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OPTIMAL STOPPING VARIABLES FOR BROWNIAN MOTION

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For a constant $\beta > \frac{1}{2}$ and W normalized Brownian motion with parameter space the nonnegative real line, the stopping variable λ defined by

$$\lambda = \sup\{t: W(s) < y_0(1+s)^{\frac{1}{2}}, 0 \leqq s < t\}$$

where y_0 is the unique positive root of

$$\int_0^\infty x^{2(\beta-1)} e^{(yx - x^2/2)}\, dx = y \int_0^\infty x^{(2\beta-1)} e^{(yx-x^2/2)}\, dx$$

is shown to be optimal in the sense that $E\{(1+\lambda)^{-\beta} W(\lambda)\}$ is equal to the supremum of $E\{(1+\tau)^{-\beta} W(\tau)\}$ over all stopping variables τ with respect to W. The values of y_0 for $\beta = 1.0, 1.5, 2.0, 2.5, 3.0, 3.5$, and 4.0 are given.

1. Introduction and summary of results. Let $\{W(t): t \in R_+\}$, where R_+ denotes the set of nonnegative real numbers, be a normalized Brownian motion, i.e. $W(0) = 0$, $E\{W(t)\} = 0$, and $E\{(W(t))^2\} = t$, defined on the probability space $(\Omega, \mathscr{F}, P)$ where the sample points in Ω are the continuous functions on R_+. Let $\mathscr{F}(t)$ denote the minimum σ-algebra of sets of $\mathscr{F}$ such that $W(s)$, $0 \leqq s \leqq t$, are $\mathscr{F}$-measurable and let T denote the set of stopping variables τ such that $\{\tau \leqq t\} \in \mathscr{F}(t)$ for all $t \in R_+$ and $P\{\tau < \infty\} = 1$. The purpose of this paper is to extend the result obtained by L. A. Shepp (1969) concerning an optimal stopping variable $\lambda \in T$ such that

$$E\{(1+\lambda)^{-1} W(\lambda)\} = \sup_{\tau \in T} E\{(1+\tau)^{-1} W(\tau)\}$$

to the case where the expected reward is $E\{(1+\tau)^{-\beta} W(\tau)\}$ for some $\tau \in T$ and constant $\beta > \frac{1}{2}$. [Shepp's case was $\beta = 1$.] The approach is different since it uses a result of A. G. Fakeev ((1970) Theorem 4, page 329) and an idea used by the author ((1974) Lemma 4 and associated material). The main result is

THEOREM. *For $\beta > \frac{1}{2}$, $\lambda = \sup\{t: W(s) < y_0(1+s)^{\frac{1}{2}}, 0 \leqq s < t\}$, where y_0 is the unique positive root of*

$$\int_0^\infty x^{2(\beta-1)} e^{(yx - x^2/2)}\, dx = y \int_0^\infty x^{(2\beta-1)} e^{(yx-x^2/2)}\, dx\,,$$

belongs to T and is optimal, i.e.

$$E\{(1+\lambda)^{-\beta} W(\lambda)\} = \sup_{\tau \in T} E\{(1+\tau)^{-\beta} W(\tau)\}\,.$$

The proof of consists of two parts. The first part establishes that an optimal stopping variable can be expressed in the form $\sup\{t: W(s) < y(1+s)^{\frac{1}{2}}, 0 \leqq s < t\}$ for some appropriate constant y. The second part determines the appropriate

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value of y using standard calculus techniques for minimizing a function and a result of D. A. Darling and A. J. F. Siegert (1953).

