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A REMARK ON THE STRONG LAW[1]

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Let $X_1, X_2, \cdots$ be a sequence of random variables, each having conditional mean zero given the past. Let V_n be the conditional variance of X_n given the past. Les $S_n = X_1 + \cdots + X_n$ and $T_n = V_1 + \cdots + V_n$. For simplicity, suppose $\sum V_i = \infty$ a.e. For which non-decreasing function ϕ does $S_n/\phi(T_n)$ necessarily lead to 0 a.e. as n increases? It is necessary and sufficient that $\int^\infty 1/\phi(t)^2\,dt < \infty$. That is, if the integral is finite, convergence to zero a.e. is guaranteed for all X_i and V_i satisfying the stated condition. If the integral is infinite, there is a sequence of independent, symmetric random variables X_i, each having variance 1, such that $S_n/\phi(n)$ oscillates between $\pm\infty$. The sufficiency is known, but a new proof is given.

Let $(\Omega, \mathscr{F}, P)$ be a probability triple, and let $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots$ be sub σ-fields of $\mathscr{F}$. Let $X_1, X_2, \cdots$ be finite functions on Ω, such that

(1) $$X_n \quad \text{is} \quad \mathfrak{F}_n\text{-measurable}$$

(2) $$E\{X_{n+1} \mid \mathfrak{F}_n\} = 0 \quad \text{a.e.}$$

This does not require X_{n+1} to be L^2 or even L^1. It does require that a.e., given $\mathfrak{F}_n$, the conditional distribution of X_{n+1} be integrable with mean 0. Let

(3) $$V_n = E\{X_{n+1}^2 \mid \mathfrak{F}_n\},$$

the conditional variance of X_{n+1} given $\mathscr{F}_n$. Let

(4) $$S_n = X_1 + \cdots + X_n \qquad \text{and} \qquad T_n = V_1 + \cdots + V_n.$$

The main result of this note, to be proved later, is

(5) THEOREM. *Suppose* (1)—(4). *Let* $A \geqq 0$, *and let* ϕ *be a positive, non-decreasing function on* $[A, \infty)$, *such that*

(6) $$\int_A^\infty 1/\phi(t)^2\,dt < \infty.$$

Then

$$\lim_{n\to\infty} S_n/\phi(T_n) = 0 \quad \text{a.e.} \quad on \quad \{\Sigma V_i = \infty\}.$$

This is known for L^2 variables; see [3] page 150. This result includes the usual strong law of large numbers for independent identically distributed L^2 variables: take $\phi(t) = t$. In a sense, (5) is sharp. Proved later will be

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(7) THEOREM. *Let* ϕ *be a positive non-decreasing function on* $[A, \infty)$, *with* $\int_A^\infty 1/\phi(t)^2\,dt = \infty$. *Then there is a sequence of independent, symmetric random variables, each having variance* 1, *such that*

(8) $$P\{\limsup_{n\to\infty} S_n/\phi(n) = \infty\} = 1 .$$

The asymptotic behavior of $S_n/\phi(T_n)$ may well depend on the underlying process, when $\int^\infty 1/\phi(t)^2\,dt = \infty$. For instance, suppose $\phi(x) = (x \log x)^{\frac{1}{2}}$, for large x. If the X_i are uniformly bounded, the law of the iterated logarithm shows $S_n/\phi(T_n) \to 0$ a.e. With this observation and (7), it is possible to construct a process satisfying (1)—(4), such that all V_i are 1, and

$$P\{S_n/(n \log n)^{\frac{1}{2}} \to 0\} = \tfrac{1}{2} .$$

The proof of (5) follows the pattern of [1]. Here is the main step. Fix one positive, measurable function ϕ on $[A, \infty)$, satisfying (6). Let

$$f(v) = \int_v^\infty 1/\phi(t)^2\,dt$$
$$Q(x, v) = [x^2/\phi(v)^2] + f(v)$$

for $v \geqq A$ and $-\infty < x < \infty$.

(9) LEMMA. *Suppose* X *is a random variable, with* $E(X) = 0$ *and* $E(X^2) = V$. *Then*

$$E\{Q(x + X, v + V)\} \leqq Q(x, v) ,$$

for $v \geqq A$ *and* $-\infty < x < \infty$.

