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WEAK CONVERGENCE OF CERTAIN VECTORVALUED MEASURES

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There are two kinds of vectorvalued measures which are involved in the theory of weakly stationary processes: orthogonal (Hilbert space valued) and multiplicative (projection-valued) measures. For both classes we show that weak convergence is equivalent with the convergence of integrals over bounded continuous functions. Moreover we prove continuity theorems for the Fourier transformation as well as for the Laplace transformation of such measures.

1. Introduction. Let X be a weakly stationary stochastic process on the real line which is continuous in quadratic mean. Then it is known that there exists a uniquely determined measure M on the Borel-sets with values in the Hilbert space L^2 over the basic probability space for which

$$X(t) = \int e^{i\lambda t}\, dM(\lambda)\,, \qquad -\infty < t < \infty\,,$$

i.e., X is the Fourier transform of M. This measure is of a special nature: it is orthogonal in the sense that $\langle M(A), M(B)\rangle = 0$ for disjoint Borel sets A, B ([4] Chapter I, Theorem 4.2). Now if one has a sequence $X_1, X_2, \cdots$ of such processes, it is natural to ask how the pointwise convergence of $\{X_n\}$ is reflected by the associated measures $\{M_n\}$. It turns out that a complete analogue of Lévy's continuity theorem holds. This is also the case for the Laplace transforms of orthogonal measures on $\mathbb{R}_+ \equiv [0, \infty)$ (or more generally on $\mathbb{R}_+{}^p$) defined by

$$Y(t) \equiv \int e^{-\lambda t}\, dM(\lambda)\,, \qquad 0 \leqq t < \infty\,.$$

A second class of vectorvalued measures involved in the theory of stationary processes are the socalled "multiplicative measures", i.e., projectionvalued σ-additive mappings $\mathscr{M}$ on the Borel-sets with $\mathscr{M}(A) \circ \mathscr{M}(B) = \mathscr{M}(A \cap B)$ for all A, B. By ([2] page 512ff) to each stationary process X corresponds a uniquely determined one-parameter-group $\{U_t : -\infty < t < \infty\}$ of unitary operators on the Hilbert space $[X(\mathbb{R})]$ (closed linear hull of $X(\mathbb{R})$) with the property

$$U_t X(0) = X(t)\,, \qquad -\infty < t < \infty$$

and the mapping $U(t \rightsquigarrow U_t)$ is the Fourier transform of a uniquely determined multiplicative measure on $\mathbb{R}$ (Stone's theorem). So the convergence of multiplicative measures becomes also interesting and we shall prove that in this case, too, weak convergence is the appropriate convergence-concept.

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2. Weak convergence of orthogonal measures. Let Z be a metric space with Borel-sets $\mathscr{A}$ and H a complex Hilbert space. oca $(\mathscr{A}, H)$ denotes the set of all H-valued orthogonal measures on $\mathscr{A}$ and $C(Z)$ the space of bounded continuous complexvalued functions on Z.

DEFINITION 1. A sequence $\{M_1, M_2, \cdots\} \subset$ oca $(\mathscr{A}, H)$ is said to converge weakly to $M \in$ oca $(\mathscr{A}, H)$—in symbols: $M_n \Rightarrow M$ — iff $M_n(A) \to M(A)$ for all $A \in \mathscr{A}$ with $M(\partial A) = 0$.

First we want to show that weak convergence is equivalent with the convergence of the integrals over the functions in $C(Z)$.

THEOREM 1. *Let* $\{M_0, M_1, M_2, \cdots\}$ *be a sequence from* oca $(\mathscr{A}, H)$. *Then* $M_n \Rightarrow M_0$ iff $\int f\,dM_n \to \int f\,dM_0$ *for all* $f \in C(Z)$.

