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ALMOST SURE CONVERGENCE OF BRANCHING PROCESSES IN VARYING AND RANDOM ENVIRONMENTS

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Using martingale theory, this note proves that Galton–Watson branching processes are a.s. convergent even when the assumption of homogeneity of reproduction is dropped.

1. Introduction and summary. Consider a branching process $\langle Z_n \rangle_{n=0}^{\infty}$, $Z_0 = 1$, where the usual independence assumptions are maintained, cf. Harris [4] page 3 ff., but the time homogeneity of reproduction is not necessarily at hand. Such processes are called Galton–Watson processes in varying environments in Jagers [6]. Let

$$\varphi_n(s) = \sum_{i=0}^{\infty} p_{ni} s^i$$

be the probability generating function (pgf) governing the number of children of an individual in the nth generation. Then the pgf of Z_n is $f_n = \varphi_0 \circ \cdots \circ \varphi_{n-1}$. The following result is due to Church [2].

THEOREM. *Suppose $\varphi_n \not\equiv 1$ for all n. Then there exists a random variable Z_∞ such that $Z_n \to_{\mathscr{D}} Z_\infty$, and $P(Z_\infty = 0$ or $\infty) < 1$ if and only if $\sum_{n=0}^{\infty} (1 - p_{n1}) < \infty$.*

For a proof, see also Athreya and Karlin [1] page 1509 ff., where a process as above appears as the realizations of a branching process with random environment, B.P.R.E., conditioned on some outcome of that environment.

The purpose of this note is to prove that convergence in distribution, $\to_{\mathscr{D}}$, may be strengthened to almost sure convergence. (If some $\varphi_n \equiv 1$, this statement becomes trivial, because then $P(Z_i = 0$ for $i \geqq n + 1) = 1$). This also proves that, almost surely, the realizations of a B.P.R.E. are convergent. The reader familiar with Heyde [5], should easily detect the influence of the ideas of that paper.

2. Proof of the almost sure convergence.

The case $\sum_{n=0}^{\infty} (1 - p_{n1}) < \infty$. Let $\varepsilon(s) = s$ and let $||\cdot||$ denote the supremum norm on $[0, 1]$. Then

$$||\varphi_n - \varepsilon|| = \sup_{0 \leqq s \leqq 1} |\sum_{i \neq 1} p_{ni} s^i - (1 - p_{n1})s| \leqq 2(1 - p_{n1}),$$

because $\sum_{i \neq 1} p_{ni} s^i$ is nonnegative and increasing. Consequently, $\sum_{n=0}^{\infty} ||\varphi_n - \varepsilon|| < \infty$. Take n_0 sufficiently large, so that $\sum_{n=n_0}^{\infty} ||\varphi_n - \varepsilon|| < \frac{1}{4}$. The functions φ_n have inverses, and they are defined at least on the interval $[\frac{1}{4}, 1]$ for $n \geqq n_0$.

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Let $a_{n_0} = \frac{1}{2}$ and define a_n recursively for $n > n_0$ by $\varphi_{n-1}(a_n) = a_{n-1}$. This definition makes sense, and since $\sum_{n=n_0}^{\infty} |a_n - a_{n+1}| = \sum_{n=n_0}^{\infty} |\varphi_n(a_{n+1}) - a_{n+1}| \leqq \sum_{n=n_0}^{\infty} ||\varphi_n - \varepsilon|| \leqq \frac{1}{4}$, we have $a_n \to a$, where $\frac{1}{4} \leqq a \leqq \frac{3}{4}$.

We now observe that $\langle a_n{}^{Z_n} \rangle_{n=n_0}^{\infty}$ is a martingale, adapted to the sequence of σ-fields $\mathscr{B}_n = \sigma(Z_0, \cdots, Z_n)$. Indeed,

$$E[a_{n+1}^{Z_{n+1}} \,||\, \mathscr{B}_n] = E[a_{n+1}^{Z_{n+1}} \,||\, Z_n] = [\varphi_n(a_{n+1})]^{Z_n} = a_n{}^{Z_n} .$$

Since $0 \leqq a_n{}^{Z_n} \leqq 1$, a martingale convergence theorem, (Doob [3] Theorem 4.1, page 319) implies $a_n{}^{Z_n} \to_{\text{a.s.}}$ some $Y \Rightarrow Z_n \to_{\text{a.s.}} (\ln a)^{-1} \ln Y = Z_\infty$. Church's Theorem tells us that $P(Z_\infty = 0 \text{ or } \infty) < 1$; it is left to the reader to construct a sequence of pgf's φ_n such that $P(1 \leqq Z_\infty < \infty)$ and $P(Z_\infty = \infty)$ are both strictly positive. Hint: let $\varphi_n(s) = (1 - (1 + n)^{-2})s + (1 + n)^{-2}s^{k_n}$, where k_n is a rapidly increasing sequence of positive integers.

