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THE MARTINTOTE

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A martintote is a random sequence such that the asymptotic behavior of the process distribution, conditioned with respect to the past, remains the same along the sequence. In this respect the conditional distributions of a martintote behave similarly to the conditional expectations of a martingale. We give an optional sampling theorem for martintotes and a class of examples.

1. Introduction. Let $(\Omega, \mathscr{F}, P)$ be a probability space, and let $(X_n, \mathscr{F}_n)$, $n = 1, 2, \cdots$ denote a sequence of real-valued random variables X_n and σ-fields generated by $X_1, \cdots, X_n$.

Definition. A sequence $(X_n, \mathscr{F}_n)$, $n = 1, 2, \cdots$ is a *martintote* if for each n and $A \in \mathscr{F}_n$,

$$\lim_{y\to\infty} [P(X_{n+1} > y \mid A) - P(X_n > y \mid A)]/P(X_1 > y) = \mu(A)/P(A)\,, \tag{1}$$

where μ is a bounded nonnegative, finitely additive function on $\bigcup \mathscr{F}_n$, absolutely continuous with respect to P.

The inspiration for the word *martintote* is apparent if we write: a sequence $(X_n, \mathscr{F}_n)$, $n = 1, 2, \cdots$ is a martingale if for each n and $A \in \mathscr{F}_n$,

$$E(X_{n+1} \mid A) - E(X_n \mid A) = 0\,. \tag{2}$$

In (1) the asymptotic property plays the role assumed by the expected value in (2). A defect in the analogy is that the right side of condition (1) is not necessarily zero. Were it zero, the definition would exclude sums of independent identically distributed random variables, which often satisfy (1) with $\mu = P$. We achieve further generality at the expense of the martingale analogy by allowing the limit in (1) to depend on A. Consideration of dependence on n as well is deferred. The martintote, as defined by (1), should be compared with a sequence $(X_n, \mathscr{F}_n)$ such that $E(X_{n+1} - X_n \mid A)$, $A \in \mathscr{F}_n$, is the restriction of a fixed bounded measure μ to $\mathscr{F}_n$. For such a sequence, if $A \in \mathscr{F}_m$, $m < n$,

$$\textstyle\int_A X_n\, dP - \int_A X_m\, dP = (n - m)\mu(A)\,, \tag{3}$$

while for the martintote a condition equivalent to (1) is, if $A \in \mathscr{F}_m$, $m < n$,

$$\lim_{y\to\infty} [P(X_n > y, A) - P(X_m > y, A)]/P(X_1 > y) = (n - m)\mu(A)\,. \tag{4}$$

The main result is Theorem 1, an optional sampling theorem. It says that a

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martintote observed at a sequence of stopping times is again a martintote. Theorem 2, an optional sampling theorem for processes satisfying (3) is a modification of Doob's martingale sampling theorem and is included for comparison with Theorem 1. A class of processes which are martintotes appears in Section 3, followed by an explicit example.

A previous paper [4] concerned a sequence X_n of sums of independent identically distributed random variables Z_i such that $P(Z_1 > y)$ is regularly varying at ∞, stopped at a random time m with $Em < \infty$. A condition similar to (5) of this paper was shown to be necessary and sufficient that

$$P(X_m > y) \sim P(Z_1 > y)Em .$$

This relation may be regarded as an asymptotic analogue of Wald's classical equation

$$E(X_m) = E(X_1)Em .$$

The martingale sampling theorem as viewed in Theorem 2 is a version of Wald's equation for martingales. Theorem 1 gives the analogous asymptotic relation for martintotes.

The results presented here restrict the possibility of embedding random sequences in other random sequences. Similar results can be obtained in the continuous parameter case.

Possible applications appear in optimal stopping problems where the aim is to minimize not the expected value of a functional of a process but instead the probability that the functional is large. Such problems arise wherever overloading may occur in a system.

2. The randomly stopped martintote. A stopping time for a sequence $(X_n, \mathscr{F}_n)$ is a function $\mathbf{m}$ from Ω to the positive integers such that $\{\omega \in \Omega : \mathbf{m}(\omega) \leqq n\} \in \mathscr{F}_n$, for each n.

