The Annals of Probability
1974, Vol. 2, No. 2, 349-353

RESULTS ON PROBABILITIES OF MODERATE DEVIATIONS[1]

By R. Michel

University of Cologne

The convergence rate problem for probabilities of moderate deviations is completely solved by giving a necessary and sufficient condition for the existence of the absolute moment of order c^2+2, $c>0$, in terms of probabilities of moderate deviations. Furthermore, a result on the rate of convergence of probabilities of moderate deviations is given under a weaker moment condition than in Rubin and Sethuraman (1965).

1. Introduction. The purpose of this paper is to improve a theorem of Rubin and Sethuraman [5], where the asymptotic behavior of probabilities of moderate deviations, i.e. of probabilities of the form

$$P^n\{\mathbf{x}\in X^n: |\sum_{i=1}^n f(x_i)| > c(n\log n)^{\frac{1}{2}}\}$$

is considered under appropriate moment conditions on f.

Furthermore, we shall prove a theorem on the convergence of

$$\sum_{n=1}^\infty n^{(c_0{}^2/2)-1}(\log n)^{(c_0{}^2/2)+1}P^n\{\mathbf{x}\in X^n: |\sum_{i=1}^n f(x_i)| > c(n\log n)^{\frac{1}{2}}\}$$

for $c > c_0 > 0$, which improves Theorem 3 of Davis [3]. Our Theorem 2 completely solves the convergence rate problem for probabilities of moderate deviations.

2. Results on moderate deviations. Let $(X, \mathscr{A}, P)$ be a probability space and $f: X\to\mathbb{R}$ an $\mathscr{A}$-measurable function.

$P^n\,|\,\mathscr{A}^n$ denotes the independent product of n identical components $P\,|\,\mathscr{A}$.

Let

$$\Phi(t) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^t \exp[-\tfrac{1}{2}r^2]\,dr$$
$$N_{(0,\sigma^2)}(-\infty, t) = \Phi(t\sigma^{-1})\,.$$

For any measurable g, $P(g) = \int g\,dP$.

In the following we shall use the inequality

$$|c(2\pi\log n)^{\frac{1}{2}}n^{c^2/2}\Phi(-c(\log n)^{\frac{1}{2}}) - 1| \leqq c^{-2}(\log n)^{-1}\,, \tag{2.1}$$

which is standard and follows from Feller (Lemma 2, page 175).

Theorem 1. *If $P(f)=0$, $P(f^2)=1$, and $P(|f|^{c_0{}^2+2})<\infty$ for some $c_0>0$, then*

$$\sum_{n=1}^\infty n^{(c_0{}^2/2)-1}(\log n)^{(c_0{}^2/2)+1}|P^n\{\mathbf{x}\in X^n: |\sum_{i=1}^n f(x_i)| > c(n\log n)^{\frac{1}{2}}\} - 2\Phi(-c(\log n)^{\frac{1}{2}})|$$

converges for all $c \geqq c_0$.

Received September 11, 1972; revised July 2, 1973.

[1] Research supported by a grant of the Deutsche Forschungsge meinschaft.

AMS 1970 *subject classifications.* Primary 60F99.

Key words and phrases. Moderate deviations, asymptotic behavior, rate of convergence.

By (2.1),

$$\sum_{n=1}^{\infty} n^{(c_0^2/2)-1}(\log n)^{(c_0^2/2)+1}\Phi(-c(\log n)^{\frac{1}{2}})$$

converges for all $c > c_0$ and diverges for $c = c_0$. This combined with Theorem 1 implies that the series in Theorem 2 below converges for all $c > c_0'$ and diverges for $c = c_0$, if $P(f) = 0$, $P(f^2) = 1$, and $P(|f|^{c_0^2+2}) < \infty$.