PROOF. Let $M_n \Rightarrow M_0$, $f \in C(Z)$ and $\varepsilon > 0$; $\alpha \equiv \sup_{n \geq 0} ||M_n(Z)||$ is finite. We can assume that f is realvalued and choose an interval (a, b) which contains $f(Z)$. We put $m_n(A) \equiv ||M_n(A)||^2$ and we know the existence of $t_0 = a < t_1 < \cdots < t_k = b$ with $t_j - t_{j-1} < \varepsilon'$ and $m_0{}^f(\{t_j\}) = 0, j = 1, \cdots, k$, where $\varepsilon' \equiv \varepsilon/4\alpha$. Now $Z = \sum_{j=1}^k A_j$, where $A_j \equiv f^{-1}((t_{j-1}, t_j])$ and $M_0(\partial A_j) = 0$.

Let $g \equiv \sum_{j=1}^k t_j I_{A_j}$; then $||g - f||_\infty < \varepsilon'$ and hence

$$||\textstyle\int f\,dM_n - \int f\,dM_0|| \leqq 2\varepsilon'\alpha + \sum_{j=1}^k |t_j|\,||M_n(A_j) - M_0(A_j)||$$

which is clearly $< \varepsilon$ if n is large enough.

Now we assume $\int f\,dM_n \to \int f\,dM_0$ $\forall f \in C(Z)$. It follows that $\int |f|^2\,dm_n \to \int |f|^2\,dm_0$ and hence, according to ([1] Theorem 2.1), that $m_n \Rightarrow m_0$. Let $\mathscr{G} \equiv \{A \in \mathscr{A}: m_0(\partial A) = 0\}$ be the algebra of m_0-continuity sets. If $F \in \mathscr{G}$ then $m_0(F) = m_0(\bar{F})$ so

$$||M_0(\bar{F}) - M_0(F)|| = ||M_0(\bar{F} - F)|| = (m_0(\bar{F} - F))^{\frac{1}{2}} = 0$$

and hence $||M_n(\bar{F}) - M_n(F)|| = (m_n(\bar{F} - F))^{\frac{1}{2}}$ converges to zero as $n \to \infty$. Let $G_\alpha = \{z \in Z: \text{dist}\,(z, \bar{F}) < \alpha\}$ and choose $\alpha_k \downarrow 0$ with $G_{\alpha_k} \in \mathscr{G}$ $\forall k$. $\bar{F}$ and $G^c_{\alpha_k}$ being disjoint implies there exists $f_k \in C(Z)$ for which $I_{\bar{F}} \leqq f_k \leqq I_{G_{\alpha_k}}$. To $\varepsilon > 0$ choose k such that $m_0(G_{\alpha_k} - \bar{F}) < (\varepsilon/3)^2$; this implies $m_n(G_{\alpha_k} - \bar{F}) < (\varepsilon/3)^2$ $\forall n \geqq n_0$. Then

$$\begin{aligned}
||M_n(F) - M_0(F)||&\\
&\leqq ||M_n(F) - M_n(\bar{F})|| + ||M_n(\bar{F}) - M_0(\bar{F})|| + ||M_0(\bar{F}) - M_0(F)||\\
&\leqq (m_n(\bar{F} - F))^{\frac{1}{2}} + ||M_n(\bar{F}) - \textstyle\int f_k\,dM_n||\\
&\qquad + ||\textstyle\int f_k\,dM_n - \int f_k\,dM_0|| + ||\int f_k\,dM_0 - M_0(\bar{F})||\\
&\leqq (m_n(\bar{F} - F))^{\frac{1}{2}} + (m_n(G_{\alpha_k} - \bar{F}))^{\frac{1}{2}} + ||\textstyle\int f_k\,dM_n - \int f_k\,dM_0||\\
&\qquad + m_0(G_{\alpha_k} - \bar{F})^{\frac{1}{2}}\\
&< \varepsilon
\end{aligned}$$

for all n sufficiently large. Thus $M_n(A) \to M_0(A)$ $\forall A \in \mathscr{G}$ and this proves $M_n \Rightarrow M_0$ $\square$

Now we shall show that for $Z = \mathbb{R}^p$ the space $C(Z)$ in Theorem 1 can be reduced to the set of complex exponentials.