The case $\sum_{n=0}^{\infty}(1 - p_{n1}) = \infty$. Let $q = \lim_{n\to\infty} f_n(0)$. If $q = 1$, there is nothing to prove, because then $P(Z_k \to 0 \text{ as } k \to \infty) = \lim_{n\to\infty} P(Z_n = 0) = \lim_{n\to\infty} f_n(0) = 1$. Hence suppose $q < 1$. The pgf of Z_∞ is $\equiv q$ on $[0, 1)$, and it is a consequence of Church's Theorem above that

$$\sup_{k \geqq n} \sup_{0 \leqq s \leqq \frac{1}{2}} |f_k(s) - q| = t_n \downarrow 0$$

as $n \to \infty$. Take n_0 so large that $q + 2t_n \leqq 1$ for $n \geqq n_0$. Let a_n be defined by $f_n(a_n) = q + 2t_n$ for $n \geqq n_0$. Obviously, $a_n \geqq \frac{1}{2}$. Furthermore,

$$f_n(\varphi_n(a_{n+1})) = f_{n+1}(a_{n+1}) = q + 2t_{n+1} \leqq q + 2t_n = f_n(a_n) ,$$

implying that $\varphi_n(a_{n+1}) \leqq a_n$. Let the sequence g_n be defined by

$$\begin{aligned} g_n(i) &= 0 , \quad & i &= 0 \\ &= a_n{}^i , \quad & i &> 0 . \end{aligned}$$

$\langle g_n(Z_n) \rangle_{n=n_0}^{\infty}$ is a supermartingale, adapted to $\langle \mathscr{B}_n \rangle_{n=n_0}^{\infty}$. Indeed,

$$\begin{aligned} E[g_{n+1}(Z_{n+1}) \,||\, Z_0, \cdots, Z_n] &= E[g_{n+1}(Z_{n+1}) \,||\, Z_n] \\ &= E[a_{n+1}^{Z_{n+1}} - 1_{\{0\}}(Z_{n+1}) \,||\, Z_n] \\ &= \varphi_n(a_{n+1})^{Z_n} - \varphi_n(0)^{Z_n} \leqq a_n{}^{Z_n} - 1_{\{0\}}(Z_n) = g_n(Z_n) , \end{aligned}$$

because $\varphi_n(a_{n+1}) \leqq a_n$, and $\varphi_n(0)^i \geqq 1_{\{0\}}(i)$. Again we know that $g_n(Z_n) \to_{\text{a.s.}}$ some $Y_1 \geqq 0$, see [3] Theorem 4.1, page 324. But

$$E[g_n(Z_n)] = f_n(a_n) - f_n(0) \leqq 3t_n$$

which tends to 0 as $n \to \infty$. Hence Y_1 is 0 a.s. We are interested in realizations of $\langle Z_n \rangle_{n=0}^{\infty}$ which do not become extinct. For such a realization, $a_n{}^{Z_n} \to 0$. But since $\inf_{n \geqq n_0} a_n \geqq \frac{1}{2}$, Z_n is forced to tend to infinity.

If we are interested in B.P.R.E., the population development is governed by a random sequence $\langle \zeta_n \rangle_{n=0}^{\infty}$ of probability measures on $\{0, 1, \cdots\}$. For such process we obtain that all realizations are convergent a.s., and $P(Z_n \to 0 \text{ or } \infty) < 1$ if and only if $P(\sum_{n=0}^{\infty}(1 - \zeta_{n1}) < \infty \text{ and } \zeta_{n0} < 1 \text{ for all } n) > 0$. For examle, if

$\langle \zeta_{n1} \rangle_{n=0}^{\infty}$ is stationary and ergodic, then $P(Z_n \to 0 \text{ or } \infty) = 1$ if the trivial case $P(\zeta_{n1} = 1) = 1$ is omitted.

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