Hence, our Theorem 1 and Theorem 2 of Davis ([3] page 2020) lead to a complete solution of the convergence rate problem for the probability of moderate deviations:

THEOREM 2. *Assume that* $P(f^2) = 1$. *Let* $c_0 > 0$ *be given. Then*

(i) $P(f) = 0$ *and* $P(|f|^{c_0^2+2}) < \infty$ *if and only if*

$$\sum_{n=1}^{\infty} n^{(c_0^2/2)-1}(\log n)^{(c_0^2/2)+1}P^n\{\mathbf{x} \in X^n : |\sum_{i=1}^{n} f(x_i)| > c(n \log n^{\frac{1}{2}}\}$$

converges for all $c > c_0$.

(ii) $P(f) = 0$ *and* $P(|f|^{c_0^2+2}) < \infty$ *imply the divergence of the series in* (i) *for* $c = c_0$.

Using the result of Theorem 2 (i) together with Proposition 1 of Davis ([3] page 2022) we obtain the following generalization of Theorem 3 of Davis, page 2023 (where $c_0^2 \geqq 1$ is assumed to be an integer, and where the exponent of $(\log n)$ is $(c_0^2 - 1)/2$):

THEOREM 3. *If* $P(f) = 0$, $P(f^2) = 1$, *and* $P(|f|^{c_0^2+2}) < \infty$ *for some* $c_0 > 0$, *then*

$$\sum_{n=1}^{\infty} n^{(c_0^2/2)-1}(\log n)^{(c_0^2/2)+1}P^n\{\mathbf{x} \in X^n : \sup_{k \geq n} |(k \log k)^{-\frac{1}{2}} \sum_{i=1}^{k} f(x_i)| > c\}$$

converges for all $c > c_0$.

When the moment generating function exists Cramér has shown that

$$P^n\{\mathbf{x} \in X^n : |\sum_{i=1}^{n} f(x_i)| > n\alpha_n\} \sim \frac{2}{\alpha_n(2\pi n)^{\frac{1}{2}}} \exp[-\tfrac{1}{2}n\alpha_n^2]$$

if $n\alpha_n^3 \to 0$, $n\alpha_n^2 \to \infty$. For moderate deviations, i.e. for the case

$$\alpha_n = c\left(\frac{\log n}{n}\right)^{\frac{1}{2}}$$

Rubin and Sethuraman ([5] page 332, Theorem 4) have established the above result under the much less restrictive moment condition

$$P(|f|^q) < \infty \qquad \text{for some} \quad q > c^2 + 2\,.$$

We obtain a result on the rate of convergence under the condition

$$P(|f|^{c^2+2}) < \infty\,,$$

namely,

THEOREM 4. *If* $P(f) = 0$, $P(f^2) = 1$, *and* $P(|f|^{c^2+2}) < \infty$ *for some* $c > 0$, *then* *there exists a constant $d > 0$ such that for all $n \in \mathbb{N}$,*

$$\left| \frac{P^n\{\mathbf{x} \in X^n : |\sum_{i=1}^n f(x_i)| > c(n \log n)^{\frac{1}{2}}\}}{(2/c(2\pi)^{\frac{1}{2}})n^{-c^2/2}(\log n)^{-\frac{1}{2}}} - 1 \right| \leqq d(\log n)^{-\beta/2-\frac{1}{2}} ,$$

where $\beta = \min(1, c^2)$.

The proof of Theorem 4 follows easily from (2.2) together with Chebychev's inequality, (2.11), and (2.1).

The bound given in Theorem 4 cannot be improved in general concerning the rate of convergence. To see this, assume that f is distributed as $N_{(0,1)}$. Then,

$$P^n\{\mathbf{x} \in X^n : |\sum_{i=1}^n f(x_i)| > c(n \log n)^{\frac{1}{2}}\} = 2\Phi(-c(\log n)^{\frac{1}{2}}) .$$

Furthermore, by Feller (Problem 1, page 193),

$$|c(2\pi \log n)^{\frac{1}{2}} n^{c^2/2}\Phi(-c(\log n)^{\frac{1}{2}}) - 1| \geqq c^{-2}(\log n)^{-1}(1 - 3c^{-2}(\log n)^{-1}) .$$

PROOF OF THEOREM 1. In the proof we shall use ideas related to those of Rubin and Sethuraman.