THEOREM 2 (Continuity-theorem). *Let* $\{M_n : n \in N\} \subset \text{oca}(\mathscr{B}^p, H)$ *and* $X_n(t) \equiv \int e^{i\lambda \cdot t}\, dM_n(\lambda)$. *If* $X_0(t) \equiv \lim X_n(t)$ *exists* $\forall t \in \mathbb{R}^p$ *and if the limit function is continuous at the origin, then there exists an* $M_0 \in \text{oca}(\mathscr{B}^p, H)$ *with Fourier transform* X_0 *and* $M_n \Rightarrow M_0$. *Moreover the convergence of* X_n *to* X_0 *is uniform on compact subsets of* $\mathbb{R}^p$.

PROOF. Let $m_n(A) \equiv ||M_n(A)||^2$ and $B_n(t) \equiv \int e^{i\lambda \cdot t}\, dm_n(\lambda) = \langle X_n(t), X_n(0)\rangle$. $B_0(t) \equiv \langle X_0(t), X_0(0)\rangle = \lim B_n(t)$ exists and is continuous at the origin and thus by Lévy's continuity-theorem there exists an $m_0 \in \text{ca}_+(\mathbb{R}^p)$ with $B_0(t) = \int e^{i\lambda \cdot t}\, dm_0$ and $m_n \Rightarrow m_0 \cdot \langle X_n(t), X_n(s)\rangle = B_n(t-s)$ implies $\langle X_0(t), X_0(s)\rangle = B_0(t-s)$; moreover X_0 is continuous, for

$$\begin{aligned} ||X_0(t+h) - X_0(t)||^2 &= 2B_0(0) - B_0(h) - B_0(-h) \\ &\to 0\,, \qquad\qquad h \to 0\,. \end{aligned}$$

This is sufficient for the existence of an $M_0 \in \text{oca}(\mathscr{B}^p, H)$ with Fourier transform X_0. Clearly $m_0(A) = ||M_0(A)||^2$ $\forall A \in \mathscr{B}^p$. $S \equiv \{a \in \mathbb{R}^p : m_0(\{x \in \mathbb{R}^p : x_j = a_j\}) = 0\ \forall j = 1, \cdots, p\}$ is a dense subset of $\mathbb{R}^p$. We shall now first show that $M_n((a, b]) \to M_0((a, b])$ for $a, b \in S$, $a < b$. Let $\varepsilon > 0$; choose $\varepsilon' > 0$ such that $\varepsilon'(5 + 2\alpha) + 2\varepsilon'^2 < \varepsilon$ where $\alpha^2 \equiv \sup_{n \geq 0} m_n(\mathbb{R}^p)$, and $T > \max\{|a_1|, \cdots, |a_p|, |b_1|, \cdots, |b_p|\} + 1$ such that $\sup_{n \geq 0} m_n(\mathbb{R}^p - K) < \varepsilon'^2$ where $K \equiv \{x \in \mathbb{R}^p : |x_j| \leq T\ \forall j = 1, \cdots, p\}$. There exist continuous functions $f_\nu : \mathbb{R}^p \to [0, 1]$ with the property that $f_\nu(x) = 1$ $\forall x \in [a, b]$ and $f_\nu(x) = 0$ if $\text{dist}(x_j, [a_j, b_j]) \geq \nu^{-1}$ for at least one $j \leq p$ $(\nu = 1, 2, \cdots)$. We fix ν and find a product g of trigonometric polynomials with period $2T$ and $\sup_{x \in K} |f_\nu(x) - g(x)| < \varepsilon'$. The periodicity of g implies $||f_\nu - g||_\infty \leq 2 + \varepsilon'$. Now we have

