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ON CONVERGENCE IN r-MEAN OF SOME FIRST PASSAGE TIMES AND RANDOMLY INDEXED PARTIAL SUMS

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Let S_n, $n = 1, 2, \cdots$ denote the partial sums of i.i.d. random variables with positive, finite mean and with a finite moment of order r, $1 \leqq r < 2$. Let Z_n, $n = 1, 2, \cdots$ denote the partial sums of i.i.d. random variables with a finite moment of order r, $0 < r < 2$, and with mean 0 if $1 \leqq r < 2$. Let $N(c) = \min\{n; S_n > c\}$, $c \geqq 0$. Theorem 1 states that $N(c)$, (suitably normalized), tends to 0 in r-mean as $c \to \infty$. The first part of that proof follows by applying Theorem 2, which generalizes the known result $E|Z_n|^r = o(n)$, as $n \to \infty$ to randomly indexed partial sums.

1. Introduction. Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables with expectation θ, $0 < \theta < \infty$, set $S_0 = 0$ and let $S_n = \sum_{k=1}^n X_k$, $n \geqq 1$.

Define the first passage time

$$N = N(c) = \min\{n; S_n > c\}, \qquad c \geqq 0. \tag{1.1}$$

THEOREM 1. *If* $E|X|^r < \infty$, $1 \leqq r < 2$, *then*

$$c^{-1} \cdot E|N - c/\theta|^r \to 0, \qquad \text{as } c \to \infty. \tag{1.2}$$

The first part of the proof turns out to be a special case of the following, more general, result.

THEOREM 2. *Let* $Y_1, Y_2, \cdots$ *be* i.i.d. *random variables, suppose that* $E|Y|^r < \infty$, $0 < r < 2$, *and let* $EY = 0$ *if* $1 \leqq r < 2$. *Set* $Z_0 = 0$ *and* $Z_n = \sum_{k=1}^n Y_k$, $n \geqq 1$. *Let* $\{\tau(c), c \geqq 0\}$ *be a non-decreasing family of stopping times such that* $E\tau(c) < \infty$ *and* $E\tau(c) \nearrow \infty$, *as* $c \to \infty$. *Then*

$$E|Z_{\tau(c)}|^r = o(E\tau(c)), \qquad \text{as } c \to \infty. \tag{1.3}$$

If moreover, $c^{-1} \cdot E\tau(c) \to \mu$, *as* $c \to \infty$, *where* μ *is a positive constant, then*

$$E|Z_{\tau(c)}|^r = o(c), \qquad \text{as } c \to \infty. \tag{1.4}$$

REMARK 1. In, [6] Theorem 2.8, it has been proved that, under the conditions of Theorem 1,

$$c^{-1/r} \cdot (N - c/\theta) \to_{\text{a.s.}} 0, \qquad \text{as } c \to \infty. \tag{1.5}$$

Theorem 1 states that convergence in r-mean also holds.

REMARK 2. In [7], Pyke and Root have proved that if $Y_1, Y_2, \cdots$ are defined as in Theorem 2, then

$$E|Z_n|^r = o(n), \qquad \text{as } n \to \infty. \tag{1.6}$$

Theorem 2 generalizes (1.6) to the case of randomly indexed partial sums.

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2. Proofs.

PROOF OF THEOREM 2. First let $1 \leqq r < 2$. As in [6], define $\tau_n(c) = \min\{\tau(c), n\}$, $n \geqq 1$, and $U_m = \sum_{k=1}^{m} Y_k \cdot I\{\tau_n(c) \geqq k\}$, $m = 1, 2, \cdots, n$, where $I\{\cdot\}$ denotes the indicator function of the set in braces. In particular, $U_n = Z_{\tau_n(c)}$. Since $\{Z_n, \sigma\{Y_1, \cdots, Y_n\}\}_{n=1}^{\infty}$ is a martingale, it follows from Doob [5] Theorem 2.1, page 300, that $\{|U_m|^r, \sigma\{Y_1, \cdots, Y_m\}\}_{m=1}^{n}$ is a submartingale and $E|U_m|^r \leqq E|U_n|^r \leqq E|Z_n|^r < \infty$. Thus by the Burkholder–Davis inequalities, (see [1] Theorem 9, and [4] Theorem 1), there exists a constant $B_r > 0$, depending only on r, such that

$$E|U_n|^r \leqq B_r \cdot E|\textstyle\sum_{k=1}^{\tau_n(c)} Y_k^2|^{r/2} . \tag{2.1}$$

