The Annals of Probability
1974, Vol. 2, No. 2, 202–230

ON THE FUNCTIONAL FORM OF THE LAW OF THE ITERATED LOGARITHM FOR THE PARTIAL MAXIMA OF INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VARIABLES[1]

BY MICHAEL J. WICHURA

University of Chicago

Let k be a positive integer, let $X, X_1, X_2, \cdots$ be i.i.d. random variables, and let $m_n^{(k)}$ be the kth largest of $X_1, \cdots, X_n$. Let $(M_n^{(k)}(t))_{0<t<\infty}$ be the random process defined by $M_n^{(k)}(t) = m_{[nt]}^{(k)}$. $M_n^{(k)}$ takes values in the space D of non-decreasing right-continuous functions on $(0, \infty)$. Let D be endowed with the usual topology of weak convergence. We show that if X is uniformly distributed over $[-1, 0]$, then wp 1 the sequence $(M_n^{(k)}/(\log_2 n/n))_{n\geqq 3}$ is relatively compact in D and its limit points coincide with $\{x \in D: x(t) \leqq 0$ for all t, and $\int x(t)\,dt \geqq -1\}$. Also, we show that if X is exponential with mean 1, then wp 1 the sequence $((M_n^{(k)} - \log n)/\log_2 n)_{n\geqq 3}$ is relatively compact in D and its limit points coincide with $\{x \in D: x(t) \geqq 0$ for all t, and $\lambda_k(x) \leqq 1\}$; here $\lambda_k(x) = \sup(\sum_{p<q} x(t_p) + kx(t_q))$, with the supremum being taken over all finite systems of points $\{t_p\}_{p\leqq q}$ over which x is strictly increasing. Extensions of and corollaries to these results are given.

1. Introduction. Let X_n, $n \geqq 1$, be i.i.d. random variables with consecutive partial sums S_n. Let H_n be the random polygonal line with vertices at the points $(j/n, S_j/(2n \log_2 n)^{\frac{1}{2}})$, $0 \leqq j \leqq n$. Strassen (1964) profoundly generalized the classical law of the iterated logarithm by showing that when the X_n's have mean zero and unit variance, the sequence (H_n) is wp 1 relatively compact in the topology of uniform convergence, and has as its limit points the class of absolutely continuous functions on $[0, 1]$ which vanish at 0 and whose derivatives lie in the unit ball of $L_2([0, 1])$ under Lebesgue measure. Strassen first proved the corresponding result for Brownian motion, and then used the well-known Skorohod imbedding theorem to deduce the discrete-time version.

We are concerned here with similar results for the partial maxima of the X_n's, or more generally, with the kth order statistics $m_n^{(k)}$, $n \geqq k$, of (2.1), k being a fixed positive integer. These results are described in Sections 2 and 3. Analogues of Skorohod's imbedding theorem can be formulated using the two-dimensional Poisson process (cf. Pickands (1971)) or the extremal processes of Dwass (1964). However we have found it simplest to establish our results directly for uniformly distributed variables (Sections 6, 7, and 8), and then to use (Section 4) a probability integral transformation to handle the non-uniform case. In Section 5 we

Received July 24, 1972; revised June 15, 1973.

[1] This research was carried out in the Department of Statistics, University of Chicago, under partial sponsorship of Research Grant No. GP 25911 from the Division of Mathematical Sciences of the National Science Foundation, and Grant No. N00014-67-A-0285-0009 of The Statistics Branch, Office of Naval Research.

AMS 1970 *subject classifications*. Primary 60F15; Secondary 60G17, 60J75.

Key words and phrases. Law of the iterated logarithm, extreme value theory, partial maxima.

give some general tools for establishing limit theorems of the type considered here.

Our results extend most of the known "weak" forms of the iterated logarithm laws for the partial maxima of i.i.d. random variables (cf. Kiefer (1972) and de Haan and Hordijk (1972)). A different sort of generalization of the weak laws has been given by Barndorff-Nielsen, (1961) and (1963); he derived the corresponding "strong" forms, giving criteria to distinguish between "upper class" and "lower class" functions. Robbins and Siegmund (1972) rediscovered the strong laws and used their martingale techniques to compute some boundary-crossing probabilities for extremal processes. They also remarked that Motoo's (1959) argument could be adapted to give a simple proof of the strong laws. This point has been elaborated on by Frankel (1972), and by Wichura (1973b), who showed that a refinement of Motoo's argument yields information on boundary-crossing probabilities. We refer to Vervaat (1974) for an overview of recent activity regarding limit results for sample maxima and record values.

We close this section by establishing some terminology, notation, and conventions. If ψ is a non-decreasing right-continuous function on $(-\infty, \infty)$, the *inverse function*, denoted $\psi^{\sim}$, is defined by

$$\psi^{\sim}(s) = \inf\{t : s < \psi(t)\}, \tag{1.1}$$

so that $\psi^{\sim}$ is also right-continuous and non-decreasing. A similar definition holds if ψ is non-increasing. A function ϕ which is *regularly varying* at ∞ with exponent ρ, i.e. for which

$$\lim_{t\to\infty} \psi(ct)/\psi(t) = c^{\rho} \tag{1.2}$$

for all $c > 0$, will be said to belong to the class $\mathscr{RV}_{\infty}(\rho)$. For ϕ nonzero in a right (resp. left) neighborhood of 0, $\phi \in \mathscr{RV}_0(\rho)$ means that (1.2) holds with $\lim_{t\to\infty}$ replaced by $\lim_{t\downarrow 0}$ (resp. $\lim_{t\uparrow 0}$). Given a reference probability space $(\Omega, \mathscr{A}, P)$, we use the phrases *with probability one* (wp 1) and *almost all* in connection with subsets of Ω which contain a $\mathscr{A}$-set of P-probability one; if $(\Omega, \mathscr{A}, P)$ is complete (as it may, of course, be assumed to be), these usages agree with the ordinary ones. For $i \geqq 2$, the *iterated logarithm* $\log_i t$ means $\log(\log_{i-1} t)$; in writing such expresssions, we assume t is large enough for them to be defined.

I_A denotes the *indicator function* of a set A. $[u]$ denotes the *greatest integer* in the number u. The abbreviations iff and i.o. have their ordinary meanings: if and only if, and infinitely often.

2. The uniform case. Throughout this paper, we shall let k denote a fixed, but arbitrary positive integer. Let X_n, $n \geqq 1$, be independent random variables, each of which is uniformly distributed over the interval $[-1, 0]$. For $n \geqq k$, put

$$m_n^{(k)} = k\text{th} \quad \text{largest of} \quad X_1, \cdots, X_n. \tag{2.1}$$

There are two (weak) laws of the iterated logarithm for the sequence $(m_n^{(k)})$, namely

(2.2) $$\liminf_n m_n^{(k)}/(\log_2 n/n) = -1 \quad \text{wp } 1$$

and

(2.3) $$\limsup_n (-\log|m_n^{(k)}| - \log(n))/\log_2 n = 1/k \quad \text{wp } 1$$

(cf. Kiefer (1972) pages 234–235). The first law may be rewritten in the form

(2.4) $$\begin{aligned} \Pr\{m_n^{(k)} \leqq \theta \log_2 n/n \ \text{ i.o.}\} &= 1, \qquad \text{if } \theta > -1 \\ &= 0, \qquad \text{if } \theta < -1 \end{aligned}$$

and is therefore a statement about how "small" the values of the $m_n^{(k)}$'s can be. Similarly, the second may be rewritten in the form

(2.5) $$\begin{aligned} \Pr\{m_n^{(k)} \geqq -1/(n \log^{\zeta} n) \ \text{ i.o.}\} &= 1, \qquad \text{if } \zeta < 1/k \\ &= 0, \qquad \text{if } \zeta > 1/k \end{aligned}$$

and is thus a statement about how "large" the values of the $m_n^{(k)}$'s can be. We note that (2.5) implies

(2.6) $$\limsup_n m_n^{(k)}/(\log_2 n/n) = 0 \quad \text{wp } 1,$$

while (2.4) implies

(2.7) $$\liminf_n (-\log|m_n^{(k)}| - \log(n))/\log_2 n = 0 \quad \text{wp } 1.$$

In what follows, we shall give analogues of (2.2)—(2.6) and (2.3)—(2.7) for the random processes $M_n^{(k)} \equiv (M_n^{(k)}(t))_{0<t<\infty}$ defined (for $n \geqq 1$) by

(2.8) $$\begin{aligned} M_n^{(k)}(t) &= m_{[nt]}^{(k)}, \qquad \text{if } k/n \leqq t < \infty \\ &= m_k^{(k)}, \qquad \text{if } 0 < t < k/n. \end{aligned}$$

Let D denote the space of non-decreasing, right-continuous real-valued functions on $(0, \infty)$. The Lévy distance between x_1 and x_2 in D is taken to be

(2.9) $$d(x_1, x_2) = \inf\{\varepsilon : x_1^*(u-\varepsilon) - \varepsilon \leqq x_2^*(u) \leqq x_1^*(u+\varepsilon) + \varepsilon, \text{ for } -\infty < u < \infty\}$$

(2.10) $$= \sup\{l_u/2^{\frac{1}{2}} : -\infty < u < \infty\},$$

where

$$\begin{aligned} x_i^*(u) &= -\infty, && \text{if } u < 0 \\ &= x_i(0+), && \text{if } u = 0 \\ &= x_i(u/(1-u)), && \text{if } 0 < u < 1 \\ &= \infty, && \text{if } 1 \leqq u \end{aligned}$$

and where l_u denotes the distance between the graphs of x_1^* and x_2^*, measured along the line $\{(\xi, \eta) : \xi + \eta = u\}$. Under d, D is a separable metric space, in which $x_n \to x$ (i.e. $d(x_n, x) \to 0$) iff $x_n(t) \to x(t)$ for all continuity points t of x. The coordinate mapping $\pi_t : x \to x(t)$ is thus uppersemicontinuous for each t.

A subset $C \subset D$ is relatively compact iff it is uniformly bounded off of neighborhoods of 0 and ∞ in the sense that

$$(2.11) \qquad \sup_{x \in C} \sup_{\delta \leqq t \leqq 1/\delta} |x(t)| < \infty$$

for each $\delta \in (0, \infty)$.

We are now going to describe two compact subsets of D. Put $D_- = \{x \in D: x(t) \leqq 0, \text{ for all } t\}$, and let $\kappa: D_- \to [-\infty, 0]$ be defined by

$$(2.12) \qquad \kappa(x) = \int_0^\infty x(t)\, dt \, ;$$

by Fatou's lemma, κ is uppersemicontinuous. Put

$$(2.13) \qquad K = \{x \in D_-: \kappa(x) \geqq -1\} \, .$$

Notice that $x \in K$ implies $x(t) \geqq -1/t$ for each t. This and the uppersemicontinuity of κ imply that K is relatively compact and closed in D_-, and thus is compact in D.