$$\begin{aligned} ||\textstyle\int f_\nu\, dM_n - \int f_\nu\, dM_0|| &\leq ||\textstyle\int (f_\nu - g)\, dM_n|| + ||\textstyle\int g\, dM_n - \int g\, dM_0|| \\ &\quad + ||\textstyle\int (f_\nu - g)\, dM_0|| \end{aligned}$$

where by assumption the middle term is $\leq \varepsilon'$ $\forall n \geq n_0(\varepsilon')$. Moreover

$$\begin{aligned} ||\textstyle\int (f_\nu - g)\, dM_n|| &\leq \varepsilon'(m_n(K))^{\frac{1}{2}} + (2 + \varepsilon')\varepsilon' \\ &\leq \varepsilon'(2 + \alpha) + \varepsilon'^2\,, \qquad\qquad n \geq 0 \end{aligned}$$

so we conclude $||\int f_\nu\, dM_n - \int f_\nu\, dM_0|| < \varepsilon$ $\forall n \geq n_0(\varepsilon')$, i.e., we know that $\int f_\nu\, dM_n \to \int f_\nu\, dM_0$ $\forall \nu \in \mathbb{N}$. For $\delta > 0$ let $B_\delta \equiv \{x \in \mathbb{R}^p : \text{dist}(x, \partial[a, b]) \leq \delta\}$. For every $\varepsilon > 0$ there exists $\delta > 0$ with $m_0(B_\delta) < \varepsilon^2$ and $\tilde{\delta} \in (0, \delta)$ with $m_0(\partial B_{\tilde{\delta}}) = 0$ which implies that $m_n(\beta_{\tilde{\delta}}) < \varepsilon^2$ for n large enough, say $n \geq n_0$. Choose $\nu \in \mathbb{N}$ such that $f_\nu - I_{(a,b]}$ is zero on $\mathbb{R}^p - B_{\tilde{\delta}}$, then

$$||\textstyle\int f_\nu\, dM_n - M_n((a, b])||^2 = \int |f - I_{(a,b]}|^2\, dm_n \leq m_n(B_{\tilde{\delta}}) < \varepsilon^2$$

$\forall n \geqq n_0$ and for $n = 0$. Hence if $n \geqq n_0$ and so large that $||\int f_\nu \, dM_n - \int f_\nu \, dM_0|| < \varepsilon$, we have

$$||M_n(a, b]) - M_0((a, b])|| \leqq ||M_n((a, b]) - \int f_\nu \, dM_n|| + ||\int f_\nu \, dM_n - \int f_\nu \, dM_0|| + ||\int f_\nu \, dM_0 - M_0((a, b])|| < 3\varepsilon ,$$

i.e., $M_n((a, b]) \to M_0((a, b])$. It is easy to see that this convergence holds also if some of the components of b and a are $+\infty$ or $-\infty$ resp. Because of the $\cap$-stability of $\{(a, b]: a, b \in S, a < b\} \cup \{\emptyset\}$ we have $M_n(G) \to M_0(G)$ for all $G \in \tilde{\mathscr{G}}$, the algebra generated by this system. $\tilde{\mathscr{G}}$ is a subset of $\mathscr{G} \equiv \{A \in \mathscr{B}^p: m_0(\partial A) = 0\}$ which by his part is also an algebla. Clearly $\tilde{\mathscr{G}}$ generates the Borel-sets and so to $A \in \mathscr{G}$ and $\varepsilon > 0$ we find a $G \in \tilde{\mathscr{G}}$ with $m_0(G \triangle A) < \varepsilon^2$; $G \triangle A \in \mathscr{G}$ implies $m_n(G \triangle A) < \varepsilon^2 \ \forall n \geqq n_0(\varepsilon)$. If in addition n is so large that $||M_n(G) - M_0(G)|| < \varepsilon$, we have

$$\begin{aligned} ||M_n(A) - M_0(A)|| &\leqq ||M_n(A) - M_n(G)|| + ||M_n(G) - M_0(G)|| \\ &\quad + ||M_0(G) - M_0(A)|| \\ &= (m_n(G \triangle A)^{\frac{1}{2}} + ||M_n(G) - M_0(G)|| + (m_0(G \triangle A))^{\frac{1}{2}} \\ &< 3\varepsilon , \end{aligned}$$

i.e., $M_n(A) \to M_0(A)$. This proves $M_n \Rightarrow M_0$.

