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FUNCTIONALS OF CRITICAL MULTITYPE BRANCHING PROCESSES[1]

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Let $\mathbf{Z}(t) = (Z_1(t), \cdots, Z_k(t))$, $t \geqq 0$, be a critical k-type, continuous time, Markov branching process. It is known that $\mathbf{Z}(t)/t$, conditioned on $\mathbf{Z}(t) \neq 0$, converges in distribution to $\mathbf{v}W$, where $\mathbf{v}$ is a vector determined by the mean matrix of the process, and W is an exponentially distributed random variable. Thus if $\boldsymbol{\xi}$ is any fixed vector, then $(\boldsymbol{\xi} \cdot \mathbf{Z}(t))/t$, conditioned on non-extinction, converges to $(\boldsymbol{\xi} \cdot \mathbf{v})W$. If $\boldsymbol{\xi}$ is orthogonal to $\mathbf{v}$ then t is not the right normalizing factor. We prove that in this case:

(a) $\{(\boldsymbol{\xi} \cdot \mathbf{Z}(t))/(\mathbf{u} \cdot \mathbf{Z}(t))^{\frac{1}{2}} \mid \mathbf{Z}(t) \neq \mathbf{0}\}$ converges in distribution to a normal random variable, and

(b) $\{(\boldsymbol{\xi} \cdot \mathbf{Z}(t))/t^{\frac{1}{2}} \mid \mathbf{Z}(t) \neq \mathbf{0}\}$ converges in distribution to a Laplacian random variable.

1. Introduction. In this note we prove a limit theorem for linear functionals of a critical, k-type ($2 \leqq k < \infty$), continuous time Markov branching processes

$$\mathbf{Z}(t) = (Z_1(t), \cdots, Z_k(t)),$$

$Z_i(t)$ being the number of type i particles at time t.

It is well known that $\mathbf{Z}(t)/t$, conditioned on $\mathbf{Z}(t) \neq \mathbf{0}$, converges in distribution to $\mathbf{v}W$; where $\mathbf{v}$ is a vector determined by the mean matrix of the process (to be defined below), and W is an exponentially distributed random variable.[2] Hence if $\boldsymbol{\xi}$ is any fixed vector, then $(\boldsymbol{\xi} \cdot \mathbf{Z}(t))/t$, conditioned on non-extinction, converges to $(\boldsymbol{\xi} \cdot \mathbf{v})W$. However, if $\boldsymbol{\xi}$ is orthogonal to $\mathbf{v}$, then this only tells us that $(\boldsymbol{\xi} \cdot \mathbf{Z}(t))/t \to 0$ in distribution, and it is natural to ask whether some normalizing function smaller than t leads to a non-degenerate limit law.

We will prove two results in this direction; one for a random normalization, and the other for a deterministic one. Namely, we show that if $(\boldsymbol{\xi} \cdot \mathbf{v}) = 0$ then

$$\{(\boldsymbol{\xi} \cdot \mathbf{Z}(t))/(\mathbf{u} \cdot \mathbf{Z}(t))^{\frac{1}{2}} \mid \mathbf{Z}(t) \neq \mathbf{0}\}$$

converges in distribution to a normally distributed random variable ($\mathbf{u}$ is a fixed vector to be specified), and that

$$\{(\boldsymbol{\xi} \cdot \mathbf{Z}(t))/t^{\frac{1}{2}} \mid \mathbf{Z}(t) \neq \mathbf{0}\}$$

converges in distribution to a Laplacian random variable.[3]

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[2] For background and references related to this result, see Sections V.7 and V.8 of [1]. Any results quoted in this paper and not referred to a specific source can be found in [1], together with the relevant bibliographical credits. Our notation will also conform to [1].

[3] The second result was proved for the discrete case by a direct but lengthy argument in the technical report [4], which has not been published. The present proof is much simpler and is in the spirit of similar results for the super-critical case in [1], [2], [3].