THEOREM 1. *Let $(X_n, \mathscr{F}_n)$ be a martintote, and let $\mathbf{m}(n)$ be a non-decreasing sequence of stopping times for $(X_n, \mathscr{F}_n)$ such that*

$$\lim_{i\to\infty} \mu(\mathbf{m}(n) > i) = 0 ,$$

$$\text{(5)} \qquad \liminf_{N\to\infty} \lim_{y\to\infty} [P(X_{\mathbf{m}(n)} > y, \mathbf{m}(n) > N, A) - P(X_N > y, \mathbf{m}(n) > N, A)]/P(X_1 > y) = 0$$

for $A \in \mathscr{F}_{\mathbf{m}(n)}$. Then $(X_{\mathbf{m}(n)}, \mathscr{F}_{\mathbf{m}(n)})$ is a martintote in the sense that if $A \in \mathscr{F}_{\mathbf{m}(n)}$,

$$\text{(6)} \qquad \lim_{y\to\infty} [P(X_{\mathbf{m}(n+1)} > y, A) - P(X_{\mathbf{m}(n)} > y, A)]/P(X_1 > y) = E_\mu(\mathbf{m}(n+1) - \mathbf{m}(n), A) .$$

The expected value in (6) is calculated with respect to the measure μ which appears in (1) and (4). The σ-field $\mathscr{F}_{\mathbf{m}(n)}$ is generated by $X_{\mathbf{m}(1)}, \cdots, X_{\mathbf{m}(n)}$.

If we define a submartintote to be a sequence $(X_n, \mathscr{F}_n)$ such that $\leqq$ replaces $=$ in (1) or (4), then an alternate form of Theorem 1 says that a submartintote viewed at a sequence of stopping times satisfying (5) is again a submartintote. The proof is essentially the same.

THEOREM 2. *Let* $(X_n, \mathscr{F}_n)$ *be a sequence such that if* $A \in \mathscr{F}_m$, $m < n$,

$$\int_A X_n \, dP - \int_A X_m \, dP = (n - m)\mu(A) ,$$

where μ *is a bounded measure on* $\bigcup \mathscr{F}_n$, *and let* $\mathbf{m}(n)$ *be a non-decreasing sequence of stopping times for* $(X_n, \mathscr{F}_n)$ *such that*

$$\liminf_{N\to\infty} \int_{A \cap (\mathbf{m}(n) > N)} X_{\mathbf{m}(n)} - X_N \, dP = 0 \tag{7}$$

for $A \in \mathscr{F}_{\mathbf{m}(n)}$. *Then* $(X_{\mathbf{m}(n)}, \mathscr{F}_{\mathbf{m}(n)})$ *has the property: for* $A \in \mathscr{F}_{\mathbf{m}(n)}$,

$$\int_A X_{\mathbf{m}(n+1)} \, dP - \int_A X_{\mathbf{m}(n)} \, dP = E_\mu(\mathbf{m}(n+1) - \mathbf{m}(n), A) .$$

The proof is similar to that of [1], pages 85–86, and is omitted. The hypothesis of Doob's optional sampling theorem for martingales, which could also be used in Theorem 2, is for $n = 1, 2, \cdots$

$$E|X_{\mathbf{m}(n)}| < \infty , \quad \text{and} \tag{8}$$

$$\liminf_{N\to\infty} \int_{\mathbf{m}(n) > N} |X_N| \, dP = 0 . \tag{9}$$

Since $(m(n) > N) \downarrow \phi$, (8) implies

$$\lim_{N\to\infty} \int_{\mathbf{m}(n) > N} X_{\mathbf{m}(n)} = 0 ,$$

and (7) follows from (8) and (9). Why does not Theorem 1 have a hypothesis analogous to (8), (9)? For the martintote a statement analogous to (8) would be, for $n = 1, 2, \cdots$

$$\lim_{y\to\infty} P(X_{\mathbf{m}(n)} > y)/P(X_1 > y) < \infty . \tag{10}$$

But (10) does *not* imply that

$$\lim_{N\to\infty} \lim_{y\to\infty} P(X_{\mathbf{m}(n)} > y, \mathbf{m}(n) > N)/P(X_1 > y) = 0 .$$

An example is constructed in [3]. The hypothesis (5) of Theorem 1 which is analogous to (7) is, then, not very different from the hypothesis of Doob's optional sampling theorem, and the difference cannot be avoided. All these hypotheses are satisfied by bounded stopping times and in other cases may be difficult to verify.