Next, put $D_+ = \{x \in D: x(t) \geqq 0, \text{ for all } t\}$, and let $\lambda_k: D_+ \to [0, \infty]$ be defined by

$$(2.14) \qquad \lambda_k(x) = \sup \left(\sum_{p<q} x(t_p) + kx(t_q)\right)$$

where the supremum is taken over all finite systems $\{t_p\}_{p \leqq q}$ of points in $(0, \infty)$ satisfying

$$(2.15) \qquad t_1 < t_2 < \cdots < t_q \quad \text{and} \quad x(t_1) < x(t_2) < \cdots < x(t_q) \, .$$

Put

$$(2.16) \qquad L_k = \{x \in D_+: \lambda_k(x) \leqq 1\} \, .$$

Each $x \in L_k$ is a step function which is bounded between 0 and $1/k$ and has only finitely many jumps, but not all such step functions are in L_k. For example, $x \in L_k$ and $x(0+) \geqq 1/(k+1)$ imply that x is constant. Also, $x \in L_k$ and $x(0+) = 1/(k+2)$ imply that x has at most one jump, which moreover, must have size $\leqq 1/(k(k+2))$; there is no constraint on the point at which the jump may occur. The value of $\lambda_k(x)$ is unchanged if in (2.16) we require that each t_p be a continuity point of x; it follows that λ_k is lowersemicontinuous. Thus L_k is closed in D_+, and being uniformly bounded, is compact in D.

Let the processes ${}_*H_n^{(k)} \equiv ({}_*H_n^{(k)}(t))_{0<t<\infty}$ and ${}^*H_n^{(k)} \equiv ({}^*H_n^{(k)}(t))_{0<t<\infty}$, $n \geqq 3$, be defined by

$$(2.17) \qquad {}_*H_n^{(k)}(t) = M_n^{(k)}(t)/(\log_2 n/n)$$

$$(2.18) \qquad {}^*H_n^{(k)}(t) = (-\log |M_n^{(k)}(t)| - \log (n))/\log_2 n$$

(cf. (2.8)). Here are our main results; the first extends (2.2) and (2.6), the second, (2.3) and (2.7):

THEOREM 1A. Wp 1, *the sequence* $({}_*H_n^{(k)})$ *is relatively compact in* D, *and the set of its limit points coincides with* K.

THEOREM 1B. Wp 1, *the sequence* $({}^*H_n^{(k)})$ *is relatively compact in* D, *and the set of its limit points coincides with* L_k.

Theorem 1A will be proved in Section 6, Theorem 1B in Section 7. Our results yield limit theorems for various functionals of ${}_*H_n^{(k)}$ and ${}^*H_n^{(k)}$. Many of these are based on the following simple lemma, which is analogous to the so-called mapping theorem in weak-convergence (cf. Billingsley (1968) page 34); see also Wichura (1973a)—condition (2.17) there was stated incorrectly, and should be changed to read like (2.19) below.

LEMMA 2.1. *Let* S *and* S' *be metric spaces. Let* C *be a compact subset of* S, *and suppose that* (H_n) *is a sequence of* S-*valued random variables which* wp 1 *is relatively compact and has* C *for its set of limit points. Let* T_n, $n \geqq 1$, *and* T *be mappings from* S *to* S'. *Suppose that for almost all sample points* ω, *one has*

$$T_{n_j}(H_{n_j}(\omega)) \to T(x) \tag{2.19}$$

whenever $x \in C$, $n_j \uparrow \infty$, *and* $H_{n_j}(\omega) \to x$. *Then* wp 1 *the sequence* $(T_n(H_n))$ *is relatively compact in* S' *and has* $C' \equiv T(C)$ *as its set of limit points. Moreover, the following statement is true for almost all sample points* ω: *for each subsequence* (n_j) *for which* $T_{n_j}(H_{n_j}(\omega))$ *converges to a point* c' *in* C', *the distance in* S *from* $H_{n_j}(\omega)$ *to the set* $\{c \in C : T(c) = c'\}$ *tends to zero.*

The somewhat complicated continuity condition involving (2.19) holds if, e.g., $T_n(x_n) \to T(x)$ whenever $x_n \to x \in C$. Here are some consequences of Theorem 1, each of which follows from Lemma 2.1 or a simple modification of it.

I. Let ν_n, $n \geqq 1$, and ν be signed measures on the Borel sets of $(0, \infty)$, and suppose that each of these measures gives finite mass to sets bounded away from ∞. Set

$$v_n(t) = \nu_n((0, t)) \qquad \text{and} \qquad v(t) = \nu((0, t)) \tag{2.20}$$

for $0 < t < \infty$. We will say that the ν_n's are M_2-convergent with limit ν (cf. Skorohod (1956) Section 2.2.10) if

$$\begin{aligned} \lim_n \inf_{a \leqq t \leqq b} v_n(t) &= \inf_{a \leqq t \leqq b} v(t) \qquad \text{and} \\ \lim_n \sup_{a \leqq t \leqq b} v_n(t) &= \sup_{a \leqq t \leqq b} v(t) \end{aligned} \tag{2.21}$$

for all continuity points $a < b$ of v. For nonnegative measures, (2.21) holds iff $v_n(t) \to v(t)$ for all continuity points t of v, that is, iff ν_n converges to ν weakly. In general (2.21) holds if ν_n^+ (resp. ν_n^-) converges weakly to ν^+ (resp. ν^-); this is a consequence of the fact that, for left-continuous functions on $(0, \infty)$ having right limits everywhere, $\xi_n \to_{M_2} \xi$ and $\eta_n \to_{M_2} \eta$ imply $\xi_n + \eta_n \to_{M_2} \xi + \eta$ if ξ and η have no common points of discontinuity. On the other hand, the weak convergence of the positive and negative parts of the ν_n's is not necessary for (2.21), as can be seen by setting $v_n(t) = n^{-1} \sin(nt)$, $t > 0$.

COROLLARY 2.1. *Suppose that the measures* ν_n *are* M_2-*convergent to* ν *in the sense* *of* (2.21). *Suppose further that*

(2.22) $\lim_{\delta\downarrow 0} \limsup_n \int_0^\delta 1/t|\nu_n|(dt) = 0$ and $\lim_{\delta\uparrow\infty} \limsup_n \int_\delta^\infty \log_2 t/t|\nu_n|(dt) = 0$,

where $|\nu_n| = \nu_n^+ + \nu_n^-$ *denotes the total variation of* ν_n. *Then*

(2.23) wp 1 *the sequence* $(\int {}_*H_n^{(k)}\, d\nu_n)$ *is relatively compact with* $[\gamma, \Gamma]$ *as limit points*;

here

(2.24) $\gamma = \inf\{v_*(t)/t\colon 0 < t < \infty\}$ *and* $\Gamma = \sup\{v^*(t)/t\colon 0 < t < \infty\}$

with (cf. (2.20))

$$v_*(t) = \min(-v(t), -v(t+)) \quad \textit{and} \quad v^*(t) = \max(-v(t), -v(t+))$$

for $0 < t < \infty$.

There is some additional information available about the shape of the function ${}_*H_n^{(k)}$ when $\int {}_*H_n^{(k)}\, d\nu_n$ is near γ or Γ. Suppose, e.g., that $\gamma < 0$. Put $B = \{t \in (0, \infty)\colon v_*(t)/t > \gamma\}$ and let V_* be the set of x in K such that $\int x(t)\, dt = -1$ and such that $\int_B dx(t) = 0$. Then wp 1, for large n, $\int {}_*H_n^{(k)}\, d\nu_n$ is near γ only if ${}_*H_n^{(k)}$ is near the set V_* (cf. the end of Lemma 2.1). This is also true when $\gamma = 0$, provided the requirement that $\int x(t)\, dt = -1$ is dropped from the definition of V_*. Dual statements hold concerning Γ.

For example when $\nu = \nu_n$, $n \geqq 1$, is a unit mass at the point $u \in (0, \infty)$ one has

(2.25) $\gamma = -1/u$ and $\Gamma = 0$

(compare (2.2) and (2.6)); the set V_* above consists of the single function $x_u \equiv -I_{(0,u)}/u$. When ν_n places mass $1/n$ at the points j/n, $1 \leqq j \leqq n$, the first half of condition (2.22) fails; moreover, so does the conclusion (2.23), for in this case $\int {}_*H_n^{(k)}\, d\nu_n = \sum_{j\leqq n} m_j^{(k)}/\log_2 n$ is known (when $k = 1$) to be almost surely asymptotic to $-k \log n/\log_2 n$ (cf. Grenander (1965)). It would be of interest to know if (2.23) were true with ${}_*H_n^{(k)}(t)$ replaced by, say, ${}_*H_n^{(k)}(t) - k/([nt] + 1)$.

II. As a consequence of (2.25), we have

(2.26) $m_n^{(k)}/(\log_2 n/n) \leqq c$

infinitely often wp 1 for each c in $(-1, 0)$ (this is, of course, also implied by (2.2)). This result can be sharpened, as follows. Let $f_{p,c}$ be the random variable which records the proportion of integers n between k and p such that (2.26) holds. Put

$$e(c) = \exp(-(|c|^{-1} - 1)), \qquad f(c) = 1 - e(c),$$

and let x_c be the function in D given by

$$(2.27) \qquad \begin{aligned} x_c(t) &= c/e(c), && \text{if } 0 < t < e(c), \\ &= c/t, && \text{if } e(c) \leqq t < 1, \\ &= 0, && \text{if } 1 \leqq t < \infty. \end{aligned}$$

COROLLARY 2.2. Wp 1 *one has*

$$0 = \liminf_p f_{p,c} \quad \textit{and} \quad \limsup_p f_{p,c} = f(c)\,; \tag{2.28}$$

moreover, for large p, $f_{p,c}$ *close to* $f(c)$ *implies that* ${}_*H_p^{(k)}$ *is near* x_c.

III. For $\sigma \in (0, \infty)$, let

$$\begin{aligned} z_\sigma &= \min\{n : m_n^{(k)} \geqq -\sigma\}\,, && \text{if} \quad m_k^{(k)} > -\sigma \\ &= 0\,, && \text{if} \quad m_k^{(k)} \leqq -\sigma \end{aligned}$$

be the first passage time through level $-\sigma$, and define $Z_t = (Z_t(s))_{0<s<\infty}$ by

$$Z_t(s) = z_{st}/(t^{-1}\log_2 t^{-1})\,.$$

The random processes Z_t take values in $-D \equiv \{-x : x \in D\}$, and their almost sure behavior as $t \downarrow 0$ is given by

COROLLARY 2.3. Wp 1, *the net* (Z_t) *is relatively compact in* $-D$ *as* $t \downarrow 0$, *and its limit points coincide with* $-K$.

IV. The mode of convergence on D which we have been using, i.e., pointwise convergence at continuity points, leaves something to be desired, in that it does not clearly specify the behavior of a convergent sequence over neighborhoods of 0 and of ∞. However, by making use of the interrelations among the ${}_*H_n$'s, one can deduce that they are relatively compact under a stronger mode of convergence, which does not suffer from the above-mentioned defect.

Let Δ be the class of bounded functions on $(0, \infty)$ which are right continuous and have left-limits everywhere. The *graph*, Γ_y, of a function y in Δ is the subset of $(0, \infty) \times (0, \infty)$ consisting of all pairs (t, β) such that β belongs to the closed interval whose endpoints are $y(t-)$ and $y(t)$. A *parameterization* of Γ_y is a one-to-one continuous mapping

$$s \to (\tau(s), \eta(s))$$

of $(0, \infty)$ onto Γ_y such that τ is non-decreasing. Following Skorohod (1956) we take the M_1-*distance* between y_1 and y_2 in Δ to be

$$\rho(y_1, y_2) = \inf\left(\sup\{(|\tau_2(s) - \tau_1(s)| + |\eta_2(s) - \eta_1(s)|) : 0 < s < \infty\}\right)$$

where the infimum is taken over all parameterizations (τ_1, η_1) of y_1 and (τ_2, η_2) of y_2. The corresponding metric topology on Δ is called the M_1-*topology*.