The following tabulation of values of y_0 for given values of β were calculated using the Computer Research Center facilities at the Brigham Young University:

β	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y_0	0.8399···	0.6388···	0.5372···	0.4728···	0.4272···	0.3928···	0.3655···

2. Proof of the theorem. The fact that $E\{\sup_{t\in R_+}(1+t)^{-\beta}|W(t)|\}$ is finite follows from Lemma 2 of L. H. Walker (1974) since

$$\int_0^\infty (1+t)^{-2\beta}\,dt = \frac{1}{2\beta-1} < \infty .$$

Hence, A. G. Fakeev's Theorem 4 ((1970) page 329) can be used to define a function λ as $\lambda = \sup\{t\colon (1+s)^{-\beta}W(s) < \nu_s,\ 0 \leqq s < t\}$ where

$$\nu_t = \operatorname{ess\,sup}_{\tau\in T_t} E\{(1+\tau)^{-\beta}W(\tau)\,|\,\mathscr{F}(t)\}$$

[A. G. Fakeev (1970) Theorem 1 page 326] and $T_t = \{\max(t,\tau)\colon \tau\in T\}$ with the property that

$$\int_{\{\lambda<\infty\}}(1+\lambda)^{-\beta}W(\lambda)\,dP + \int_{\{\lambda=\infty\}}\limsup_{t\to\infty}(1+t)^{-\beta}W(t)\,dP = \sup_{\tau\in T} E\{(1+\tau)^{-\beta}W(\tau)\} .$$

Observe that the above results do not insure that $\lambda\in T$; that is, $P\{\lambda=\infty\}$ may be positive. Note that

$$\limsup_{t\to\infty}(1+t)^{-\beta}W(t) = \limsup_{t\to\infty}\left(\frac{(2t\log\log t)^{\frac12}}{(1+t)^{\beta}}\,\frac{W(t)}{(2t\log\log t)^{\frac12}}\right) = 0$$

by the law of the iterated logarithm for Brownian motion; hence, the second integral on the left-hand side of the above equation equals zero.

By following exactly the same procedure as described in L. H. Walker ((1974) Lemma 4 and associated material) with only a small change in notation, one can establish that $\lambda = \sup\{t\colon W(s) < f(s),\ 0\leqq s<t\}$ where f is a function on R_+ to the real line which is characterized by the property

$$\begin{aligned}\sup_{\tau\in T} E\{(1+t+\tau)^{-\beta}(y+W(\tau))\} &> (1+t)^{-\beta}y \qquad \text{when}\quad y<f(t)\,,\\ &\leqq (1+t)^{-\beta}y \qquad \text{when}\quad y\geqq f(t)\,,\end{aligned}$$

with the interpretation that when $f(t)$ is $+\infty$ or $-\infty$ the appropriate inequality is deleted from the above expression. Consideration of the stopping variable $\tau=1$ shows that $f(t)\geqq 0$ for all $t\in R_+$, i.e.

$$E\{(2+t)^{-\beta}(y+W(1))\} = (2+t)^{-\beta}y > (1+t)^{-\beta}y \qquad \text{for all}\quad y<0 .$$

Because of the scaling property of W, namely that $W(s)$ and $t^{\frac12}W(s/t)$ have the same distribution function,

$$\begin{aligned}\sup_{\tau \in T} E\{(1+t+\tau)^{-\beta}(y+W(\tau))\} \\ &= (1+t)^{\frac{1}{2}-\beta} \sup_{\tau \in T} E\left\{\left(1+\frac{\tau}{1+t}\right)^{-\beta}\left(\frac{y}{(1+t)^{\frac{1}{2}}}+W\left(\frac{\tau}{1+t}\right)\right)\right\} \\ &= (1+t)^{\frac{1}{2}-\beta} \sup_{\tau \in T} E\{(1+\tau)^{-\beta}((1+t)^{-\frac{1}{2}}y+W(\tau))\}.\end{aligned}$$

Comparison of both sides of the above expression with $(1+t)^{-\beta}$ quickly establishes that $f(t) = f(0)(1+t)^{\frac{1}{2}}$. If $f(0)$ is finite, then λ, which can now be expressed as

$$\lambda = \sup\{t : W(s) < f(0)(1+s)^{\frac{1}{2}}, 0 \leqq s < t\},$$

is finite with probability one by the law of the iterated logarithm for Brownian motion. Hence, $\lambda \in T$. The other possibility for $f(0)$ is that it is infinite; however, then $P\{\lambda = \infty\} = 1$ and

$$\sup_{\tau \in T} E\{(1+\tau)^{-\beta}W(\tau)\} = \int_{\{\lambda<\infty\}} (1+\lambda)^{-\beta}W(\lambda)\, dP = 0.$$