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2. Results. The mean matrix of the process is

$$M(t) = \{m_{ij}(t); i, j = 1, \cdots, k\} = \{E(Z_j(t) \mid \mathbf{Z}(0) = \mathbf{e}_i)\},$$

where $\mathbf{e}_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ is the unit vector with a 1 in the ith coordinate. It is easy to check that

$$M(t + s) = M(t)M(s), \qquad t, s \geqq 0,$$

and one can write

$$M(t) = \exp\{At\},$$

where A is the infinitesimal generator of the semigroup $\{M(t); t \geqq 0\}$. We assume that the process is positively regular, i.e., that all elements of $M(t_0)$ are positive for some t_0, $0 < t_0 < \infty$. Then there is a simple eigenvalue λ_1 of A which is real, and is larger than the real part of any other eigenvalue. Let $\mathbf{u}$ and $\mathbf{v}$ be the right and left eigenvectors associated with this maximal eigenvalue. Denote by A_t the event that $\mathbf{Z}(t) \neq \mathbf{0}$, by $\{X \mid A_t\}$ the random variable X conditioned on A_t, and by

$$P\{- \mid A_t\}, \qquad E\{- \mid A_t\}, \qquad P\{-; A_t\}, \qquad E\{-; A_t\}$$

the conditional probability and expectation, given A_t, and restricted to A_t, respectively.

THEOREM 1. *If $\mathbf{Z}(t)$ has finite second moments, $\boldsymbol{\xi}$ is real, and $\boldsymbol{\xi} \cdot \mathbf{v} = 0$, then*

$$\{(\boldsymbol{\xi} \cdot \mathbf{Z}(t))/(\mathbf{u} \cdot \mathbf{Z}(t))^{\frac{1}{2}} \mid A_t\}$$

converges in distribution to a normally distributed random variable with zero mean, and finite, nonzero variance.

PROOF. Due to the additive property of the branching process, we can write

$$(1) \qquad \mathbf{Z}(t + s) = \textstyle\sum_{i=1}^{k} \sum_{j=1}^{Z_i(t)} \mathbf{Z}^{(ij)}(t, s), \qquad s \geqq 0,\ t \geqq 0,$$

where $\mathbf{Z}^{(ij)}(t, s) =$ the population vector at time $t + s$ of the process descendent from the jth type i particle alive at time t.

Let $X(t) = \mathbf{u} \cdot \mathbf{Z}(t)$ and $Y(t) = \boldsymbol{\xi} \cdot \mathbf{Z}(t)$. Then by using (1) we can write

$$(2) \qquad Y(t + s) = \textstyle\sum_{i=1}^{k} \sum_{j=1}^{Z_i(t)} [\boldsymbol{\xi} \cdot \mathbf{Z}^{(ij)}(t, s) - (M(s)\boldsymbol{\xi})_i] + (M(s)\boldsymbol{\xi}) \cdot \mathbf{Z}(t)$$

and hence

$$(3) \qquad \frac{Y(t + s)}{(X(t))^{\frac{1}{2}}} = \sum_{i=1}^{k} \left(\frac{Z_i(t)}{X(t)}\right)^{\frac{1}{2}} \left\{\frac{1}{(Z_i(t))^{\frac{1}{2}}} \sum_{j=1}^{Z_i(t)} \eta_{ij}(t, s)\right\} + \frac{(M(s)\boldsymbol{\xi}) \cdot \mathbf{Z}(t)}{(X(t))^{\frac{1}{2}}},$$

where $\eta_{ij}(t, s) = \boldsymbol{\xi} \cdot \mathbf{Z}^{(ij)}(t, s) - (M(s)\boldsymbol{\xi})_i$. The idea of the proof is to condition on A_{t+s}, apply the central limit theorem to the independent, identically distributed random variable $\eta_{ij}(s)$, and show that if $(\boldsymbol{\xi} \cdot \mathbf{v}) = 0$ then the last term in (3) goes to zero in probability.