Now let w be a continuous mapping from $(0, \infty)$ to $(0, \infty)$ such that

$$w(t) = o(t) \quad \text{as} \quad t \downarrow 0 \qquad \text{and} \qquad w(t) = O(t/\log_2 t) \quad \text{as} \quad t \uparrow \infty\,. \tag{2.29}$$

Put

$$D_w = \{x \in D : wx \in \Delta\}$$

and for x_1, x_2 in D_w set

$$d_w(x_1, x_2) = \rho(wx_1, wx_2)\,.$$

The random functions ${}_*H_n$ take values in D_w because of (2.2)—(2.6). Also, $K \subset D_w$ since $x(t) \geqq -1/t$ for all t and all x in K.

COROLLARY 2.4. Wp 1, *the sequence* $({}_*H_n)$ *is relatively compact in* D_w *under the metric* d_w, *and its limit points coincide with the set* K.

The proofs of the above corollaries are given in Section 8. We close this section with a result which is intermediate between Theorem 1A and 1B. From (2.4) and (2.5), it follows that the almost sure $\liminf_n$ and $\limsup_n$ of

$$(-\log |m_n^{(k)}| - \log (n))/\log_3 n$$

are -1 and ∞ respectively; this was brought to my attention by Wim Vervaat. To formulate a functional analogue of this result we introduce the space D^* of non-decreasing right-continuous functions $x: (0, \infty) \to (-\infty, \infty]$. Convergence of x_n to x in D^* is taken to be convergence at continuity points of x. A subset C of D^* is relatively compact iff it is uniformly bounded from below off of neighborhoods of 0 and ∞ (cf. (2.11)). The techniques used to prove Theorems 1A and 1B also yield

THEOREM 1C. *Put* ${}^\bullet H_n^{(k)} = (-\log |M_n^{(k)}| - \log (n))/\log_3 n$. *Then* wp 1 *the sequence* $({}^\bullet H_n^{(k)})$ *is relatively compact in* D^*, *and its limit points coincide with*

$$J \equiv \{x \in D^* : x(t) \geqq -1 \text{ for } 0 < t < \infty\} . \tag{2.30}$$

In view of the rather uninteresting nature of the derived set J, we shall not pursue this result in the sequel.

3. Extension to the non-uniform case. Suppose now that the random variables X_n, $n \geqq 1$, are i.i.d. as in Section 2, but not necessarily uniformly distributed on $[-1, 0]$. We are going to deduce analogues of the strong limit theorems of Section 2 by means of a probability integral transformation. To see what is involved, set $\mathscr{F} = 1 - F$, where F is the common distribution function of the X_n's, and put $\mathscr{G} = -\mathscr{F}$; notice that $\mathscr{G}^\sim(u) = \mathscr{F}^\sim(-u)$, where $\mathscr{F}^\sim$ and $\mathscr{G}^\sim$ denote respectively the inverse of $\mathscr{F}$ and of $\mathscr{G}$ (cf. (1.1)). Also, set $\mathscr{R} = -\log (\mathscr{F})$ and notice $\mathscr{R}^\sim(u) = \mathscr{F}^\sim(e^{-u})$ and $\mathscr{F}^\sim(t) = \mathscr{R}^\sim(\log (1/t))$. Let U_n, $n \geqq 1$, be i.i.d. random variables, each uniformly distributed on $[-1, 0]$. The sequence (X_n) can and will be represented by $(\mathscr{G}^\sim(U_n))$. By monotoneity, the kth-largest of $X_1, \cdots, X_n$ is then $\mathscr{G}^\sim$(kth-largest of $U_1, \cdots, U_n$), whence (cf. (2.8))

$$M_n^{(k)} \equiv M_n^{(k)}(X) = \mathscr{G}^\sim(M_n^{(k)}(U)) , \tag{3.1}$$

where we let $\mathscr{G}^\sim$ act on functions in D by letting it act on each coordinate. Using the same kind of convection for $\mathscr{R}^\sim$, we find from (3.1) that

$$M_n^{(k)} = \mathscr{R}^\sim(\log (n) - \log_3 n - \log |{}_*H_n^{(k)}(U)|) \tag{3.2}$$

$$M_n^{(k)} = \mathscr{R}^\sim(\log (n) + (\log_2 n)^*H_n^{(k)}(U)) . \tag{3.3}$$

In order to get strong limit theorems for functions of the form $(M_n^{(k)} - b_n)/a_n$, we are led, via the mapping Lemma 2.1 and the "pointwise" nature of the convergences in Theorems 1A and 1B, to consider distributions F such that, for suitably chosen numbers α_t and β_t,

$$\lim_{t\to\infty} (\mathscr{R}^\sim(t + \xi f(t)) - \beta_t)/\alpha_t \equiv l(\xi) \tag{3.4}$$

exists at all continuity points ξ of some non-decreasing, nonconstant function $l\colon (-\infty, \infty) \to (-\infty, \infty)$; we take $f(t) = 1$ in conjunction with (3.2) and $f(t) = \log(t)$ in conjunction with (3.3).

The characterization of functions $\mathscr{R}^{\sim}$ satisfying (3.4) can be given conveniently in terms of the asymptotic theory for non-decreasing functions worked out by L. de Haan (see, e.g. de Haan (1970)) and others. We shall quote a few results from this theory, and refer the reader to Vervaat (1974) for a more detailed exposition. It turns out that the only possible limits l in (3.4) are, up to changes of scale and location on the target space of l, the functions

$$\text{(3.5)}\qquad \begin{aligned} &\text{(i)} \quad l(\xi) = -e^{-\alpha\xi}\,, \quad \alpha > 0\,;\\ &\text{(ii)} \quad l(\xi) = e^{\alpha\xi}\,, \quad \alpha > 0\,; \qquad \text{and}\\ &\text{(iii)} \quad l(\xi) = \xi\,. \end{aligned}$$

In case (i), one has $\mathscr{R}^{\sim}(\infty) = \sup\{t\colon F(t) < \infty\} \equiv t_F < \infty$ and

$$\text{(3.6)}\qquad (\mathscr{R}^{\sim}(t + \xi f(t)) - \mathscr{R}^{\sim}(\infty))/(\mathscr{R}^{\sim}(\infty) - \mathscr{R}^{\sim}(t)) \to -e^{-\alpha\xi}$$

for all real ξ. In what follows we shall assume in this case that $t_F = 0$; this can always be achieved by a change of location. In case (ii), $t_F = \infty$ and

$$\text{(3.7)}\qquad \mathscr{R}^{\sim}(t + \xi f(t))/\mathscr{R}^{\sim}(t) \to e^{\alpha\xi}$$

for all real ξ. Finally, in case (iii), one has

$$\text{(3.8)}\qquad (\mathscr{R}^{\sim}(t + \xi f(t)) - \mathscr{R}^{\sim}(t))/(\mathscr{R}^{\sim}(t + f(t)) - \mathscr{R}^{\sim}(t)) \to \xi$$

for all real ξ. We shall write $\mathscr{R}^{\sim} \in \Delta(f)$ to mean (3.8) holds; similarly $\mathscr{R}^{\sim} \in \Gamma^{\alpha}(f)$ means (3.7) holds, and $\mathscr{R}^{\sim} \in \Gamma^{-\alpha}(f)$ means (3.6) holds, along with our convention that $t_F = 0$. The classes $\Gamma^{-\alpha}(f)$ and $\Gamma^{\alpha}(f)$ are connected via the relation $\phi \in \Gamma^{-\alpha}(f)$ iff $\psi \in \Gamma^{\alpha}(f)$, when $\phi(t)\psi(t) = -1$ for all large t. The normalizing constants used in (3.6), (3.7) and (3.8) are unique up to an equivalence; indeed, given that (3.4) holds, one has $(\mathscr{R}^{\sim}(t + \xi f(t)) - b_t)/a_t \to l(\xi)$ for all ξ iff

$$\text{(3.9)}\qquad a_t/\alpha_t \to 1 \qquad \text{and} \qquad (b_t - \beta_t)/\alpha_t \to 0\,.$$

For purposes of comparison with our strong limit theorems, it will be instructive to recall the corresponding weak limit properties, at least for the sequence of partial maxima $(m_n^{(1)})$ (cf. (2.1)). In this connection the distribution function F of the X_n's is said to belong to the domain of attraction of a nondegenerate distribution function Q, written $F \in \mathscr{D}(Q)$, if there exist numbers $a_n > 0$ and b_n, $n \geqq 1$, such that

$$\text{(3.10)}\qquad \Pr_F\{(m_n^{(1)} - b_n)/a_n \leqq t\} = F(a_n t + b_n)^n \to Q(t)$$

at all continuity points t of Q. Only the following types of distribution functions have domains of attraction:

$$\text{(3.11)}\qquad \begin{aligned} \psi_\alpha\colon t &\to \exp(-|t|^{\alpha})\,, && \text{if} \quad t \leqq 0\\ &\to 1\,, && \text{if} \quad t < 0\,; \end{aligned}$$

$$(3.12)\qquad \phi_\alpha : t \to 0\,, \qquad \text{if } t < 0$$
$$\to \exp(-t^{-\alpha})\,, \qquad \text{if } t \geqq 0\,;$$

$$(3.13)\qquad \Lambda : t \to \exp(-e^{-t})\,, \qquad \text{for } -\infty < t < \infty\,.$$

In (3.11) and (3.12), α is an arbitrary positive number.

One has $F \in \mathscr{D}(\psi_\alpha)$ iff t_F is finite, say 0 for convenience, and $\mathscr{R}^\sim \in \Gamma^{-1/\alpha}(1)$ (1 denoting the function $f(t) = 1$ for all t), or equivalently, $\mathscr{F}^\sim \in \mathscr{R}\mathscr{V}_0(1/\alpha)$, or $\mathscr{F} \in \mathscr{R}\mathscr{V}_0(\alpha)$. In this case, (3.10) holds with

$$(3.14)\qquad b_n = 0 \quad\text{and}\quad a_n = -\mathscr{R}^\sim(\log n) = -\mathscr{F}^\sim(1/n)\,.$$

Similarly, one has $F \in \mathscr{D}(\phi_\alpha)$ iff $t_F = \infty$ and $\mathscr{R}^\sim \in \Gamma^{1/\alpha}(1)$ or equivalently, $\mathscr{F}^\sim \in \mathscr{R}\mathscr{V}_0(-1/\alpha)$, or $\mathscr{F} \in \mathscr{R}\mathscr{V}_\infty(-\alpha)$. In this case, (3.10) holds with

$$(3.15)\qquad b_n = 0 \quad\text{and}\quad a_n = \mathscr{R}^\sim(\log n) = \mathscr{F}^\sim(1/n)\,.$$

Finally, one has $F \in \mathscr{D}(\Lambda)$ iff $\mathscr{R}^\sim \in \Delta(1)$; in this case (3.10) holds with

$$(3.16)\qquad b_n = \mathscr{R}^\sim(\log n) = \mathscr{F}^\sim(1/n) \quad\text{and}$$
$$a_n = \mathscr{R}^\sim(\log(n) + 1) - \mathscr{R}^\sim(\log n)\,.$$

Various necessary and sufficient conditions for $F \in \mathscr{D}(\Lambda)$ have been given, e.g., by de Haan (1970) and Marcus and Pinsky (1969); these conditions are rather complicated. A simple sufficient condition, due to von Mises and slightly extended by Gnedenko, is that F be twice differentiable in a left neighborhood of t_F, and that

$$(3.17)\qquad \lim_{t \uparrow t_F} (1/i)'(t) = 0\,;$$

here (and in what follows) $'$ denotes differentiation and

$$i : t \to F'(t)/\mathscr{F}(t) = \mathscr{R}'(t)$$

is the so-called intensity function. Condition (3.17) implies (3.10) with

$$(3.18)\qquad b_n = \mathscr{F}^\sim(1/n) \quad\text{and}\quad a_n = (\mathscr{R}^\sim)'(\log n) = 1/i(b_n)\,.$$

In discussing our strong limit theorems for the processes $M_n^{(k)}$, we shall treat the cases of $\mathscr{R}^\sim \in \Gamma^{-\alpha}(f)$, $\Gamma^\alpha(f)$, and $\Delta(f)$ for $f = 1$ and for $f = \log$. One could, of course, give analogous results for $\sigma(M_n^{(k)})$, where σ is some nondecreasing function, e.g., by assuming $\sigma(\mathscr{R}^\sim) \in \Delta(f)$. We shall not do this, except for one case motivated by Theorem 1B, namely with $\sigma(t) = \log(t)$ (or $-\log|t|$) and $f = \log$.