PROOF OF THEOREM 1. Let $A \in \mathscr{F}_{\mathbf{m}(n)}$. Let $D_j = A \cap \{\mathbf{m}(n) = j\}$. Then $D_j \in \mathscr{F}_j$ (see [1] page 85) and $A = \bigcup D_j$.
For any N,

$$\begin{aligned} P(X_{\mathbf{m}(n+1)} > y, A) - P(X_{\mathbf{m}(n)} > y, A) \\ = \sum_{j=1}^N [P(X_{\mathbf{m}(n+1)\wedge N} > y, D_j) - P(X_j > y, D_j)] \\ + P(X_{\mathbf{m}(n+1)} > y, \mathbf{m}(n+1) > N, A) \\ - P(X_N > y, \mathbf{m}(n+1) > N, A) . \end{aligned}$$

The hypothesis (5) allows us to restrict attention to the first sum. Add and subtract $\sum_{i=j+1}^{N-1} P(X_i > y, \mathbf{m}(n+1) \geqq i, D_j)$ to obtain:

$$\sum_{j=1}^N \sum_{i=j+1}^N P(X_i > y, \mathbf{m}(n+1) \geqq i, D_j) - P(X_{i-1} > y, \mathbf{m}(n+1) \geqq i, D_j) .$$

Now divide by $P(X_1 > y)$. As $y \to \infty$, since X_n is a martintote the result is

$$\begin{aligned}\textstyle\sum_{j=1}^{N} \sum_{i=j+1}^{N} \mu(\mathbf{m}(n+1) \geqq i, D_j) \\ &= \textstyle\sum_{i=1}^{N} \mu(\mathbf{m}(n+1) \geqq i, \mathbf{m}(n) \leqq i-1, A) \\ &= \textstyle\sum_{j=1}^{N} \sum_{i=j+1}^{N} \mu(\mathbf{m}(n+1) - \mathbf{m}(n) \geqq i-j, \mathbf{m}(n) = j) \\ &= \textstyle\sum_{k=1}^{N} \mu(\mathbf{m}(n+1) - \mathbf{m}(n) \geqq k)\,.\end{aligned}$$

The hypothesis implies that the final term of this sum goes to 0 as N goes to ∞. Summation by parts shows that the limit of the sum, whether finite or not, is $E_\mu(\mathbf{m}(n+1) - \mathbf{m}(n))$.

3. A class of examples. Let Z_n denote the increment $X_n - X_{n-1}$ of a random sequence $(X_n, \mathscr{F}_n)$, with $X_0 = 0$.

DEFINITION. If X and Y are random variables such that $\limsup_{x,y\to\infty} P(X > x, Y > y)/P(X > x)P(Y > y) < \infty$, we will say that the pair (X, Y) is of *bounded dependence*. If each of the pairs (X_{n-1}, Z_n), $(|X_{n-1}|, Z_n)$, $(X_{n-1}, |Z_n|)$ is of bounded dependence, $n = 1, 2, \cdots$, we will say that the random sequence $(X_n, \mathscr{F}_n)$ has *increments of bounded dependence*.

A non-increasing function f *varies regularly* if $\lim_{x\to\infty} f(rx)/f(x)$ exists and is finite for every $r > 0$. In this case the limit is r^{-p} for some p in $[0, \infty)$. Feller (1966) gave a simple proof that if Z_1 and Z_2 are independent random variables such that the functions $P(Z_i > y)$, $i = 1, 2$, are regularly varying and $\lim_{y\to\infty} P(Z_2 > y)/P(Z_1 > y) = 1$, then $\lim_{y\to\infty} P(Z_1 + Z_2 > y)/P(Z_1 > y) = 2$. The following lemma can be proved by induction using Feller's argument. We omit the proof.