First, let $V = \exp\{i\theta Y(t + s)/(X(t))^{\frac{1}{2}}\}$. By expressing $E\{V; A_{t+s}\}$ as

$$E\{V; A_t\} - E\{V; A_t A_{t+s}^c\} = E\{V \mid A_t\}P\{A_t\} - P\{A_t A_{t+s}^c\},$$

we see that

$$E\{V \mid A_{t+s}\} = [P\{A_t\}/P\{A_{t+s}\}][E\{V \mid A_t\} - P\{A_{t+s}^c \mid A_t\}].$$

But for fixed s, $P\{A_{t+s}\}/P\{A_t\} = P\{A_{t+s} \mid A_t\} \to 1$ as $t \to \infty$. Hence

(4) $$\lim_{t\to\infty} E\{V \mid A_{t+s}\} = \lim_{t\to\infty} E\{V \mid A_t\}.$$

Now return to (3) and condition both sides on A_t. Observe that $\{\eta_{ij}(t, s) \mid A_t\}$, $j = 1, \cdots, Z_i(t)$ are independent, identically distributed random variables with mean 0 and finite variance, whose distributions are independent of t. Let $\sigma_i^2(s, \boldsymbol{\xi}) = \operatorname{Var}\{\eta_{ij}(t, s) \mid A_t\} = \operatorname{Var}\{\boldsymbol{\xi} \cdot \mathbf{Z}(s) \mid \mathbf{Z}(0) = \mathbf{e}_i\}$.

Since $\{\mathbf{Z}_i(t) \mid A_t\} \to_d \infty$ and $\{Z_i(t)/X(t) \mid A_t\} \to_d v_i$ (where $\to_d$ denotes convergence in distribution), we see that

(5) $$\sum_{i=1}^k \left(\frac{Z_i(t)}{X(t)}\right)^{\frac{1}{2}} \left\{\frac{1}{(Z_i(t))^{\frac{1}{2}}} \sum_{j=1}^{Z_i(t)} \eta_{ij}(t, s)\right\} \to_d \sum_{i=1}^k v_i^{\frac{1}{2}} N_i(s),$$

where $N_i(s)$ are independent $N(0, \sigma_i^2(\mathrm{s}, \boldsymbol{\xi}))$ variables. The exact expression for $\sigma_i^2(s, \boldsymbol{\xi})$ need not concern us, it being sufficient to know that

(6) $$\lim_{s\to\infty} \sigma_i^2(s, \boldsymbol{\xi}) = \sigma_i^2(\boldsymbol{\xi}) \neq 0 \qquad \text{for} \quad \boldsymbol{\xi} \neq \mathbf{0}, \quad \boldsymbol{\xi} \cdot \mathbf{v} = 0.$$

This is in turn a consequence of the fact that for any $\boldsymbol{\phi} \in R_k$ (see expression (4.16) of [3])

(7) $$E\{(\boldsymbol{\phi} \cdot \mathbf{Z}(t))^2 \mid \mathbf{Z}(0) = \mathbf{e}_i\} - (M(t)\boldsymbol{\phi})_i^2 = \sigma_i^2(t, \boldsymbol{\phi}) = \sum_{j=1}^k \int_0^t m_{ij}(\tau)(M(t-\tau)\boldsymbol{\phi})' C_j'(0)(M(t-\tau)\boldsymbol{\phi})\, d\tau,$$

where $C_j(0)$ is a positive definite matrix representing the infinitesimal variances and covariances for a process starting at $\mathbf{e}_j$. When $\boldsymbol{\phi} \cdot \mathbf{v} = 0$, one can show (see proposition (1) of [3]) that there exist constants $b > 0$ and γ depending on $\boldsymbol{\phi}$, such that

(8) $$\sup_t M(t)\boldsymbol{\phi} e^{bt} t^{-\gamma} = r(\boldsymbol{\phi}) < \infty,$$

thus making the integrand in (7) integrable in $[0, \infty)$. Also, since the process is critical, $m_{ij}(t) \to u_i v_i$ as $t \to \infty$. Thus, by the dominated convergence theorem, we may conclude that

(9) $$\lim_{t\to\infty} \sigma_i^2(t, \boldsymbol{\phi}) = u_i \sum_{j=1}^k v_j \int_0^\infty (M(t)\boldsymbol{\phi})' C_j'(0)(M(t)\boldsymbol{\phi})\, dt \equiv \sigma_i^2(\boldsymbol{\phi}), \quad \text{say}.$$

It is clear that $\sigma_i^2(\boldsymbol{\phi}) = 0$ iff $M(t)\boldsymbol{\phi} = 0$ for all t, or equivalently that $\boldsymbol{\phi} = 0$. Thus, $\boldsymbol{\phi} \neq 0$ implies $\sigma_i^2(\boldsymbol{\phi}) \neq 0$. Also, it is clear from the above expression for $\sigma_i^2(\boldsymbol{\phi})$ that it goes to zero if $\boldsymbol{\phi} \to 0$.