In stating the following results, we have adopted the convention of writing, e.g.,

$$x_n \to C$$

to mean that sequence (x_n) is relatively compact and has the set C for its limit points.

I. $\mathscr{R}^\sim \in \Gamma^{-\alpha}(1)$ or $\mathscr{R}^\sim \in \Gamma^\alpha(1)$. Suppose first that $\mathscr{R}^\sim \in \Gamma^{-\alpha}(1)$, or equivalently, $t_F = 0$ and $\mathscr{F}^\sim \in \mathscr{R}\mathscr{V}_0(\alpha)$. Here we have

THEOREM $2A_1$. *If* $\mathscr{R}^{\sim} \in \Gamma^{-\alpha}(1)$, *then* wp 1

$$M_n^{(k)}/(-\mathscr{F}^{\sim}(\log_2 n/n)) \to -|K|^{\alpha} = \{x \in D_- : \textstyle\int |x(t)|^{1/\alpha}\, dt \leqq 1\} \tag{3.19}$$

in D.

Theorem 1A is actually a special case of Theorem $2A_1$. With appropriate changes, the corollaries of Theorem 1A go over to the present setting. For example, after replacing $\log_2 n/n$ in (2.26) by $-\mathscr{F}^{\sim}(\log_2 n/n)$, (2.28) becomes

$$\limsup_p f_{p,c} = 1 - \exp(-(|c|^{-1/\alpha} - 1)) \quad \text{wp } 1$$

for $c \in [-1, 0]$. If $\mathscr{F}^{\sim}$ varies "sufficiently regularly," then

$$-\mathscr{F}^{\sim}((1/n)\log_2 n) \sim a_n(\log_2 n)^{\alpha}$$

where a_n is given by (3.14). This is the situation, e.g., if for some positive C and finite β

$$F(t) \sim C|t|^{1/\alpha}(\log(1/|t|))^{\beta} \tag{3.20}$$

as $t \uparrow 0$, or equivalently,

$$\mathscr{F}^{\sim}(u) \sim -C^{-\alpha}u^{\alpha}(\alpha \log(1/u))^{-\alpha\beta}$$

as $u \downarrow 0$. In most situations of interest, one has (3.20) with $\beta = 0$; this is the case, e.g., when $-X_1$ is distributed as Beta $(1/\alpha, \delta)$ (with δ arbitrary) or Gamma $(1/\alpha, \delta)$ (with the scale parameter δ being arbitrary).

Similar results hold when $\mathscr{R}^{\sim} \in \Gamma^{\alpha}(1)$, or equivalently, $t_F = \infty$ and $\mathscr{F}^{\sim} \in \mathscr{R}\mathscr{V}_0(-\alpha)$, as is the case when X_1 has a Pareto distribution with parameter $1/\alpha$, or (for $\alpha = 1$) a Cauchy distribution. Recall that the space D^* was defined prior to (2.30).

THEOREM $2A_2$. *If* $\mathscr{R}^{\sim} \in \Gamma^{\alpha}(1)$, *then* wp 1

$$M_n^{(k)}/\mathscr{F}^{\sim}(\log_2 n/n) \to |K|^{-\alpha} = \{x \in D_+^* : \textstyle\int x(t)^{-1/\alpha}\, dt \leqq 1\} \tag{3.21}$$

in D^*.

In particular the almost sure limit points of $m_n^{(k)}/\mathscr{F}^{\sim}(\log_2 n/n)$ coincide with $[1, \infty]$, and for $c \in [1, \infty]$ one has

$$\limsup_p p^{-1} \textstyle\sum_{n \leq p} I_{(-\infty, c]}(m_n^{(k)}/\mathscr{F}^{\sim}(\log_n n/n)) = 1 - \exp(-(c^{1/\alpha} - 1))$$

wp 1.

II. $\mathscr{R}^{\sim} \in \Delta(1)$. Here we have

THEOREM $2A_3$. *If* $\mathscr{R}^{\sim} \in \Delta(1)$, *then* wp 1

$$\begin{aligned}(M_n^{(k)} - \mathscr{F}^{\sim}(\log_2 n/n))/(\mathscr{F}^{\sim}(e^{-1}\log_2 n/n) - \mathscr{F}^{\sim}(\log_2 n/n))\\ \to -\log|K| = \{x \in D^* : \textstyle\int e^{-x(t)}\, dt \leqq 1\}\end{aligned} \tag{3.22}$$

in D^*.

Under the Gnedenko–von Mises condition (3.17), this simplifies via (3.9) to

$$(3.23) \qquad (M_n^{(k)} - \mathscr{F}^{\sim}(\log_2 n/n))/(1/i(\mathscr{F}^{\sim}(\log_2 n/n))) \to -\log|K| .$$

Under the slightly stronger condition that

$$(3.24) \qquad \lim_{t\uparrow t_F} (\log_2 \mathscr{R}(t))^2(1/i)'(t) = c$$

for some $c \in (-\infty, \infty)$, (3.23) in turn gives rise to

$$(3.25) \qquad (M_n^{(k)} - b_n)/a_n + \log_3 n - c/2 \to -\log|K|$$

where a_n and b_n are given by (3.18).

Condition (3.24) is easily seen to be satisfied with $c = 0$ when F is normal, lognormal, Gamma, or Weibull. Examples of distribution functions for which (3.24) holds with $c \neq 0$ are obtained by taking

$$\mathscr{F}_c(t) = t^{-(\log_3 t)^2/c}$$

for $c > 0$, and $\mathscr{F}_c(t) = \mathscr{F}_{|c|}(-1/t)$ for $c < 0$; in the latter case, $t_F = 0$. Some discrete distributions satisfy the hypothesis of Theorem $2A_3$. For example, if X_1 takes on only nonnegative integral values and $\Pr\{X_1 = j\} = \gamma \exp(-cj^\beta)$ for positive γ, c, and β, then $\mathscr{R}^{\sim} \in \Delta(1)$ iff $\beta < 1$; notice that $\beta = 1$ corresponds to the geometric distribution.

Applying (3.25) when F is the standard normal distribution function, we find, after setting

$$\alpha_n = 1/(2\log(n))^{\frac{1}{2}} \qquad \text{and} \qquad \beta_n = (2\log(n) - \log_2 n - \log 4\pi)^{\frac{1}{2}}$$

and using (3.9), that the almost sure limit points of

$$(m_n^{(k)} - \beta_n)/\alpha_n + \log_3 n$$

coincide with $[0, \infty]$, and that the proportion of integers $n \leqq p$ for which

$$m_n^{(k)} \leqq (c - \log_3 n)/(2\log(n))^{\frac{1}{2}} + (2\log(n) - \log_2 n - \log 4\pi)^{\frac{1}{2}}$$

has an almost sure lim sup of $1 - \exp(-(e^c - 1))$, for $0 \leqq c \leqq \infty$.

III. $\mathscr{R}^{\sim} \in \Gamma^{-\alpha}(\log)$ or $\mathscr{R}^{\sim} \in \Gamma^{\alpha}(\log)$. Suppose first that $\mathscr{R}^{\sim} \in \Gamma^{-\alpha}(\log)$; recall our convention that $t_F = 0$ in this case. We have

THEOREM $2B_1$. *If* $\mathscr{R}^{\sim} \in \Gamma^{-\alpha}(\log)$, *then* wp 1

$$(3.26) \qquad M_n^{(k)}/(-\mathscr{F}^{\sim}(1/n)) \to -\exp(-\alpha L_k)$$

in D.

It is known (cf. de Haan (1972)) that a necessary and sufficient condition for $\mathscr{R}^{\sim} \in \Gamma^{-\alpha}(\log)$ is that (for $t < 0$)

$$(3.27) \qquad \mathscr{F}(t) = \zeta(t)^{-\log_2(\zeta(t))/\alpha}$$

where $\zeta \in \mathscr{R}\mathscr{V}_0(1)$. Sufficient conditions are

$$(3.28) \qquad \lim_{t\uparrow 0} (\log \mathscr{R}(t))/(ti(t)) = -\alpha$$

and the slightly stronger

$$\text{(3.29)} \qquad \lim_{t \uparrow 0} (\log \mathscr{R}(t))(1/i)'(t) = -\alpha$$

(compare with (3.24)).

Similar results hold when $\mathscr{R}^{\sim} \in \Gamma^{\alpha}(\log)$; everything above carries over with the minus signs deleted, and all appearances of 0 replaced by ∞. In particular, the analogue of Theorem $2B_1$ is

THEOREM $2B_2$. *If* $\mathscr{R}^{\sim} \in \Gamma^{\alpha}(\log)$, *then* wp 1

$$\text{(3.30)} \qquad M_n^{(k)}/\mathscr{F}^{\sim}(1/n) \to \exp(\alpha L_k)$$

in D.

IV. $\mathscr{R}^{\sim} \in \Delta(\log)$. It is known (cf. de Haan (1972)) that $\mathscr{R}^{\sim} \in \Delta(\log)$ iff $\mathscr{R}^{\sim}(t) = V(t/\log t)$ for some $V \in \Delta(1)$, or equivalently, iff the mapping $u \to \mathscr{R}^{\sim}(u \log (u))$ is in $\Delta(1)$.

THEOREM $2B_3$. *If* $\mathscr{R}^{\sim} \in \Delta(\log)$, *then* wp 1

$$\text{(3.31)} \qquad (M_n^{(k)} - \mathscr{F}^{\sim}(1/n))/(\mathscr{F}^{\sim}(1/(n \log n)) - \mathscr{F}^{\sim}(1/n)) \to L_k$$

in D.

Under a condition a little stronger than the Gnedenko-von Mises condition (3.17), namely

$$\text{(3.32)} \qquad \lim_{t \uparrow t_F} (\log \mathscr{R}(t))(1/i)'(t) = 0$$

(compare also (3.24) and (3.29)), we get

$$\text{(3.33)} \qquad (M_n^{(k)} - b_n)/(a_n \log_2 n) \to L_k$$

where a_n and b_n are given by (3.18). Condition (3.32) is satisfied by the normal, lognormal, Gamma, and Weibull distributions.

Theorem $2B_3$ and its corollary (3.33) extend to the functional setting some results due to Pickands (1967), de Haan and Hordijk (1972), and Resnick and Tomkins (1973). When $\mathscr{R}^{\sim} \in \Delta(1) \cap \Delta(\log)$, (3.22) and (3.31) specify the "bottom" and "top" behavior of the $M_n^{(k)}$'s in the spirit of Theorems 1A and 1B.