To simplify our notations we shall use in the following $d > 0$ and $\alpha > 0$ as generic constants not depending on $n \in \mathbb{N}$.

(i) Let f_n denote f truncated at $r(n \log n)^{\frac{1}{2}}$, where $r = (1/4c)\min(1, c_0^2)$.

We remark that, with this truncation,

$$\exp\left[c\left(\frac{\log n}{n}\right)^{\frac{1}{2}} f_n\right] \leqq n^{c_0^2/4} \qquad c_0^2 < 1$$
$$\leqq n^{\frac{1}{4}} \qquad c_0^2 \geqq 1 .$$

(This will be used in the proof of (2.4)).

We have

$$\begin{aligned} &|P^n\{\mathbf{x} \in X^n : |\textstyle\sum_{i=1}^n f(x_i)| > c(n \log n)^{\frac{1}{2}}\} \\ (2.2) \qquad &\quad - P^n\{\mathbf{x} \in X^n : |\textstyle\sum_{i=1}^n f_n(x_i)| > c(n \log n)^{\frac{1}{2}}\}| \\ &\qquad \leqq nP\{x \in X : |f(x)| > r(n \log n)^{\frac{1}{2}}\} . \end{aligned}$$

As

$$\sum_{n=1}^\infty n^{c_0^2/2}(\log n)^{(c_0^2/2)+1} P\{x \in X : |f(x)| > r(n \log n)^{\frac{1}{2}}\}$$

converges by Lemma 1 of Davis ([3] page 2017), it suffices to prove the assertion for the truncated variables f_n.

To this aim, let

$$g_n = \exp\left[c\left(\frac{\log n}{n}\right)^{\frac{1}{2}} f_n\right] \quad \text{and} \quad \beta_n = P(g_n) .$$

Define $\tilde{Q}_n | \mathscr{A}$ by $\tilde{Q}_n(A) = \beta_n^{-1}P(1_A g_n)$, $A \in \mathscr{A}$, and let $Q_n | \mathscr{A}^n$ be the independent product of n identical components $\tilde{Q}_n | \mathscr{A}$.

With $\mu_n = \tilde{Q}_n(f_n)$ and $c_n = c(\log n)^{\frac{1}{2}} - n^{\frac{1}{2}}\mu_n$ we have

$$\begin{aligned} (2.3) \qquad &P^n\{\mathbf{x} \in X^n : \textstyle\sum_{i=1}^n f_n(x_i) > c(n \log n)^{\frac{1}{2}}\} \\ &\quad = \beta_n^{\,n} \exp[-c(n \log n)^{\frac{1}{2}}\mu_n] \textstyle\int_{c_n}^\infty \exp[-c(\log n)^{\frac{1}{2}} t] F_n(dt) \end{aligned}$$

where $F_n \mid \mathscr{B}$ is the measure induced by $Q_n \mid \mathscr{A}^n$ and

$$\mathbf{x} \to n^{-\frac{1}{2}} \sum_{i=1}^n (f_n(x_i) - \mu_n) .$$

The reduction in (2.3) is standard and has been used by Cramér [2] and Bahadur and Rao [1].

(ii) Obviously,

$$\begin{aligned} |P(f_n)| &\leqq dn^{-\frac{1}{2}-\alpha} \\ 0 \leqq 1 - P(f_n^2) &\leqq dn^{-\alpha} \qquad (2.4) \\ P(|f_n|^3 g_n) &\leqq dn^{\frac{1}{2}-\alpha} . \end{aligned}$$

Hence, by a Taylor expansion of $s \to \exp[s]$ about $s = 0$,

$$\begin{aligned} \left|\beta_n - 1 - c^2 \frac{\log n}{2n}\right| &\leqq dn^{-1-\alpha} \\ \left|\mu_n - c\left(\frac{\log n}{n}\right)^{\frac{1}{2}}\right| &\leqq dn^{-\frac{1}{2}-\alpha} \qquad (2.5) \\ |\sigma_n^2 - 1| &\leqq dn^{-\alpha} , \end{aligned}$$

where $\sigma_n^2 = \tilde{Q}_n(f_n^2) - \mu_n^2$.