From (7), (8), and (9) we can further conclude that

(10) $$L_i(\gamma) \equiv \sup_t E\{|\boldsymbol{\eta} \cdot \mathbf{Z}(t)|^2 \mid \mathbf{Z}(0) = \mathbf{e}_i\} < \infty,$$

and

(11) $$L_i(\boldsymbol{\eta}) \to 0 \qquad \text{as} \quad |\boldsymbol{\eta}| \to 0.$$

By Chebychev's inequality and the fact that $\mathbf{Z}(t) = \mathbf{0}$ on A_t^c, we have

(12) $$P\{|(M(s)\boldsymbol{\xi}) \cdot \mathbf{Z}(t)| > \varepsilon t^{\frac{1}{2}} \mid \mathbf{Z}(0) = \mathbf{e}_i, A_t\} \leqq E\{|(M(s)\boldsymbol{\xi}) \cdot \mathbf{Z}(t)|^2 \mid \mathbf{Z}(0) = \mathbf{e}_i\}/\varepsilon^2 t P\{A_t\}.$$

Since $tP\{A_t\} \to$ constant > 0, we can conclude from (10) and (12) that

(13) $$\sup_t P\{|(M(s)\boldsymbol{\xi}) \cdot \mathbf{Z}(t)| \leqq \varepsilon t^{\frac{1}{2}} \mid \mathbf{Z}(0) = \mathbf{e}_i, A_t\} \leqq cL_i(M(s)\boldsymbol{\xi})$$

for some positive constant c. Now when $\boldsymbol{\xi} \cdot \mathbf{v} = 0$, $M(s)\boldsymbol{\xi} \to 0$ as $s \to \infty$, so by taking s large and applying (11) and (13), we see that we can make $\{(M(s)\boldsymbol{\xi}) \cdot \mathbf{Z}(t)/t^{\frac{1}{2}} \mid A_t\}$ small in probability uniformly in t. Hence, since $\{X(t)/t \mid A_t\}$ converges to a proper distribution, we conclude that given any ε, $\delta > 0$, there is an s_0 such that

(14) $$\sup_t P\{|(M(s)\boldsymbol{\xi}) \cdot \mathbf{Z}(t)/(X(t))^{\frac{1}{2}}| > \varepsilon \mid A_t\} < \delta$$

for $s > s_0$.

Finally, note that

$$\frac{Y(t+s)}{(X(t+s))^{\frac{1}{2}}} = \frac{Y(t+s)}{(X(t))^{\frac{1}{2}}} \left(\frac{X(t)}{X(t+s)}\right)^{\frac{1}{2}},$$

and that for fixed s,

$$\{X(t)/X(t+s) \mid A_t\} \to_d 1 \qquad \text{as } t \to \infty .$$

To see the latter, note that as $t \to \infty$

$$\left\{\frac{1}{Z_i(t)} \sum_{j=1}^{Z_i(t)} Z_r^{(ij)}(t, s) \mid A_t\right\} \to_d m_{ir}(s) ,$$

and hence that

$$\left\{\frac{\sum_{r=1}^{k} u_r \sum_{j=1}^{Z_i(t)} Z_r^{(ij)}(t, s)}{u_i Z_i(t)} \middle| A_t\right\} \to_d \frac{1}{u_i} (M(s)\mathbf{u})_i = 1 .$$

Thus

$$\begin{aligned}\left\{\frac{X(t+s)}{X(t)} \middle| A_t\right\} &= \left\{\frac{\sum_r u_r Z_r(t+s)}{\sum_i u_i Z_i(t)} \middle| A_t\right\} \\ &= \left\{\frac{\sum_{i=1}^{k} \sum_{r=1}^{k} u_r \sum_{j=1}^{Z_i(t)} Z_r^{(ij)}(t, s)}{\sum_{i=1}^{k} u_i Z_i(t)} \middle| A_t\right\} \to_d 1 .\end{aligned}$$

Hence taking s large and then t large, and combining (3), (4), (5), (6) and (14), we have Theorem 1.