An interesting class of discrete distributions to which Theorem $2B_3$ is applicable are the negative binomial distributions, under which one has

$$\text{(3.34)} \qquad \Pr\{X_1 = j\} = (-1)^j \binom{-a}{j} p^a (1-p)^j\,, \qquad j = 0, 1, 2, \cdots$$

for some parameters $a > 0$ and $0 < p < 1$. Utilizing (3.9), we find that when (3.34) holds, then wp 1

$$\text{(3.35)} \qquad (M_n^{(k)} - \sigma \log n)/(\sigma \log_2 n) \to (a-1) + L_k\,,$$

with σ defined to be $1/\log(1/(1-p))$. On the other hand, the Poisson distribution with mean μ lies just outside the domain of applicability of Theorem $2B_3$, because in this case $\mathscr{R}(j) - \mathscr{R}(j-1) = \log(j) - \log(\mu) + O(1/j)$ as $j \to \infty$, so that $\mathscr{R}^{\sim} \notin \Delta(\log)$.

V. A logarithmic case.

THEOREM 3. *If either*

(3.36a) $$t_F = 0 \quad \textit{and} \quad -\log|\mathscr{R}^{\sim}| \in \Delta(\log)\,, \qquad \textit{or}$$

(3.36b) $$t_F = \infty \quad \textit{and} \quad \log \mathscr{R}^{\sim} \in \Delta(\log)\,,$$

then wp 1

(3.37) $$\log(M_n^{(k)}/\mathscr{F}^{\sim}(1/n))/\log(\mathscr{F}^{\sim}(1/(n\log(n))/\mathscr{F}^{\sim}(1/n)) \to L_k$$

in D.

Actually under (3.36b), the left-hand side of (3.37) may be undefined at time points t in a neighborhood of 0; however, since $m_n^{(k)} \to \infty$ wp 1, this neighborhood shrinks to 0, and we may and do ignore it.

Condition (3.36a) is satisfied if $\mathscr{F}^{\sim} \in \mathscr{RV}_0(\alpha)$ (equivalently, if $\mathscr{F} \in \mathscr{RV}_0(1/\alpha)$), and in this case (3.37) implies

(3.38) $$-\log(|M_n^{(k)}|/|\mathscr{F}^{\sim}(1/n)|)/(\alpha \log_2 n) \to L_k$$

wp 1. Similarly (3.36b) holds if $\mathscr{F}^{\sim} \in \mathscr{RV}_0(-\alpha)$, and in this case

(3.39) $$\log((M_n^{(k)})^+/\mathscr{F}^{\sim}(1/n))/(\alpha \log_2 n) \to L_k$$

wp 1. Combining (3.19) and (3.38) we get bottom and top analogues of Theorem 1A and 1B for the case of $\mathscr{F}^{\sim} \in \mathscr{RV}_0(\alpha)$; similarly for (3.21) and (3.39) in the case that $\mathscr{F}^{\sim} \in \mathscr{RV}_0(-\alpha)$.

The validity of (3.37) is by no means restricted to the regularly varying case. For example, (3.36b) holds for

(3.40) $$\mathscr{F}(t) = e^{-(\log t)^{\gamma}/\delta}$$

for any positive γ and δ; the regularly varying case has $\gamma = 1$.

4. Proofs for Section 3. Throughout this section, we shall write $\mathscr{S}$ for $\mathscr{R}^{\sim}$.

(a) The method of proof of Theorems 2 and 3. The main theorems in Section 3 are proved by combining Theorems 1A and 1B, the mapping Lemma 2.1, the representations (3.2) and (3.3), and the following

LEMMA 4.1. *Suppose* (3.4) *holds, i.e., that as* $t \to \infty$

(4.1) $$(\mathscr{S}(t + \xi f(t)) - \beta_t)/\alpha_t \to l(\xi)$$

for all continuity points ξ *of* l. *Then* $x_t \to x$ *in D implies that*

(4.2) $$(\mathscr{S}(t + x_t f(t)) - \beta_t)/\alpha_t \to l(x)$$

in D, with the understanding that the right-hand side of (4.2) *is defined by letting* l *operate on each coordinate of* x.

PROOF. We have seen (cf. (3.5)) that l must be continuous. As the functions $\mathscr{S}$ and l are non-decreasing, the convergence in (4.1) must be uniform for ξ in compact sets. This implies

$$(\mathscr{S}(t + \xi_t f(t)) - \beta_t)/\alpha_t \to l(\xi)$$

whenever $\xi_t \to \xi$ in $(-\infty, \infty)$, and as convergence in D is simply pointwise convergence at continuity points, (4.2) follows. □

(b) Proof of corollaries (3.23) and (3.25) to Theorem $2A_3$. Suppose first that the Gnedenko–von Mises condition (3.17) holds. Using

$$\mathscr{S}'(t) = 1/\mathscr{R}'(\mathscr{S}(t)) = 1/i(\mathscr{S}(t)), \tag{4.3}$$

it is easily seen that (3.17) implies

$$\lim_{t\to\infty} (\log \mathscr{S}')'(t) = 0 .$$

The mean value theorem (MVT) then implies that as $t \to \infty$

$$\mathscr{S}'(t + \theta) \sim \mathscr{S}'(t)$$

uniformly for bounded θ values, and another application of the MVT gives

$$\mathscr{S}(t + \xi) - \mathscr{S}(t) \sim \xi \mathscr{S}'(t) \tag{4.4}$$

for all real ξ. Thus $\mathscr{S} \in \Delta(1)$, and, in view of (4.4) and (3.9), (3.22) implies

$$(M_n^{(k)} - \mathscr{S}(\log(n) - \log_3 n))/(\mathscr{S}'(\log(n) - \log_3 n)) \to -\log |K| . \tag{4.5}$$

Together with (4.3), this gives (3.23).

Next suppose (3.24) holds, or equivalently,

$$\lim_{t\to\infty} (\log_2 t)^2 (\log \mathscr{S}')'(t) = c . \tag{4.6}$$

The MVT gives

$$\mathscr{S}'(t - \theta \log_2 t) \sim \mathscr{S}'(t) \tag{4.7}$$

uniformly for bounded θ values, and this, a second-order Taylor expansion, and (4.6) imply

$$\begin{aligned} &\frac{\mathscr{S}(t - \log_2 t) - \mathscr{S}(t)}{\mathscr{S}'(t)} + \log_2 t - \frac{c}{2} \\ &\qquad = \frac{(\log_2 t)^2 \mathscr{S}''(t - \theta_t \log_2 t)}{2\mathscr{S}'(t)} - \frac{c}{2} \to 0 . \end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8) with (4.5), and using (3.9), we arrive at

$$(M_n^{(k)} - \mathscr{S}(\log n))/\mathscr{S}'(\log n) + \log_3 n - c/2 \to -\log |K| ,$$

which is the same as (3.25).

(c) A remark concerning conditions (3.28) and (3.29). That (3.28) implies $\mathscr{S} \in \Gamma^{-\alpha}(\log)$ is a consequence of the MVT. That (3.29) implies (3.28) is essentially proved in de Haan and Hordijk (1972), page 1193.

(d) A remark concerning (3.33). Condition (3.32) is equivalent to

$$\lim_{t\to\infty} (\log t)(\log \mathscr{S}')'(t) = 0$$

and thus (3.33) follows from (3.31) by virtue of (3.9) and the MVT.

(e) Proof of (3.35). It is enough to show that under (3.24)

$$\mathscr{S}(u) = \sigma(u + (a - 1 + o(1)) \log (u))$$

as $u \to \infty$, or that

$$\mathscr{R}(j) = j/\sigma - (a - 1 + o(1)) \log (j)$$

as $j \to \infty$ through the integers, or that

$$\mathscr{R}(j) - \mathscr{R}(j - 1) = 1/\sigma - (a - 1 + o(1))/j \tag{4.9}$$

as $j \to \infty$. But

$$\mathscr{R}(j) - \mathscr{R}(j - 1) = -\log (1 - r_j)$$

where

$$r_j = f_j/(\sum_{l \geqq j} f_l)$$

with

$$f_l = \Pr\{X_1 = l\} = a(a + 1) \cdots (a + (l - 1))p^a(1 - p)^l/l!$$

for $l = 0, 1, 2, \cdots$. Thus

$$1/r_j = 1 + \sum_{m=1}^{\infty} s_{m;j} ,$$

where

$$s_{m;j} = (\prod_{l=1}^{m} (1 + (a - 1)/(j + l)))q^m$$

with $q = 1 - p$.

To carry the argument further, we suppose that $a > 1$ (the case of $a < 1$ is treated in a similar manner, and will be omitted). Then

$$s_{m;j} \leqq (q(1 + (a - 1)/j))^m .$$

Moreover, if δ is any positive number slightly less than 1, some simple estimates (such as $(1 + x) \geqq e^{\delta x}$ for positive x near 0) show that

$$s_{m;j} \geqq (q(1 + \delta^2(a - 1)/j))^m$$

provided j is sufficiently large and $m \leqq j^{\frac{1}{2}}$. After summing on m, letting $j \to \infty$, and letting δ approach 1, we arrive at (4.9).

(f) A remark on (3.39). From Karamata's representation theorem (Feller (1971) page 282), it follows that $\mathscr{F}^{\sim}$ being of regular variation of exponent $-\alpha$ at 0 implies

$$\mathscr{F}^{\sim}((1/n)(1/\log n))/\mathscr{F}^{\sim}(1/n) = (1/\log n)^{-\alpha+o(1)}$$

as $n \to \infty$. Hence (3.39) follows from (3.37) and (3.9).

5. How to identify limit points. The following lemma is a useful aid for identifying the limit points of a random sequence.

LEMMA 5.1. *Let $(\Omega, \mathscr{A}, P)$ be a probability space. Let S be a separable metric space and let H_n, $n \geqq 1$, be mappings from Ω to S.*

(a) *Let C be a subset of S. Suppose there exists a sequence (n_j) tending to ∞ such that*

(5.1) *{the sequence $(H_n)_{n \geqq 1}$ has a limit point in C}; $\subset$ {the sequence $(H_{n_j})_{j \geqq 1}$ has a limit point in C}* wp 1 *and such that each x in C has a neighborhood N_x such that*

(5.2) $$P\{H_{n_j} \in N_x \text{ for infinitely many } j\} = 0 .$$

Then

(5.3) $$P\{\text{the sequence } (H_n) \text{ has no limit points in } C\} = 1 .$$

(b) *Let B be a subset of S, B_0 a dense subset of B. Suppose that for each x in B_0 there is a neighborhood base $(N_j)_{j \geq 1}$ at x such that*

(5.4) $$P\{H_n \in N_j \text{ for infinitely many } n\} = 1$$

for each j. Then

(5.5) $$P\{\text{every point of } B \text{ is a limit point of } (H_n)\} = 1 .$$

PROOF. (a) As C is itself separable, there exist countably many points x_i, $i \geqq 1$, in C with neighborhoods N_{x_i} satisfying (5.2), such that

$$C \subset \bigcup_{1 \leqq i < \infty} \text{ interior } (N_{x_i}) .$$

But then

$$\{(H_{n_j}) \text{ has a limit point in } C\} \subset \bigcup_{1 \leqq i < \infty} \{H_{n_j} \in N_{x_i} \text{ for infinitely many } j\} ,$$

so (5.3) holds by (5.1).

(b) Let $\{x: i \geqq 1\}$ be countable dense subset of B_0. Letting d be a distance function for S, one has

$$\bigcap_{x \in B} \{\liminf_n d(H_n, x) = 0\} = \bigcap_{1 \leqq i < \infty} \{\liminf_n d(H_n, x_i) = 0\} .$$

The set on the right has probability one by (5.4). □

Typically, one would use some form of the first and second Borel–Cantelli lemmas to verify (5.2) and (5.4) respectively. When S is the space D (or D_-) of Section 2, one can use the lemma below to check condition (5.1).

