infty G(x)\, dx$. Define $A(x) = -\int_x^\infty G(u)\, du$. Under the hypothesis $\int_0^\infty |G(x)|\, dx < +\infty$, which is guaranteed by $G(x) = o(x^{-\alpha})$ $(x \to +\infty)$, it is easy to verify that A is of bounded variation on $[0, \infty]$, that

$$t^{-2}(\psi(t) - t\mu_1^*) = \int_0^\infty \sin txA(x)\,dx$$

and that

$$A(x) = \frac{2}{\pi}\int_0^\infty \sin tx \cdot t^{-2}(\psi(t) - t\mu_1^*)\,dt\,.$$

Further it is easily shown that, for $T > 0$,

$$\int_0^T x^{\alpha-2}A(x)\,dx = \frac{2}{\pi}\int_0^\infty t^{-2}(\psi(t) - t\mu_1^*)\int_0^T x^{\alpha-2}\sin tx\,dx\,dt\,.$$

Now for $\alpha = 2$,

$$\int_0^T A(x)\,dx = \frac{2}{\pi}\int_0^\infty t^{-3}(\psi(t) - t\mu_1^*)(1 - \cos tT)\,dt$$

$$\to \frac{2}{\pi}\int_0^\infty t^{-3}(\psi(t) - t\mu_1^*)\,dt \qquad (T \to +\infty)\,,$$

by the Riemann–Lebesgue lemma, since by hypothesis $t^{-3}(\psi(t) - t\mu_1^*) \in L(0, \infty)$. For $1 < \alpha < 2$,

$$\int_0^T x^{\alpha-2}A(x)\,dx + \frac{2}{\pi}\Gamma(\alpha - 1)\cos\frac{\alpha\pi}{2}\int_0^\infty t^{-\alpha-1}(\psi(t) - t\mu_1^*)\,dt$$

$$= \frac{2}{\pi}\int_0^\infty t^{-\alpha-1}(\psi(t) - t\mu_1^*)\int_{Tt}^\infty x^{\alpha-2}\sin x\,dx\,dt$$

$\to 0$ as $T \to +\infty$ by dominated convergence, since the inner integral is a bounded function of t. Thus for $1 < \alpha \leqq 2$,

$$\int_0^\infty x^{\alpha-2}A(x)\,dx = -\frac{2}{\pi}\Gamma(\alpha - 1)\cos\frac{\alpha\pi}{2}\int_0^\infty t^{-\alpha-1}(\psi(t) - t\mu_1^*)\,dt\,,$$

and the theorem follows on integrating the left hand side by parts, remembering that $G(x) = o(x^{-\alpha})\,(x \to +\infty)$. Finally, for $2 < \alpha < 3$ define $A_1(x) = \int_x^\infty A(u)\,du$; then it is easy to see that

$$t^{-3}(\psi(t) - t\mu_1^*) = \int_0^\infty \cos txA_1(x)\,dx\,.$$

This transform can be inverted, and proceeding exactly as before, the theorem follows.

REMARKS. (i) Theorem 3 can be extended in an obvious way to values of $\alpha > 1$, α not an odd integer.

(ii) It is easy to give a sufficient condition for $t^{-\alpha-1}(\psi(t) - t\mu_1^*) \in L(0, 1)$ for $1 < \alpha < 3$. In fact, $\psi(t) - t\mu_1^* = o(t^\beta)$ $(t \to 0+)$, $\beta > \alpha$, is sufficient for this, and by Remark (i) following Theorem 2, $G(x) = o(x^{-\beta})$ $(x \to +\infty)$ is sufficient for $\psi(t) - t\mu_1^* = o(t^\beta)$ $(t \to 0+)$ $(\beta > 1)$. Of course in this case μ_α^* exists for $\alpha < \beta$, and Theorem 3 is just a formula for calculation of μ_α^*.

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