LEMMA. *Let $(X_n, \mathscr{F}_n)$ be a random sequence with increments Z_n of bounded dependence such that the functions $P(Z_n > y)$ are regularly varying. Suppose that*

$$\lim_{y\to\infty} P(Z_n > y)/P(X_1 > y) = 1\,, \qquad n = 1, 2, \cdots\,.$$

Then

$$\lim_{y\to\infty} P(X_n > y)/P(X_1 > y) = n\,, \qquad n = 1, 2, \cdots\,.$$

THEOREM 3. *Let $(X_n, \mathscr{F}_n)$ be a random sequence with increments Z_n of bounded dependence such that the functions $P(Z_n > y)$ are regularly varying. For $n = 2, 3, \cdots$, and $A \in \mathscr{F}_{n-1}$, let*

$$\lim_{y\to\infty} P(Z_n > y)/P(X_1 > y) = 1$$

and

$$\lim_{y\to\infty} P(A \mid Z_n > y) = \mu(A)\,, \qquad \textit{i.e. the limit exists}\,.$$

Then $(X_n, \mathscr{F}_n)$ is a martintote.

PROOF. Let $A \in \mathscr{F}_n$. Let us write condition (1) in the form

$$\begin{aligned}(11)\quad \lim_{y\to\infty} [P(X_{n+1} > y, X_n \leqq y, A) - P(X_n > y, X_{n+1} \leqq y, A)]/P(X_1 > y) \\ = \mu(A)\,.\end{aligned}$$

We shall show that the first quotient of (11) converges to $\mu(A)$, the second to zero. If $0 < \varepsilon < \frac{1}{2}$,

$$P(X_n + Z_{n+1} > y, X_n \leqq y, A) \geqq P(y \geqq X_n > y(1+\varepsilon), |Z_{n+1}| < y\varepsilon, A) + P(Z_{n+1} > y(1+\varepsilon), |X_n| < y\varepsilon, A) .$$

The first term on the right is zero and

$$\begin{aligned} \liminf_{y\to\infty} & P(Z_{n+1} > y(1+\varepsilon), |X_n| < y\varepsilon, A)/P(X_1 > y) \\ & \geqq \lim_{y\to\infty} P(A \mid Z_{n+1} > y(1+\varepsilon))P(Z_{n+1} > y(1+\varepsilon))/P(X_1 > y) \\ & \quad - c \lim_{y\to\infty} P(Z_{n+1} > y(1+\varepsilon))P(|X_n| \geqq y\varepsilon)/P(X_1 > y) \\ & = \mu(A)(1+\varepsilon)^{-p} . \end{aligned}$$

The c indicates use of the bounded dependence condition. In brief,

$$\liminf_{y\to\infty} P(X_{n+1} > y, X_n \leqq y, A)/P(X_1 > y) \geqq \mu(A)(1+\varepsilon)^{-p} .$$

On the other hand,

$$\begin{aligned} P(X_n + & Z_{n+1} > y, X_n \leqq y, A) \\ & \leqq P(y \geqq X_n > y(1-\varepsilon), A) + P(Z_{n+1} > y(1-\varepsilon), X_n \leqq y, A) \\ & \quad + P(y \geqq X_n > y\varepsilon, Z_n > y\varepsilon, A) . \end{aligned}$$

The A may be omitted from the first and third terms and $(X_n \leqq y)$ from the second. The lemma, used in a computation similar to the foregoing, yields

$$\begin{aligned} \limsup_{y\to\infty} & P(X_{n+1} > y, X_n \leqq y, A)/P(X_1 > y) \\ & \leqq ((1-\varepsilon)^{-p} - 1)n + \mu(A)(1-\varepsilon)^{-p} + c(\varepsilon^{-p} - 1)n \lim_{y\to\infty} P(Z_n > y\varepsilon) . \end{aligned}$$

As in the lemma of [4] we see that for all sufficiently small $\varepsilon > 0$

$$\begin{aligned} \lim_{y\to\infty} P(X_n > y, X_{n+1} \leqq y)/P(X_1 > y) & \\ & \leqq \lim_{y\to\infty} P(y < X_n \leqq y(1+\varepsilon))/P(X_1 > y) \\ & = n((1+\varepsilon)^{-p} - 1) . \end{aligned}$$

The remaining conditions in the definition of *martintote* are clearly satisfied.

COROLLARY. *If X_n is a sum of independent, identically distributed random variables Z_n such that $P(Z_n > y)$ varies regularly, then $(X_n, \mathscr{F}_n)$ is a martintote with $\mu = P$.*

The corollary together with Theorem 1, slightly extends the main result of [4].