LEMMA 5.2. *Let (H_n) be a sequence of random functions in D. Suppose there exist constants $a_{m,n}$, $b_{m,n}$, and $c_{m,n}$ such that*

(5.6a) $$H_n(t) = a_{m,n} H_m(b_{m,n} t) + c_{m,n}$$

for all m, n, and t, and such that

(5.6b) $$a_{m,n} \to 1 , \qquad b_{m,n} \to 1 , \qquad \text{and} \qquad c_{m,n} \to 0$$

whenever m and n tend to ∞ in such a way that $n/m \to 1$. Then (5.1) holds for any sequence (n_j) satisfying

(5.7) $$n_j \to \infty \qquad \text{and} \qquad n_j/n_{j-1} \to 1 .$$

Roughly speaking condition (5.6) says that the H_n's change slowly with n. We have stated the condition in form convenient for our purposes; it should be clear through that it can be modified in many ways.

6. Proof of Theorem 1A. In this section we will simply write H_n for ${}_*H_n^{(k)}$ (defined by (2.17)). H_n takes all its values in D_-, so we may and will replace D by

D_- throughout. For convenience, the proof will be divided into several lemmas.

The first of these concerns the topology of D_-. Suppose $x \in D_-$. Let $\mathscr{N}(x)$ denote the class of subsets of D_- of the form

$$(6.1) \qquad \begin{aligned} N(\mathbf{t}; \mathbf{G}) &\equiv N(t_1, \cdots, t_q; G_1, \cdots, G_q) \\ &= \{y \in D_- : y(t_p) \in G_p, 1 \leqq p \leqq q\}, \end{aligned}$$

where

(a) $t_1 < \cdots < t_q$ is a finite system of continuity points of x,

(b) the G_p's are open subintervals of $(-\infty, 0]$ which increase with p, in the sense that for each p, either $G_p = G_{p+1}$ or $\sup(G_p) \leqq \inf(G_{p+1})$, and

(c) $x(t_p) \in G_p$, $1 \leqq p \leqq q$.

LEMMA 6.1. *There exist sets* $N_m = N(\mathbf{t}^{(m)}, \mathbf{G}^{(m)})$ *in* $\mathscr{N}(x)$ *for* $m \geqq 1$, *such that*

$$(6.2) \qquad \{N_m : m \geqq 1\} \quad \textit{is a neighborhood base at} \quad x$$

$$(6.3) \qquad \kappa_m \equiv \textstyle\sum_p (t_p^{(m)} - t_{p-1}^{(m)}) \sup(G_p^{(m)}) \to \kappa(x) \qquad \textit{as} \quad m \to \infty .$$

We recall that $\kappa(x)$ is defined by (2.11); by convention $t_0^{(m)} = 0$.

PROOF. Let $s_1, s_2, s_3, \cdots$ be an enumeration of a countable dense subset of the points of continuity of x in $(0, \infty)$. For each m, let $t^{(m)} = (t_1^{(m)}, \cdots, t_m^{(m)})$ be the points $s_1, \cdots, s_m$ arranged in increasing order. Choose intervals $G_p^{(m)}$ $(1 \leqq p \leqq m)$ such that $N_m \equiv N(\mathbf{t}^{(m)}, \mathbf{G}^{(m)}) \in \mathscr{N}(x)$, and such that each $G_p^{(m)}$ has length less than $1/m$. Elementary convergence considerations show that (6.2) holds.

To show (6.3), let $x_m \in D_-$ be the function which takes the value $\sup(G_p^{(m)})$ over the interval $[t_{p-1}^{(m)}, t_p^{(m)})$ $(= (0, t_1^{(m)})$, if $p = 1)$ and the value 0 over $[t_m^{(m)}, \infty)$. Then $x_m \geqq x$, and so

$$(6.4) \qquad \kappa_m = \kappa(x_m) \geqq \kappa(x)$$

for each m. On the other hand, $x_m \to x$; indeed, for any j and any $\varepsilon > 0$, we can find i and l such that $s_i < s_j < s_l$ and $x(s_l) - x(s_i) < \varepsilon$, which implies $\limsup_m |x_m(s_j) - x(s_j)| < \varepsilon$. The uppersemicontinuity of κ then implies $\kappa(x) \geqq \limsup_m \kappa(x_m)$; together with (6.4), this gives (6.3). □

LEMMA 6.2. *Let* $N(\mathbf{t}, \mathbf{G})$ *be a set of the form* (6.1). *Put* $g_p = \sup(G_p)$. *Then*

$$(6.5) \qquad \Pr\{H_n \in N(\mathbf{t}, \mathbf{G})\} = 1/(\log n)^{(1+o(1))\sum_{1 \leqq p \leqq q}(t_p - t_{p-1})|g_p|}$$

as $n \to \infty$.

PROOF. For any finite collections $(A_j)_{j \in J}$ and $(B_j)_{j \in J}$ of events such that $A_j \subset B_j$ for each $j \in J$, one has

$$\Pr\left(\textstyle\bigcap_{j \in J}(B_j - A_j)\right) = \textstyle\sum_I (-1)^{\operatorname{card}(J-I)} P\left(\left(\bigcap_{i \in I} B_i\right) \cap \left(\bigcap_{j \in J-I} A_j\right)\right),$$

where the sum is taken over all subsets I of J. In view of the form of the set $N(\mathbf{t}, \mathbf{G})$, it therefore suffices to show that

$$(6.6) \qquad \Pr\{H_n(t_p) \leqq g_p, 1 \leqq p \leqq q\} = \exp\left(-(1 + o(1)) \textstyle\sum_p (t_p - t_{p-1})|g_p| \log_2 n\right).$$

Moreover, because the sample paths of H_n lie in D_-, we need only consider the case in which

$$g_1 < g_2 < \cdots < g_q < 0 . \tag{6.7}$$

For each Borel set B of $(0, \infty) \times (-\infty, 0]$, set

$$\mu_n(B) = \sum_{1 \leq i < \infty} I_B((i/n, X_i/(\log_2 n/n))) ,$$

where I_B is the indicator function of B, and $X_1, X_2, \cdots$ are the i.i.d. uniform random variables of Section 2. When B is a "block," that is, the product of two finite intervals, an easy calculation based on the binomial probability formula shows that for each integer $a \geqq 0$, one has

$$\Pr\{\mu_n(B) = a\} \sim (|B|^a(\log_2 n)^a/a!)e^{-|B|\log_2 n} \tag{6.8}$$

as $n \to \infty$; here $|B|$ denotes the area of B. Moreover, for any finite collection $(B_j)_{j \in J}$ of disjoint blocks, the multinomial probability formula may be used to show

$$\Pr\{\mu_n(B_j) = a_j, j \in J\} \sim \prod_{j \in J} \Pr\{\mu_n(B_j) = a_j\} \tag{6.9}$$

for every choice of a_j, $j \in J$.

Because of (6.7), the event $\{H_n(t_p) \leqq g_p, 1 \leqq p \leqq q\}$ can be rewritten as

$$\{\mu_n(0, t_p] \times (g_p, 0]) \leqq k - 1; 1 \leqq p \leqq q\}$$

and this in turn can be expressed as a finite disjoint union of events of the form

$$\{\mu_n(B_j) = a_j, j \in J\}$$

where for $(B_j)_{j \in J}$ we take the collection $(t_{p-1}, t_p] \times (g_\rho, g_{\rho+1}]$; $p \leqq \rho \leqq q$, $1 \leqq p \leqq q$ (with $g_{\rho+1} = 0$). These blocks have a combined area of

$$\sum_p (t_p - t_{p-1})|g_p|$$

and so (6.6) follows from (6.8) and (6.9). □

LEMMA 6.3. Wp 1, (H_n) *has no limit points in* $C \equiv \{x \in D_- : \kappa(x) < -1\}$.

PROOF. We shall make use of part (a) of Lemma 5.1, with $n_j \equiv [e^{j/\log j}]$. Since (ignoring the greatest integer function)

$$H_n(t) = ((\log_2 m/m)/(\log_2 n/n))H_m((n/m)t) . \tag{6.10}$$

Lemma 5.2 implies that (5.1) holds. Next, let $x \in C$. By lemma 6.1, x has a neighborhood $N(\mathbf{t}, \mathbf{G})$ of the form (6.1) such that $\sum_p (t_p - t_{p-1})|\sup(G_p)| > 1$. Setting $N_x = N(\mathbf{t}, \mathbf{G})$, we find that (5.2) holds, because Lemma 6.2 implies

$$\sum_j \Pr\{H_{n_j} \in N_x\} < \infty . \tag{6.11}$$

□

LEMMA 6.4. Wp 1, (H_n) *is relatively compact in* D_-.

PROOF. In view of (2.11), it suffices to show that for each $t \in (0, \infty)$ one has $H_n(t) \leqq -2/t$ i.o. with probability zero. This can be proved by an argument similar to that used in Lemma 6.3; note that Lemma 6.2 implies

$$\Pr\{H_n(t) \leqq g\} = 1/(\log n)^{(1+o(1))t|g|} . \qquad \square$$

LEMMA 6.5. *Let* $N(\mathbf{t}, \mathbf{G})$ *be a subset of* D_- *of the form* (6.1) *with* $G_p = (f_p, g_p)$, *and suppose that*

$$(6.12) \qquad \gamma \equiv \sum_{1\leqq p\leqq q} (t_p - t_{p-1})|g_p| < 1 .$$

This if $0 < \varepsilon$ *is sufficiently small and* $n_j = [\exp(j^{1+\varepsilon})]$, *one has*

$$(6.13) \qquad \Pr\{H_{n_j} \in N(\mathbf{t}, \mathbf{G}) \ \text{i.o.}\} = 1 .$$

PROOF. We shall show that (6.13) holds whenever $\varepsilon < \gamma^{-1} - 1$. Choose and fix such an ε. For each j, one has

$$(6.14) \qquad \{H_{n_j} \in N(\mathbf{t}, \mathbf{G})\} \supset A_j \cap B_j ,$$

where

$$A_j = \{\max(X_1, \cdots, X_{[n_{j-1}t_q]}) < f_1 \log_2 n_j/n_j\}$$

and

$$B_j = \bigcap_{1\leqq p\leqq q} \{k\text{th largest of } X_{[n_{j-1}t_q]+1}, \cdots, X_{[n_jt_p]} \in (\log_2 n_j/n_j)G_p\} .$$

Since $n_{j-1}/n_j \to 0$, the B_j's are (for large j) mutually independent events with

$$P(B_j) = 1/(\log n_j)^{(1+o(1))\gamma} = 1/j^{(1+o(1))(1+\varepsilon)\gamma}$$

as $j \to \infty$ (cf. the argument of Lemma 6.2); the second Borel–Cantelli lemma gives

$$(6.15) \qquad \Pr(\limsup_j B_j) = 1 .$$

On the other hand,

$$\Pr(A_j^c) = 1 - (1 + (f_1 n_{j-1} n_j^{-1} \log_2 n_j)/n_{j-1})^{[n_{j-1}t_q]}$$
$$\sim t_q|f_1| n_{j-1} n_j^{-1} \log_2 n_j \leqq (1 + \varepsilon)t_q|f_1| \log(j)/\exp((1 + \varepsilon)(j - 1)^\varepsilon)$$

so the first Borel–Cantelli lemma implies $\Pr(\limsup_j A_j^c) = 0$, or what is the same,

$$(6.16) \qquad \Pr(\liminf_j A_j) = 1 .$$

Together (6.14), (6.15), and (6.16) give (6.13). $\square$

LEMMA 6.6. Wp 1, *every point of* K *is a limit point of* (H_n).