Now to the uniform convergence of X_n. It is easy to see that as in the classical case the sequence $X_0, X_1, X_2, \cdots$ is uniformly equicontinuous. Given a compact set $K \subset \mathbb{R}^p$ and $\varepsilon > 0$ one obtains the uniform convergence on K by choosing a suitable finite δ-net for K. $\square$

For the Laplace transforms of orthogonal measures on the semigroup $\mathbb{R}_+{}^p \equiv \{x \in \mathbb{R}^p : x_i \geqq 0 \ \forall i\}$ we have a characterisation which is analogous to that of Fourier transforms: a mapping $Y\colon \mathbb{R}_+{}^p \to H$ is the Laplace transform of an element $M \in \text{oca}\,(\mathscr{B}_+{}^p, H)$ iff Y is bounded, continuous and $\langle Y(s), Y(t)\rangle$ depends only on $s + t$ ([3] Satz 2). This enables us to prove a continuity theorem for Laplace transforms.

THEOREM 3. *Let* $\{M_n : n \in \mathbb{N}\} \subset \text{oca}\,(\mathscr{B}_+{}^p, H)$ *and* $Y_n(t) \equiv \int e^{-\lambda \cdot t} \, dM_n(\lambda)$. *If* $Y_0(t) \equiv \lim Y_n(t)$ *exists* $\forall t \in \mathbb{R}_+{}^p$ *and if the limit function is continuous at the origin, then there exists an* $M_0 \in \text{oca}\,(\mathscr{B}_+{}^p, H)$ *with Laplace transform* Y_0 *and* $M_n \Rightarrow M_0$. *Moreover the convergence is uniform on compact subsets of* $\mathbb{R}_+{}^p$.

PROOF. Let $m_n(A) \equiv ||M_n(A)||^2$ and $\Gamma_n(t) \equiv \int e^{-\lambda \cdot t} \, dm_n(\lambda) = ||Y_n(t/2)||^2$. $\Gamma_0(t) \equiv ||Y_0(t/2)||^2 = \lim \Gamma_n(t)$ exists, is continuous at the origin, weakly monotone decreasing (especially bounded) and positive semi-definite by which we mean that $\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \Gamma(t_i + t_j)$ is nonnegative for all $(\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$, $(t_1, \cdots, t_n) \in (\mathbb{R}_+{}^p)^n$ and $n \in \mathbb{N}$. Let $h \in \mathbb{R}_+{}^p$; then

$$\begin{aligned} \Gamma_0(t) - \Gamma_0(t + h) &= \lim \int [e^{-\lambda \cdot t} - e^{-\lambda \cdot (t+h)}] \, dm_n(\lambda) \\ &\leqq \lim \int (1 - e^{-\lambda \cdot h}) \, dm_n(\lambda) \\ &= \lim (\Gamma_n(0) - \Gamma_n(h)) = \Gamma_0(0) - \Gamma_0(h) \end{aligned}$$

which implies the uniform right-continuity and so the continuity of Γ_0. From ([3] Satz 1) we learn the existence of $m_0 \in \mathrm{ca}_+ (\mathbb{R}_+{}^p)$ with Laplace transform Γ_0. Because of $\lim m_n(\mathbb{R}_+{}^p) = \Gamma_0(0) < \infty$ the first theorem of Helly implies that every subsequence of $\{m_n\}$ contains a convergent subsequence whose limit is m_0 by the unicity of Laplace transform, hence $m_n \Rightarrow m_0$. From