THEOREM 2. *If $\mathbf{Z}(t)$ has finite second moments, $\boldsymbol{\xi}$ is real, and $\boldsymbol{\xi} \cdot \mathbf{v} = 0$, then*

$$\{(\boldsymbol{\xi} \cdot \mathbf{Z}(t))/t^{\frac{1}{2}} \mid A_t\}$$

converges in distribution to a random variable with density function $\frac{1}{2}\gamma e^{-\gamma|x|}$, $-\infty < x < \infty$, where $\gamma > 0$.

PROOF. Let $\mathscr{F}_t$ denote the σ-field generated by $\{\mathbf{Z}(u) : u \leqq t\}$. Then by using the decomposition as at (2), and noting that conditional on $\mathscr{F}_t$ the second term there is constant, we have

(15) $$\begin{aligned}&E[\exp\{i\theta Y(t+s)\} \mid A_t] \\ &\quad = E\{\exp\{i\theta(M(s)\boldsymbol{\xi}) \cdot \mathbf{Z}(t)\} E[\exp\{i\theta \textstyle\sum_{i=1}^{k} \sum_{j=1}^{Z_i(t)} \eta_{ij}(t, s)\} \mid \mathscr{F}_t] \mid A_t\} .\end{aligned}$$

Let $\varphi_i(\theta; s) = E\exp\{i\theta\eta_{ij}(t, s)\}$ (note that this *is* independent of j and t). Then the expression at (15) equals

$$E\{\exp\{i\theta(M(s)\boldsymbol{\xi}) \cdot \mathbf{Z}(t)\} \prod_{i=1}^{k} \varphi_i^{Z_i(t)}(\theta; s) \mid A_t\}. \tag{16}$$

Now by (14) and the remarks around it, given $\varepsilon, \delta > 0$, we can find s_0 such that (14) holds with $(X(t))^{\frac{1}{2}}$ replaced by $t^{\frac{1}{2}}$, so

$$\begin{aligned} E[\exp\{i\theta Y(t+s)/(t+s)^{\frac{1}{2}}\} \mid A_t] \\ = [1 + \gamma(s, t)]E\{\textstyle\prod_{i=1}^{k} \varphi_i^{Z_i(t)}(\theta(t+s)^{-\frac{1}{2}}; s) \mid A_t\}, \end{aligned} \tag{17}$$

where $\gamma(s, t) \to 0$ as $s \to \infty$, uniformly in t. Now for fixed s, φ_i is the characteristic function of a random variable with mean 0 and finite variance $\sigma_i^2(s)$. Also, we can write the exponent of φ_i in (17) as

$$[Z_i(t)/(\mathbf{u} \cdot \mathbf{Z}(t))][(\mathbf{u} \cdot \mathbf{Z}(t))/t][t/(t+s)](t+s),$$

whence, recalling that as $t \to \infty$, $(\mathbf{u} \cdot \mathbf{Z}(t))/t$ converges in distribution to an exponentially distributed random variable U with parameter λ say, that $\{\mathbf{Z}(t) \mid A_t\} \to_d \infty$, and that $\{Z_i(t)/(\mathbf{u} \cdot \mathbf{Z}(t)) \mid A_t\} \to_d v_i$, it follows that the limit (as $t \to \infty$), of the right-hand expectation of (17) equals

$$E[\exp\{-\tfrac{1}{2}\theta^2\sigma^2(s)U\}] = (2\lambda/\sigma^2(s))/[(2\lambda/\sigma^2(s)) + \theta^2] \tag{18}$$

where $\sigma^2(s) = \sum v_i \sigma_i^2(s)$. We have already observed that $\lim_{s\to\infty} \sigma^2(s) = \sigma^2$ exists, and by arguing exactly as in the proof of (4), with

$$V(t) = \exp\{i\theta Y(t)/t^{\frac{1}{2}}\},$$

$$\lim_{t\to\infty} E\{V(t) \mid A_{t+s}\} = \lim_{t\to\infty} E\{V(t+s) \mid A_{t+s}\} = \lim_{t\to\infty} E\{V(t+s) \mid A_t\}.$$

So by taking s large, letting $t \to \infty$, and referring to (17) and (18), we arrive at the characteristic function of a random variable of the asserted form.

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