PROOF. The set of x in D_- such that $x(t) < 0$ for all t and such that $\int x(t)\,dt > -1$ is dense in K. By Lemma 6.1, each such x has a neighborhood base consisting of sets of the form $N(\mathbf{t}, \mathbf{G})$ satisfying the condition (6.12) of Lemma 6.5. In view of this and Lemma 6.4, the assertion follows from part (b) of Lemma 5.1. $\square$

Together, Lemmas 6.3, 6.4, and 6.6 yield Theorem 1A.

7. Proof of Theorem 1B. Theorem 1B is in effect a statement about the kth largest order statistics arising from a sequence of i.i.d. exponential random variables, and so in this section we will change the notation accordingly. Thus we suppose that X_n, $n \geqq 1$, are independent random variables, each exponentially distributed with mean 1, and we put

$$H_n(t) = (M_n^{(k)}(t) - \log n)/\log_2 n$$

(cf. (2.8)). Our task is to show that the sequence (H_n) is relatively compact in D and has L_k for its limit points. As in the previous section, we will divide the proof into several lemmas.

The sets $N(\mathbf{t}, \mathbf{G})$ defined by (6.1) continue to play a fundamental role. However, since we are now working in D rather than D_-, the definitions made at the beginning of Section 6 need to be modified by replacing D_- by D and $(-\infty, 0]$ by $(-\infty, \infty)$ throughout. We assume henceforth that the reader has made these changes.

LEMMA 7.1. *Let $N(t_1, \cdots, t_q; G_1, \cdots, G_q)$ be a neighborhood in D of the form* (6.1), *with the $G_p = (g_p, h_p)$'s satisfying*

$$0 < g_1 \qquad \text{and} \qquad h_q < \infty .$$

Let $\mathscr{P} = \{1\} \cup \{p: 2 \leqq p \leqq q$ and $G_{p-1} \neq G_p\}$, and let ρ be the largest element of $\mathscr{P}$. Then as $n \to \infty$,

$$\Pr\{H_n \in N(\mathbf{t}, \mathbf{G})\} = 1/(\log n)^{(1+o(1))(\sum_{p\in\mathscr{P}; p<q} g_p + kg_\rho)} . \tag{7.1}$$

PROOF. For each Borel set B of $(0, \infty) \times (0, \infty)$ set

$$\mu_n(B) = \sum_{1\leqq i<\infty} I_B((i/n, (X_i - \log n)/\log_2 n)) .$$

Suppose B is a set of the form

$$(s, t] \times G \tag{7.2}$$

where G is an interval with $g = \inf(G) > 0$ and $h = \sup(G) \leqq \infty$. Making use of the binomial probability formula, one gets

$$\Pr\{\mu_n(B) = a\} = (1 + o(1))((t - s)/(\log n)^g)^a/a!$$

$$\Pr\{\mu_n(B) > a\} \leqq \sum_{b>a} (n(t - s) + 1)^b(1/(n(\log n)^g))^b/b! = o(\Pr\{\mu_n(B) = a\})$$

for each integer $a \geqq 0$. Thus for any set A of nonnegative integers,

$$\Pr\{\mu_n(B) \in A\} \sim ((t - s)/(\log n)^g)^a/a! \tag{7.3}$$

where a is the smallest element of A. An analogous argument using the multinomial probability formula shows that

$$\Pr\{\mu_n(B_j) \in A_j, j \in J\} \sim \prod_{j\in J} \Pr\{\mu_n(B_j) \in A_j\} \tag{7.4}$$

for any finite collection $(B_j)_{j\in J}$ of disjoint sets of the form (7.2), and any sets A_j of nonnegative integers.

We shall now give the proof of (7.1) in the case that $\rho \geqq 2$. On the one hand, the event $\{H_n \in N(\mathbf{t}, \mathbf{G})\}$ contains the event $U_n \cap V_n \cap W_n$, where, with $T_p = (t_{p-1}, t_p]$,

$$U_n = \{\mu_n(T_1 \times G_1) = 1, \mu_n(T_1 \times [h_1, g_\rho]) = 0, \mu_n(T_1 \times G_\rho) = k - 1,$$
$$\mu_n(T_1 \times [h_\rho, \infty)) = 0\}$$

$$V_n = \bigcap_{p\geqq 2, p\in\mathscr{P}} \{\mu_n(T_p \times G_p) = 1 \text{ and } \mu_n(T_p \times [h_p, \infty)) = 0\}$$

$$W_n = \bigcap_{p\geqq 2, p\notin\mathscr{P}} \{\mu_n(T_p \times (g_p, \infty)) = 0\} .$$

On the other hand, $\{H_n \in N(t, G)\}$ is contained in the event

$$(\bigcap_{1 \leqq p < \rho, p \in \mathscr{P}} \{\mu_n((0, t_p] \times G_p) \geqq 1\}) \cap \{\mu_n((0, t_\rho] \times (g_\rho, \infty)) \geqq k\} .$$

Combining these observations with (7.3) and (7.4), we get (7.1). When $\rho = 1$, the argument goes through with U_n replaced by

$$\{\mu_n(T_1 \times G_1) = k, \mu_n(T_1 \times [h_1, \infty)) = 0\} . \qquad \square$$

LEMMA 7.2. *Wp 1 (H_n) has no limit points in $C \equiv \{x \in D_+ : \lambda_k(x) > 1\}$.*

PROOF. The proof is similar to that of Lemma 6.3. Equation (6.10) has a counterpart in

$$H_n(t) = (\log_2 m/\log_2 n)H_m((n/m)t) + (\log (m/n))/\log_2 n , \tag{7.5}$$

and we can get a counterpart to (6.11) in the following manner. Suppose $x \in C$. Referring to the definition of λ_k (cf. (2.14)), choose continuity points t_p, $1 \leqq p \leqq q$, of x such that

$$t_1 < t_2 < \cdots < t_q ,$$
$$0 < x(t_1) < x(t_2) < \cdots < x(t_q) ,$$
$$x(t_1) + \cdots + x(t_{q-1}) + kx(t_q) > 1 .$$

Choose intervals $G_p = (g_p, h_p)$, $1 \leqq p \leqq q$, such that

$$0 < g_1 < h_1 < \cdots < g_q < h_q \qquad \text{and} \qquad \sum_{p<q} g_p + kg_q > 1 .$$

Then with $n_j = [e^{j/\log j}]$, one has

$$\sum_j \Pr\{H_{n_j} \in N(t_1, \cdots, t_q; G_1, \cdots, G_q)\} < \infty$$

by virtue of Lemma 7.1. $\square$

LEMMA 7.3. *Wp 1, (H_n) is relatively compact in D, and all its limit points are contained in D_+.*

PROOF. In view of (2.11), we have to show that wp 1

$$0 \leqq \liminf_n H_n(t) \tag{7.6}$$

$$\limsup_n H_n(t) < \infty \tag{7.7}$$

for each t in $(0, \infty)$. An argument similar to (but easier than) that of Lemma 7.2 gives (7.7). In view of (7.5), with m set equal to $[nt]$, it suffices to show (7.6) for $t = 1$, and thus to show that wp 1

$$0 = \liminf_n H_n(1) . \tag{7.8}$$

But (7.8) is equivalent to (2.7), which we know holds because (2.2) (which implies it) is a consequence of Corollary 2.1 to Theorem 1A. $\square$

LEMMA 7.4. *Let*

$$N(\mathbf{t}, \mathbf{G}) = N(t_1, t_2, t_3, t_4, \cdots, t_{2q-1}, t_{2q}; G_1, G_1, G_2, G_2, \cdots, G_q, G_q) \tag{7.9}$$

be a neighborhood in D of the form (6.1), *with the $G_\rho = (g_\rho, h_\rho)$'s satisfying*

$$0 < g_1 < h_1 < \cdots < g_q < h_q < \infty \qquad \textit{and} \qquad \sum_{\rho<q} g_\rho + kg_q < 1 . \tag{7.10}$$

Then if $\varepsilon > 0$ *is sufficiently small and* $n_j = [\exp(j^{1+\varepsilon})]$, $j \geqq 1$, *one has*

$$\Pr\{H_{n_j} \in N(\mathbf{t}, \mathbf{G}) \text{ i.o.}\} = 1 .$$

PROOF. The proof is entirely analogous to that of Lemma 6.5. □

LEMMA 7.5. Wp 1, *each point of* L_k *is a limit point of* (H_n).

PROOF. The proof is similar to that of Lemma 6.6. The set of x in D such that $x(0+) > 0$ and $\lambda_k(x) < 1$ is dense in L_k. Each such x is a step function having only finitely many jumps, and so has a neighborhood base of sets of the form (7.9) satisfying condition (7.10). □

8. Proofs of the corollaries to Theorem 1A. Throughout this section, we will write H_n for ${}_*H_n^{(k)}$.

(a) Proof of Corollary 2.1. For convenience we will divide the proof into several lemmas. Throughout this subsection we will write $w(t)$ for t, when t is near 0, and for $t/\log_2 t$, when t is near ∞.

LEMMA 8.1. *The* M_2*-convergence of* ν_n *to* ν *and* (2.22) *imply*

$$(8.1)\qquad \lim_{\delta\downarrow 0} \textstyle\int_0^\delta 1/w(t)|\nu|(dt) = 0 \quad\text{and}\quad \lim_{\delta\uparrow\infty} \int_\delta^\infty 1/w(t)|\nu|(dt) = 0 ,$$

where $|\nu|$ *is the total variation of* ν.

PROOF. If $a = t_0 < t_1 < \cdots < t_q = b$ are continuity points of the function v defined by (2.20), then (2.21) implies

$$\textstyle\sum_{i\leqq j} |v(t_i) - v(t_{i-1})| = \lim_n \sum_{i\leqq j} |v_n(t_i) - v_n(t_{i-1})| .$$

It follows that the total variation of v over $[a, b]$ does not exceed $\liminf_n$ of the total variation of v_n over $[a, b]$. But this means that $|\nu|((a, b]) \leqq \liminf_n |\nu_n|((a, b])$. After approximating $1/w$ from below by step functions and invoking (2.22), one gets (8.1). □

LEMMA 8.2. *Suppose* ν_n, $n \geqq 1$, *and* ν *satisfy* (2.21) *and* (2.22). *Suppose also that* $x_n \to x$ *in* D_-, *and that*

$$(8.2)\qquad \begin{aligned} &\lim_{\delta\downarrow 0} \limsup_n \sup_{t<\delta} w(t)|x_n(t)| < \infty \qquad \textit{and} \\ &\lim_{\delta\uparrow\infty} \limsup_n \sup_{t\geqq\delta} w(t)|x_n(t)| < \infty . \end{aligned}$$

Then

$$(8.3)\qquad \begin{aligned} \textstyle\int v_*\, d\xi &= \textstyle\int x_*\, d\nu \leqq \liminf_n \int x_n\, d\nu_n \leqq \limsup_n \int x_n\, d\nu_n \leqq \int x^*\, d\nu \\ &= \textstyle\int v^*\, d\xi , \end{aligned}$$

where

$$(8.4)\qquad v_*(t) = \min(-v(t), -v(t+)) \quad \textit{and} \quad v^*(t) = \max(-v(t), -v(t+)) ,$$

$$(8.5)\qquad \begin{aligned} x_*(t) &= x(t) , && \textit{if } \nu(\{t\}) \leqq 0 \\ &= x(t-) , && \textit{if } \nu(\{t\}) > 0 \qquad \textit{and} \\ x^*(t) &= x(t-) , && \textit{if } \nu(\{t\}) < 0 \\ &= x(t) , && \textit{if } \nu(\{t\}) \geqq 0 , \end{aligned}$$

and ξ *is the nonnegative measure on the Borel sets of* $(0, \infty)$ *such that*

$$x(t) = -\xi((t, \infty)) \tag{8.6}$$

for all $t \in (0, \infty)$.