$$||Y_0(t+h) - Y_0(t)||^2 = \Gamma_0(2t+2h) - 2\Gamma_0(2t+h) + \Gamma_0(2t)$$

and $||Y_0(t)||^2 = \Gamma_0(2t)$ we conclude that Y_0 is continuous and bounded, finally $\langle Y_0(s), Y_0(t)\rangle = \Gamma_0(s+t)$ is only a function of $s+t$, so that we find an $M_0 \in \mathrm{oca}\,(\mathscr{B}_+{}^p, H)$ with Laplace transform Y_0. Observing that the linear hull of the functions $\lambda \rightsquigarrow e^{-\lambda\cdot t}$, $t \in \mathbb{R}_+{}^p$ is dense in $C([0, \infty]^p)$ the proof can now be finished as in the preceding theorem. □

3. **Weak convergence of multiplicative measures.** Let again Z be a metric space with metric d and Borel-sets $\mathscr{A}$, H a complex separable Hilbert space. $\mathrm{msca}\,(\mathscr{A}, H)$ denotes the set of multiplicative measures on $\mathscr{A}$, i.e., strongly σ-additive mappings $\mathscr{M}$ from $\mathscr{A}$ to the selfadjoint operators on H with the property that $\mathscr{M}(A) \circ \mathscr{M}(B) = \mathscr{M}(A \cap B)$ for all $A, B \in \mathscr{A}$. Note that $\mathscr{M} \in \mathrm{msca}\,(\mathscr{A}, H)$ implies $\{M_x : x \in H\} \subset \mathrm{oca}\,(\mathscr{A}, H)$ where $M_x(A) \equiv \mathscr{M}(A)(x)$. The separability assumption guarantees the existence of a finite positive measure $\mu_{\mathscr{M}}$ which has the same sets of measure zero as $\mathscr{M}$: choose a dense sequence $\{x_1, x_2, \cdots\}$ in H and put

$$\mu_{\mathscr{M}}(A) \equiv \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{||\mathscr{M}(A)(x_n)||^2}{1 + ||x_n||^2}\,.$$

DEFINITION 2. A sequence $\{\mathscr{M}_1, \mathscr{M}_2, \cdots\} \subset \mathrm{msca}\,(\mathscr{A}, H)$ is said to converge weakly to $\mathscr{M} \in \mathrm{msca}\,(\mathscr{A}, H)$—in symbols: $\mathscr{M}_n \Rightarrow \mathscr{M}$—iff $\mathscr{M}_n(A) \to \mathscr{M}(A)$ strongly for all $A \in \mathscr{A}$ with $\mathscr{M}(\partial A) = 0$.

THEOREM 4. *Let $\{\mathscr{M}_0, \mathscr{M}_1, \mathscr{M}_2, \cdots\}$ be a sequence from* $\mathrm{msca}\,(\mathscr{A}, H)$ *and put $M_{n,x}(A) \equiv \mathscr{M}_n(A)(x)$ for $n \geqq 0$, $x \in H$ and $A \in \mathscr{A}$. Then $\mathscr{M}_n \Rightarrow \mathscr{M}_0$* iff *$M_{n,x} \Rightarrow M_{0,x}$ for every $x \in H$.*