PROOF. Clearly,

$$E\{Q(x + X, v + V)\} = x^2/\phi(v + V)^2 + V/\phi(v + V)^2 + f(v + V) ,$$

and $x^2/\phi(v + V)^2 \leqq x^2/\phi(v)^2$. Next,

$$f(v) - f(v + V) = \int_v^{v+V} 1/\phi(t)^2\,dt$$
$$\geqq V/\phi(v + V)^2 ;$$

so

$$V/\phi(v + V)^2 + f(v + V) \leqq f(v) . \qquad \square$$

(10) PROPOSITION. *Suppose* (1)—(4) *and* (6). *Let* τ *be a stopping time*: $\{\tau = n\} \in \mathscr{F}_n$ *for all* n. *For* $v \geqq A$ *and* $-\infty < x < \infty$,

$$E\{(x + S_\tau)^2/\phi(v + T_\tau)^2\} \leqq Q(x, v) .$$

PROOF. This follows from (9), using Theorem (22) of [1]. $\square$

(11) COROLLARY. *Suppose* (1)—(4) *and* (6). *Let* $a > 0$. *Then*

$$P\{|S_n| > a\phi(v + T_n) \text{ for some } n\} \leqq Q(0, v)/a^2 .$$

PROOF. This follows from (10), using Chebychev and Lemma (30) of [1]. $\square$

(12) COROLLARY. *Suppose* (1)—(4) *and* (6). *Let* $k = 1, 2, \cdots$ *and* $a > 0$. *Then*

$$P\{|X_{k+1} + \cdots + X_{k+n}| > a\phi(T_n) \text{ for some } n \,|\, \mathscr{F}_k\} \leqq Q(0, T_k)/a^2 .$$

PROOF. Condition on $\mathscr{F}_k$, and use (11). □

I believe estimates (9)—(12) are new.

PROOF OF (5). Fix $a > 0$. Let $D = \{\Sigma V_i = \infty\}$. Let

$$D_k(a) = \{|X_{k+1} + \cdots + X_{k+n}| \leqq a\phi(T_n) \text{ for all } n\}.$$

Since $\phi(\infty) = \infty$,

$$\limsup |S_n|/\phi(T_n) \leqq a \quad \text{on} \quad D \cap D_k(a). \tag{13}$$

Fix $\varepsilon > 0$. Choose $v \geqq A$ so large that $Q(0, v)/a^2 \leqq \varepsilon$. Then choose a positive integer k so large that

$$P\{D \text{ and } T_k \geqq v\} \geqq (1 - \varepsilon)P\{D\}.$$

But

$$P\{D_k(a) \,|\, \mathscr{F}_k\} \geqq 1 - Q(0, T_k)/a^2$$
$$\geqq 1 - \varepsilon \quad \text{on} \quad \{T_k \geqq v\}.$$

So

$$P\{D_k(a) \text{ and } T_k \geqq v\} \geqq (1 - \varepsilon)P\{T_k \geqq v\}.$$

Now

$$P\{D \text{ but not } D_k(a)\} \leqq \varepsilon P\{D\} + \varepsilon P\{T_k \geqq v\} \leqq 2\varepsilon.$$

This, together with (13), shows $\limsup |S_n|/\phi(T_n) \leqq a$ on all of D apart from a set of measure 2ε. Since ε was arbitrary, $\limsup_{n\to\infty} |S_n|/\phi(T_n) \leqq a$ a.e. on D. Now let a tend to zero through a sequence. □

The next argument follows[2].

PROOF OF (7). Choose $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ so slowly that

$$\varepsilon_n^2/\phi(n)^2 < 1 \qquad \text{for all} \quad n = 1, 2, \cdots \quad \text{and} \quad \textstyle\sum_n \varepsilon_n^2/\phi(n)^2 = \infty.$$

Make X_n take the values $\pm\phi(n)/\varepsilon_n$ with chance $\frac{1}{2}\varepsilon_n^2/\phi(n)^2$ each, and the value 0 with remaining chance $1 - \varepsilon_n^2\phi(n)^2$; make the X_n independent. So $E(X_n) = 0$ and $\operatorname{Var} X_n = 1$. By way of contradiction, suppose

$$P\{\limsup S_n/\phi(n) < \infty\} > 0.$$

By the Kolmogorov 0–1 Law,

$$P\{\limsup S_n/\phi(n) < \infty\} = 1.$$

By symmetry,

$$P\{\liminf S_n/\phi(n) > -\infty\} = 1.$$

So

$$P\{\liminf S_{n-1}/\phi(n-1) > -\infty\} = 1.$$

Since ϕ is non-decreasing,

$$P\{\liminf S_{n-1}/\phi(n) > -\infty\} = 1.$$

So

$$P\{\limsup X_n/\phi(n) < \infty\} = 1.$$

But $\sum \varepsilon_n{}^2/\phi(n)^2 = \infty$, so $X_n = \phi(n)/\varepsilon_n$ for infinitely many n, and $X_n/\phi(n) = 1/\varepsilon_n \to \infty$ for these n.

REFERENCES

[1] DUBINS, L. E. and FREEDMAN, D. A. (1965). A sharper form of the Borel-Cantelli lemma and the strong law. *Ann. Math. Statist.* **36** 800-807.

[2] FREEDMAN, D. A. (1967). A remark on the law of the iterated logarithm. *Ann. Math. Statist.* **38** 598-600.

[3] NEVEU, J. (1972). *Martingales à temps discret.* Masson, Paris.

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