Consideration of the stopping variable $\tau = \sup\{t : W(s) < 1, 0 \leqq s < t\}$, i.e. stop the first time that W reaches 1, establishes that $\sup_{\tau \in T} E\{(1+\tau)^{-\beta}W(\tau)\} > 0$, which contradicts the above statement. Hence, $f(0)$ is finite. The first part of the proof is completed. It remains to characterize $f(0)$ as the unique positive root of

$$\int_0^\infty x^{2(\beta-1)}e^{(yx-x^2/2)}\, dx = y \int_0^\infty x^{(2\beta-1)}e^{(yx-x^2/2)}\, dx.$$

For τ defined by

$$\tau = \sup\{t : W(s) < y(1+s)^{\frac{1}{2}}, 0 \leqq s < t\},$$

put

$$\begin{aligned}V(y) \equiv E\{(1+\tau)^{-\beta}W(\tau)\} &= \int_0^\infty y(1+x)^{\frac{1}{2}-\beta}\, d_x P\{\tau \leqq x\} \\ &= y \int_1^\infty z^{\frac{1}{2}-\beta}\, d_z P\{\tau \leqq z-1\}.\end{aligned}$$

By the definition of τ,

$$\begin{aligned}P\{\tau \leqq z-1\} &= 1 - P\{s^{-\frac{1}{2}}W(s) < y, 1 \leqq s < z\} \\ &= 1 - P\{U(u) < y, 0 \leqq u < 2^{-1}\log z\}\end{aligned}$$

where $s = e^{2u}$ and $U(u) = e^{-u}W(e^{2u})$. In terms of the U process (which is the Uhlenbeck process), V can be expressed in the form

$$V(y) = -y \int_0^\infty e^{v(1-2\beta)}\, d_v P\{U(u) < y, 0 \leqq u < v\}.$$

The above integral is the Laplace transform, with parameter $2\beta - 1$, of the first passage time of the Uhlenbeck process across the boundary y. Hence, the results of D. A. Darling and A. J. F. Siegert ((1953) Theorem 3.1, page 627; Theorem 4.1, page 629; Example b, page 630) give

$$V(y) = D_{(1-2\beta)}(0)/e^{y^2/4}D_{(1-2\beta)}(-y)$$

where D_α is the parabolic cylinder function for parameter α. $D_\alpha(z)$ can be expressed as a contour integral in the complex plane [E. T. Whittaker and G. N.

Watson (1948) page 349; see page 244 for the definition of the contour] which is equal to

$$-\frac{\sin(\pi\alpha)\Gamma(\alpha+1)}{\pi} e^{-z^2/4} \int_0^\infty x^{-(\alpha+1)} e^{-(zx+x^2/2)}\, dx .$$

Therefore,

$$V(y) = \frac{y \int_0^\infty x^{2(\beta-1)} e^{-x^2/2}\, dx}{\int_0^\infty x^{2(\beta-1)} e^{(yx-x^2/2)}\, dx} .$$

V is nonnegative for $y \geqq 0$, zero when $y = 0$, and approaches zero as y tends to infinity. Also, V is differentiable as a function of y; hence, there must be at least one value of y where the derivative of V is zero and V is maximum. For the derivative to be zero,

$$\int_0^\infty x^{2(\beta-1)} e^{(yx-x^2/2)}\, dx = y \int_0^\infty x^{(2\beta-1)} e^{(yx-x^2/2)}\, dx .$$

The left-hand side of this equation is positive when $y = 0$ and increases with slope

$$\int_0^\infty x^{(2\beta-1)} e^{(yx-x^2/2)}\, dx ,$$

while the right-hand side is zero when $y = 0$ and increases with the greater slope

$$\int_0^\infty x^{(2\beta-1)} e^{(yx-x^2/2)}\, dx + y \int_0^\infty x^{2\beta} e^{(yx-x^2/2)}\, dx .$$

Hence, there can be at most one positive root and at this root V is maximum. The theorem is proved.

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