PROOF. Clearly there is only one direction to prove. Let $\mathscr{M}_n \Rightarrow \mathscr{M}_0$ and fix $x \in H$. We put $M_n \equiv M_{n,x}$, $m_n(A) \equiv ||M_n(A)||^2$ and define algebras $\mathscr{G}_0$ and $\mathscr{G}$ by $\mathscr{G}_0 \equiv \{A \in \mathscr{A} : \mathscr{M}_0(\partial A) = 0\}$, $\mathscr{G} \equiv \{A \in \mathscr{A} : m_0(\partial A) = 0\}$. $\mathscr{G}_0$ is a subalgebra of $\mathscr{G}$; what we want to show is that $M_n(A) \to M_0(A)\ \forall A \in \mathscr{G}$, but we know it only if A is from $\mathscr{G}_0$. Let $G \subset Z$ be open and $A_\alpha \equiv \{z \in Z : d(z, G^c) > \alpha\}$, $\alpha > 0$. Because of $\partial A_\alpha \subset \{z \in G : d(z, G^c) = \alpha\}$ and the existence of a finite measure which is equivalent to $\mathscr{M}_0$ there is a sequence $\alpha_1, \alpha_2, \cdots$ of strictly positive numbers converging to zero for which $A_{\alpha_j} \in \mathscr{G}_0\ \forall j$; we then have $G = \bigcup_{j=1}^{\infty} A_{\alpha_j}$. This has two consequences: firstly $m_n \Rightarrow m_0$ by ([1] Theorem 2.2), secondly that $\mathscr{G}_0$ generates the σ-Algebra $\mathscr{A}$. So to $B \in \mathscr{G}$ and $\varepsilon > 0$ we find an $A \in \mathscr{G}_0$ with $m_0(A \triangle B) < \varepsilon^2$; $A \triangle B \in \mathscr{G}$ implies that $m_n(A \triangle B) < \varepsilon^2$ for large n and certainly we have $||M_n(A) - M_0(A)|| < \varepsilon$, n large enough. For those n

we obtain

$$\begin{aligned}||M_n(B) - M_0(B)|| &\leqq ||M_n(B) - M_n(A)|| + ||M_n(A) - M_0(A)|| \\ &\quad + ||M_0(A) - M_0(B)|| \\ &= (m_n(A \triangle B))^{\frac{1}{2}} + ||M_n(A) - M_0(A)|| + (m_0(A \triangle B))^{\frac{1}{2}} \\ &< 3\varepsilon ,\end{aligned}$$

hence $M_n \Rightarrow M_0$ which had to be proved. □

Together with Theorem 1 we have as

COROLLARY. *Let* $\{\mathscr{M}_n : n \geqq 0\} \subset \text{msca}(\mathscr{A}, H)$. *Then* $\mathscr{M}_n \Rightarrow \mathscr{M}_0$ iff $\int f d\mathscr{M}_n \to \int f d\mathscr{M}_0$ *for all* $f \in C(Z)$.

Now again we take $Z = \mathbb{R}^p$ resp. $Z = \mathbb{R}_+{}^p$ and prove two further continuity-theorems.

THEOREM 5. *Let* $\{\mathscr{M}_n : n \in \mathbb{N}\} \subset \text{msca}(\mathscr{B}^p, H)$ *and* $U_n(t) \equiv \int e^{i\lambda \cdot t} d\mathscr{M}_n(\lambda)$. *If* $U_0(t) \equiv \lim U_n(t)$ *exists pointwise* $\forall t \in \mathbb{R}^p$ *and if the limit function is continuous at the origin, then there exists an* $\mathscr{M}_0 \in \text{msca}(\mathscr{B}^p, H)$ *with Fourier transform* U_0 *and* $\mathscr{M}_n \Rightarrow \mathscr{M}_0$.

PROOF. $U_n(t) \to U_0(t)$ implies $U_0(s + t) = U_0(s) \circ U_0(t)$ and so the continuity of U_0 on $\mathbb{R}^p$. $U_0(0)$ is the identity on $K \equiv U_0(0)(H)$ because of $U_0(0) = \lim \mathscr{M}_n(\mathbb{R}^p)$ and we claim that U_0 is a continuous grouphomomorphism with values in the unitary group of K. Now we know the existence of an $\mathscr{M}_0 \in \text{msca}(\mathscr{B}^p, K)$ with Fourier transform U_0; certainly we think of $\mathscr{M}_0$ as an element from msca $(\mathscr{B}^p, H)$. $U_n(t)(x)$ is for fixed $x \in H$ the Fourier transform of $M_{n,x}(M_{n,x}(A) \equiv \mathscr{M}_n(A)(x))$ and converges to the continuous function $U_0(t)(x)$. Theorem 2 implies $M_{n,x} \Rightarrow M_{0,x}$, hence $\mathscr{M}_n \Rightarrow \mathscr{M}_0$. □