From (2.5) we obtain by standard arguments

$$|n^{c^2/2}\beta_n^n \exp[-c(n \log n)^{\frac{1}{2}}\mu_n] - 1| \leqq dn^{-\alpha} \tag{2.6}$$

and

$$|c_n| = |c(\log n)^{\frac{1}{2}} - n^{\frac{1}{2}}\mu_n| \leqq dn^{-\alpha} . \tag{2.7}$$

Furthermore, by (2.4) and (2.5),

$$\tilde{Q}(|f_n - \mu_n|^3)\sigma_n^{-3} \leqq 8P(|f_n|^3 g_n)\beta_n^{-1}\sigma_n^{-3} \leqq dn^{-\alpha+\frac{1}{2}} .$$

Hence, by the Berry–Esséen theorem,

$$\sup_{t \in \mathbf{R}} |F_n(-\infty, t) - N_{(0,\sigma_n^2)}(-\infty, t)| \leqq dn^{-\alpha} . \tag{2.8}$$

This implies together with (2.7)

$$\left|\int_{c_n}^{\infty} \exp[-c(\log n)^{\frac{1}{2}}t](F_n - N_{(0,\sigma_n^2)})(dt)\right| \leqq dn^{-\alpha} . \tag{2.9}$$

As

$$\int_{c_n}^{\infty} \exp[-c(\log n)^{\frac{1}{2}}t]N_{(0,\sigma_n^2)}(dt) = \exp[\tfrac{1}{2}\sigma_n^2 c^2 \log n]\Phi(-c_n\sigma_n^{-1} - c\sigma_n(\log n)^{\frac{1}{2}}) ,$$

it is straightforward to show that (2.1), (2.5), and (2.7) imply,

$$\left|\int_{c_n}^{\infty} \exp[-c(\log n)^{\frac{1}{2}}t]N_{(0,\sigma_n^2)}(dt) - n^{c^2/2}\Phi(-c(\log n)^{\frac{1}{2}})\right| \leqq dn^{-\alpha} . \tag{2.10}$$

From (2.3), (2.6), and (2.8)—(2.10) we finally obtain

$$n^{c^2/2}|P^n\{\mathbf{x} \in X^n : \textstyle\sum_{i=1}^n f_n(x_i) > c(n \log n)^{\frac{1}{2}}\} - \Phi(-c(\log n)^{\frac{1}{2}})| \leqq dn^{-\alpha} . \tag{2.11}$$

As

$$\sum_{n=1}^{\infty} n^{(c_0^2/2)-1}(\log n)^{(c_0^2/2)+1} n^{-\alpha-(c^2/2)}$$

converges for $c \geqq c_0$ (recall that $\alpha > 0$), the proof of Theorem 1 is completed.

REFERENCES

[1] BAHADUR, R. R. and RANGA RAO, R. (1960). On deviations of the sample mean. *Ann. Math. Statist.* **31** 1015–1027.

[2] CRAMÉR, H. (1938). Sur un nouveau théorème limite de la probabilités. *Actualitiés Sci. Indust.* No. 736.

[3] DAVIS, J. A. (1968). Convergence rates for probabilities of moderate deviations. *Ann. Math. Statist.* **39** 2016–2028.

[4] FELLER, W. (1970). *An Introduction to Probability Theory and its Applications* **1** 3rd ed. Wiley, New York.

[5] RUBIN, H. and SETHURAMAN, J. (1965). Probabilities of moderate deviations. *Sankhyā Ser. A* **27** 325–346.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT ZU KÖLN
BRD-5000 KÖLN-LINDENTHAL, WEYERTAL 90
GERMANY