We illustrate the application of Theorem 3 to a sequence without independent increments. Let X_n be the non-decreasing Markov chain with $X_0 = 0$ and with increments $Z_n = X_n - X_{n-1}$ such that

$$\begin{aligned} P(Z_n > y \mid X_{n-1} > x) & = (1+x)/y^2(1+xy) && \text{if } x, y \geqq 1 , \\ & = 1 && \text{if } y \leqq 1 , \\ & = 2/y^2(1+y) && \text{if } x \leqq 1, y > 1 . \end{aligned}$$

Then

$$P(Z_n > y) = 2/y^2(1+y) , \qquad n = 1, 2, \cdots, \text{ and}$$

$$\lim_{x,y\to\infty} P(Z_n > y \mid X_{n-1} > x)/P(Z_n > y) = \tfrac{1}{2} .$$

Since X_n, $Z_n \geqq 1$, it follows that the process has boundedly dependent increments of regularly varying and identical distribution. To establish that X_n is a martintote we verify the last hypothesis of Theorem 3. Because of the Markov property, we need only show that $\lim_{y\to\infty} P(X_{n-1} \in C \mid Z_n > y) = \mu(C)$ exists for Borel sets of $[1, \infty)$. Let $\mathscr{G}$ denote the collection of such C for which $\mu(C)$ exists. Let C be an open set and write it as a union of disjoint open intervals, $C = \bigcup_i (a_i, b_i)$, all $a_i \geqq 1$. Then $C \in \mathscr{G}$:

$$\begin{aligned}
&\textstyle\sum_i P(X_{n-1} \in (a_i, b_i) \mid Z_n > y) \\
&\quad = \textstyle\sum_i [P(Z_n > y \mid X_{n-1} > a_i)P(X_{n-1} > a_i) - P(Z_n > y \mid X_{n-1} \geqq b_i) \\
&\qquad \times P(X_{n-1} \geqq b_i)]/P(Z_n > y) \\
&\quad = ((1 + y)/2y)[\textstyle\sum_i ((1 + a_i)/(1/y + a_i))P(X_{n-1} \in (a_i, b_i)) \qquad (12)\\
&\qquad + \textstyle\sum_i [((1 + a_i)/(1/y + a_i)) - ((1 + b_i)/(1/y + b_i))]P(X_{n-1} \geqq b_i)] \\
&\quad \to \tfrac{1}{2} \textstyle\sum_i [((1 + a_i)/a_i)P(X_{n-1} \in (a_i, b_i)) \\
&\qquad + ((b_i - a_i)/a_i b_i)P(X_{n-1} \geqq b_i)] \qquad \text{as } y \to \infty .
\end{aligned}$$

If some $b_j = \infty$, add to (12) the term $\frac{1}{2}(1 + a_j)/a_j P(X_{n-1} > a_j)$. The regrouping of terms has made it apparent that the sums converge uniformly in y.

To show that $\mathscr{G}$ contains all Borel sets we consider a monotone decreasing sequence A_m in $\mathscr{G}$ such that $A_m \downarrow A$. The sequence $\mu(A_m)$ is also monotone decreasing, $\mu(A_m) \downarrow \alpha$. Let $B_m = A_m - A$. Then

$$P(X_{n-1} \in A \mid Z_n > y) = P(X_{n-1} \in A_m \mid Z_n > y) - P(X_{n-1} \in B_m \mid Z_n > y) .$$

Given $\varepsilon > 0$, choose a such that $P(X_{n-1} \in [a, \infty)) < \varepsilon$. Choose m and y_0 such that $P(X_{n-1} \in A_m \mid Z_n > y)$ is within ε of α for $y > y_0$, and such that $P(B_m) + \lambda(B_m - [a, \infty)) < \varepsilon$, where λ is Lebesgue measure. We retain this value of m. Let C be an open set in $[1, a]$ such that $B_m \subset C \cup [a, \infty)$ and $P(C) + \lambda(C) < 2\varepsilon$. For y sufficiently large, we see from (12) that

$$\begin{aligned}
P(X_{n-1} \in B_m \mid Z_n > y) &\leqq P(X_{n-1} \in C \cup [a, \infty)) \\
&\leqq P(C) + \lambda(C) + P[a, \infty) + \varepsilon \leqq 4\varepsilon .
\end{aligned}$$

It follows that $\lim_{y\to\infty} P(X_{n-1} \in A \mid Z_n > y) = \alpha$.

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