Since $E|Y|^r < \infty$, there exists, for every $\varepsilon > 0$, an $M > 0$, such that $E(|Y|^r \cdot I\{|Y| > M\}) < \varepsilon$. As in [3], set

$$Y_k' = Y_k \cdot I\{|Y_k| \leqq M\} \quad \text{and} \quad Y_k'' = Y_k - Y_k' , \qquad k = 1, 2, \cdots .$$

By the c_r-inequalities and Wald's lemma,

$$\begin{aligned} E|\textstyle\sum_{k=1}^{\tau_n(c)} Y_k^2|^{r/2} &= E|\textstyle\sum_{k=1}^{\tau_n(c)} (Y_k')^2 + (Y_k'')^2|^{r/2} \\ &\leqq E|\textstyle\sum_{k=1}^{\tau_n(c)} (Y_k')^2|^{r/2} + E|\textstyle\sum_{k=1}^{\tau_n(c)} (Y_k'')^2|^{r/2} \\ &\leqq E|\tau_n(c) \cdot M^2|^{r/2} + E\textstyle\sum_{k=1}^{\tau_n(c)} |Y_k''|^r \\ &= M^r \cdot E(\tau_n(c))^{r/2} + E\tau_n(c) \cdot E|Y''|^r \\ &\leqq M^r \cdot E(\tau_n(c))^{r/2} + \varepsilon \cdot E\tau_n(c) \\ &\leqq M^r \cdot E(\tau(c))^{r/2} + \varepsilon \cdot E\tau(c) . \end{aligned}$$

Thus, $E|U_n|^r \leqq B_r \cdot M^r \cdot E(\tau(c))^{r/2} + B_r \cdot \varepsilon \cdot E\tau(c) < \infty$, and by Fatou's lemma,

$$E|Z_{\tau(c)}|^r \leqq B_r \cdot M^r \cdot E(\tau(c))^{r/2} + B_r \cdot \varepsilon \cdot E\tau(c) < \infty . \tag{2.2}$$

Now, let $0 < r < 1$. By applying the c_r-inequalities, Wald's lemma and Fatou's lemma as above we obtain

$$E|Z_{\tau(c)}|^r \leqq M^r \cdot E(\tau(c))^r + \varepsilon \cdot E\tau(c) < \infty \tag{2.3}$$

This proves (1.3), from which the proof of (1.4) is immediate.

PROOF OF THEOREM 1. Now, $1 \leqq r < 2$. By [6] Theorem 2.1, $EN^r < \infty$, in particular, $EN < \infty$. Furthermore, by [2], Theorem 2, $c^{-1} \cdot EN \to \theta^{-1}$, as $c \to \infty$. Therefore, an application of Theorem 2 yields, (set $Y_k = X_k - \theta$),

$$c^{-1} \cdot E|S_N - N\theta|^r \to 0 , \qquad \text{as } c \to \infty . \tag{2.4}$$

Let $||X||_r = (E|X|^r)^{1/r}$, where X is a random variable.

$$||\theta N - c||_r \leqq ||S_N - N\theta||_r + ||S_N - c||_r . \tag{2.5}$$

By [6] Theorem 2.4,

$$c^{-1/r} \cdot ||S_N - c||_r \to 0 , \qquad \text{as } c \to \infty . \tag{2.6}$$

Now, (2.4)—(2.6) yield

$$c^{-1/r} \cdot ||\theta N - c||_r \to 0 , \qquad \text{as } c \to \infty . \tag{2.7}$$

Hence (1.2) has been proved.

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