THEOREM 6. *Let* $\{\mathscr{M}_n : n \in \mathbb{N}\} \subset \text{msca}(\mathscr{B}_+{}^p, H)$ *and* $V_n(t) \equiv \int e^{-\lambda \cdot t} d\mathscr{M}_n(\lambda)$. *If* $V_0(t) \equiv \lim V_n(t)$ *exists pointwise and if the limit function is continuous at the origin, then there exists an* $\mathscr{M}_0 \in \text{msca}(\mathscr{B}_+{}^p, H)$ *with Laplace transform* V_0 *and* $\mathscr{M}_n \Rightarrow \mathscr{M}_0$.

PROOF. One can easily see that V_0 is a continuous semigrouphomomorphism with the property that $0 \leqq V_0(t) \leqq Id_H$ $\forall t \in \mathbb{R}_+{}^p$. This implies ([3] Satz 3) the existence of an $\mathscr{M}_0 \in \text{msca}(\mathscr{B}_+{}^p, H)$ with Laplace transform V_0. By theorem 3 we have $M_{n,x} \Rightarrow M_{0,x}$ $\forall x \in H$ and thus $\mathscr{M}_n \Rightarrow \mathscr{M}_0$. □

4. Application. Let $X_0, X_1, X_2, \cdots$ be a sequence of weakly stationary stochastic processes on $\mathbb{R}$ (or on $\mathbb{R}^p$), continuous in quadratic mean. The X_n are then Fourier transforms of orthogonal measures M_n on $\mathscr{B}$. Let further U_n denote the uniquely determined one-(or p-)parameter-group of unitary operators on $H_n \equiv [X_n(\mathbb{R})]$ with $U_n(t)X_n(0) = X_n(t)$, $n \geqq 0$, and let finally $\mathscr{M}_n$ be the multiplicative measure whose Fourier transform U_n is. We shall assume that X_0 is not too degenerate in comparison with $X_1, X_2, \cdots$, i.e., that $H_n \subset H_0$

$\forall n \geqq 0$. Clearly $U_n(t)$ can be regarded as operator on the separable Hilbert space H_0 by setting $U_n(t)x \equiv 0$ for $x \in H_0 \ominus H_n$. We then have

THEOREM 7. *From the following four statements*

$$(1) \qquad X_n(t) \to X_0(t) \qquad \forall t$$

$$(2) \qquad M_n \Rightarrow M_0$$

$$(3) \qquad \mathscr{M}_n \Rightarrow \mathscr{M}_0$$

$$(4) \qquad U_n(t) \to U_0(t) \qquad \forall t \text{ (pointwise)}$$

(1) *and* (2) *are equivalent*, (3) *and* (4) *are equivalent and* (1)—(2) *implies* (3)—(4).

PROOF. In view of Theorems 1, 2, 4 and 5 only the last assertion remains to be proved. Suppose that $X_n(t) \to X_0(t)$ $\forall t$ and let $x = X_0(s)$ be given. Then

$$\begin{aligned}(U_n(t) - U_0(t))(x) &= U_n(t)(X_0(s) - X_n(s)) + U_n(t)X_n(s) - U_0(t)X_0(s) \\ &= U_n(t)(X_0(s) - X_n(s)) + X_n(s+t) - X_0(s+t) \\ &\to 0 .\end{aligned}$$

Hence $U_n(t)x \to U_0(t)x$ for all $x \in H_0$. □

REFERENCES

[1] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.
[2] DOOB, J. L. (1953). *Stochastic Processes*. Wiley, New York.
[3] RESSEL, P. (1972). Laplace-Transformation nichtnegativer und vektorwertiger Maße. Submitted to *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*.
[4] ROZANOV, Y. A. (1967). *Stationary Random Processes*. Holden-Day, San Francisco.

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