PROOF. Put $C = \{t \in (0, \infty) : t$ is a continuity point of v and of $x\}$. Let $a < b$ be points in C. Let μ_n be the measure on the Borel sets of $(0, \infty)$ defined by $\mu_n(B) = \nu_n(B \cap (a, b])$, $n \geqq 1$; similarly, define μ by $\mu(B) = \nu(B \cap (a, b])$. A straightforward calculation shows that

(8.7) the μ_n's are M_2-convergent with limit μ.

Now write

$$\int x_n\, d\nu_n = \int_{(0,a]} x_n\, d\nu_n + \int x_n\, d\mu_n + \int_{(b,\infty)} x_n\, d\nu_n \tag{8.8}$$

$$\int x^*\, d\nu = \int_{(0,a]} x^*\, d\nu + \int x^*\, d\mu + \int_{(b,\infty)} x^*\, d\nu\,. \tag{8.9}$$

By (2.22), (8.1), and (8.2), the first and third terms on the right-hand side of (8.8) and (8.9) can be made arbitrarily small by choosing (as one may) a sufficiently small and b sufficiently large in C. So to establish

$$\limsup_n \int x_n\, d\nu_n \leqq \int x^*\, d\nu \tag{8.10}$$

it suffices to show

$$\limsup_n \int x_n\, d\mu_n \leqq \int x^*\, d\mu\,. \tag{8.11}$$

For this, let ξ_n be the nonnegative measure on the Borel sets of $(0, \infty)$ for which $\xi_n((t, \infty)) = -x_n(t)$ for all t. By Fubini's theorem

$$\int x_n\, d\mu_n = \int u_n\, d\xi_n$$

where $u_n(t) = -\mu_n((0, t))$. If T is a system of points $a = t_0 < t_1 < \cdots < t_q = b$ in C, then by (8.7) and the weak convergence of x_n to x, we have

$$\limsup_n \int u_n\, d\xi_n \leqq \int u_T^*\, d\xi\,,$$

where

$$\begin{aligned} u_T^*(t) &= 0\,, && \text{if } 0 < t \leqq a \\ &= \sup_{t_{p-1} \leqq \tau \leqq t_p} u(\tau)\,, && \text{if } t_{p-1} < t \leqq t_p\,, \quad 1 \leqq p \leqq q \\ &= u(b)\,, && \text{if } t_q < t < \infty \end{aligned}$$

with $u(t) = -\mu((0, t))$. Letting T grow dense in $[a, b]$, we get

$$\limsup_n \int x_n\, d\mu_n \leqq \int u^*\, d\xi\,,$$

where

$$u^*(t) = \max(u(t), u(t+)) = u(t) + \theta_t(u(t+) - u(t))$$

with

$$\begin{aligned} \theta_t &= 0\,, && \text{if } u(t) \geqq u(t+) \\ &= 1\,, && \text{if } u(t) < u(t+)\,. \end{aligned}$$

Thus

$$\begin{aligned}\int u^*(t)\xi(dt) &= \int u(t)\xi(dt) + \textstyle\sum_t \theta_t(u(t+) - u(t))(x(t) - x(t-)) \\ &= \int x\, d\mu + \textstyle\sum_t \theta_t(x(t-) - x(t))\mu(\{t\}) \\ &= \int x^*\, d\mu\end{aligned}$$

(cf. (8.5)), and so (8.11) holds.

This completes the proof of (8.10). An application of Fubini's theorem shows that $\int x^*\, d\nu = \int v^*\, d\xi$, and thus that the right half of (8.3) holds. The left half of (8.3) follows by a similar argument. □

LEMMA 8.3. Wp 1, *one has*

$$\lim_{\delta\downarrow 0} \limsup_n \sup_{t\leqq\delta} w(t)|H_n(t)| < \infty \tag{8.12}$$

$$\lim_{\delta\uparrow\infty} \limsup_n \sup_{t\geqq\delta} w(t)|H_n(t)| = 0\,. \tag{8.13}$$

PROOF. We consider first the situation as $\delta \uparrow \infty$. Here

$$w(t)H_n(t) = l_{n,t}m^{(k)}_{[nt]}/((\log_2 nt)/nt)\,,$$

where

$$l_{n,t} = \log_2 nt/((\log_2 t)(\log_2 n))\,.$$

Calculus shows that $\sup_{t\geqq\delta} l_{n,t} = l_{n,\delta}$, from which (8.13) follows because

$$\limsup_j |m_j^{(k)}|/(\log_2 j/j) < \infty \quad \text{wp } 1 \tag{8.14}$$

(cf. Lemma 6.4).

For t near 0, we have $w(t)H_n(t) = tm^{(k)}_{\max(k,[nt])}/(\log_2 n/n)$, so

$$\sup_{t\leqq\delta} w(t)|H_n(t)| \leqq k^*/\log_2 n + (\log_2 n\delta/\log_2 n)\sup_{k^*/n<t\leqq\delta}(|m^{(k)}_{[nt]}|/((\log_2 nt)/nt))\,,$$

where $k^* = \max(k, 3)$. Combining this with (8.14), one gets (8.12). □

REMARK. For future use, we note that the argument just used shows that

$$\lim_{\delta\downarrow 0} \limsup_n \sup_{t\leqq\delta} t\varepsilon(t)|H_n(t)| = 0 \tag{8.15}$$

whenever $\varepsilon(t) = o(1)$ as $t \downarrow 0$.

LEMMA 8.4. *Under the hypotheses of Corollary* 2.1, (2.23) *holds.*

PROOF. It follows easily from Theorem 1A, and Lemmas 8.2 and 8.3, that wp 1

$$\limsup_n \int H_n\, d\nu_n \leqq \sup\{\int v^*\, d\xi_x : x \in K\}\,,$$

where ξ_x is the measure corresponding to x under (8.6). But for each x in K, one has (cf. (2.24))

$$\int v^*\, d\xi_x = \int (v^*(t)/t)t\xi_x(dt) \leqq \Gamma \int t\xi_x(dt) = \Gamma|\int x(t)\, dt| \leqq \Gamma\,,$$

the next to the last step being a consequence of Fubini's theorem. Thus

$$\limsup_n \int H_n\, d\nu_n \leqq \Gamma$$

wp 1, and a similar argument shows

$$\liminf_n \int H_n\, d\nu_n \geqq \gamma\,.$$

Moreover, since K is convex and since the functions $-I_{(0,t)}/t$ are in K for each t, Theorem 1A and Lemmas 8.2 and 8.3 imply that wp 1 every convex linear combination of the numbers $-v(t)/t$, t being a continuity point of ν, is a limit point of the sequence $(\int H_n\, d\nu_n)$. Consequently, the limit points must include $[\gamma, \Gamma]$, wp 1. □

(b) Proof of Corollary 2.2. For $-1 < c < 0$, define $T^c\colon D \to [0, 1]$ by

$$T^c(x) = \int_0^1 I_{(-\infty,c]}(tx(t))\, dt = \mu\{t\colon x(t) \leqq c/t\}$$

where μ denotes Lebesgue measure on $[0, 1]$. By Fatou's lemma, T^c is uppersemicontinuous on D.

LEMMA 8.5. *One has*

$$\sup_{x\in K} T^c(x) = f(c) \equiv 1 - \exp(-(|c|^{-1} - 1)) \,. \tag{8.16}$$

Moreover, $x \in K$ *realizes the supremum in* (8.16) iff $x = x_c$, *defined by* (2.27).

PROOF. Since T^c is uppersemicontinuous and K is compact, we can find an $x \in K$ such that

$$T^c(x) = \sup_{y\in K} T^c(y) \,. \tag{8.17}$$

Since $\kappa(x) \equiv \int x(t)\, dt \geqq -1$, there must be some $t \in (0, 1)$ for which $x(t) > c/t$. From this and (8.17) it follows that

$$x(t) \geqq c/t \qquad \text{for all} \quad t \in (0, 1) \tag{8.18}$$

$$x(t) = 0 \qquad \text{for} \quad t \in [1, \infty) \,. \tag{8.19}$$

Put $\mathscr{E} = \{t < 1 : x(t) = c/t\}$. $\mathscr{E}$ is nonempty by (8.17); let s be its infimum. We must have $s > 0$, for otherwise we could find $s_j \in \mathscr{E}$, $j \geqq 1$, with $s_{j+1} \leqq s_j/2$, which would imply $\kappa(x) \leqq \sum_j c/2 = -\infty$. The right continuity of x implies that $s \in \mathscr{E}$; combining this with (8.17) gives

$$x(r) = c/s \tag{8.20}$$

for $0 < r \leqq s$.

Now suppose that

$$x(v) > c/v \tag{8.21}$$

for some $v \in (s, 1)$. Put $u = \sup\{t \in \mathscr{E}\colon s \leqq t < v\}$. Then $u \leqq v$ and $x(u-) = c/u$. Moreover, (8.17) and (8.20) imply that $x(u) = x(v)$; in particular, $x(u) > c/u$ and so $s < u$. Let Δs be a very small positive number and define $y \in D$ as follows:

$$\begin{aligned} y(\tau) &= c/(s + \Delta s)\,, && \text{for} \quad 0 < \tau < s + \Delta s \\ &= x(\tau)\,, && \text{for} \quad s + \Delta s \leqq \tau < u \\ &= c/u\,, && \text{for} \quad u \leqq \tau < u + \Delta u \\ &= x(\tau)\,, && \text{for} \quad u + \Delta u \leqq \tau < \infty\,; \end{aligned}$$

here Δu is chosen so that

$$\kappa(x) = \kappa(y) \,. \tag{8.22}$$

Clearly $y \in K$. Condition (8.22) implies that $s(|c|/s^2)\Delta s$ is essentially no bigger than $(|c|/u)\Delta u$, and thus that $T^c(y) \geqq T^c(x) - \Delta s + \Delta u > T^c(x)$. Thus (8.17) implies the impossibility of (8.21):

We now know (cf. (8.18)) that $x(v) = c/v$ for $v \in (s, 1)$. Combining this with (8.19) and (8.20) gives $\kappa(x) = c(1 - \log s)$; (8.17) then implies $s = \exp(-(|c|^{-1} - 1))$. □

Now define $T_p^c: D \to [0, 1]$ by

$$T_p^c(x) = (p - k + 1)^{-1} \sum_{k \leqq n \leqq p} I_{(-\infty, c]}((n/p)x(n/p)) .$$

Although condition (2.19) does not hold, the following modification of it does:

LEMMA 8.6. *Let $x_p \to x$ in D, and suppose that $-1 < b < c < 0$. Then*

$$T^b(x) \leqq \liminf_p T_p^c(x_p) \leqq \limsup_p T_p^c(x_p) \leqq T^c(x) . \tag{8.23}$$

PROOF. Define y_p $(p \geqq 1)$ as follows:

$$\begin{aligned} y_p(\tau) &= x_p(1/p) && \text{if } 0 < \tau \leqq 1/p \\ &= x_p(n/p) && \text{if } n/p \leqq \tau < (n+1)/p , \quad 1 \leqq n \leqq p \\ &= x_p(\tau) , && \text{if } 1 \leqq \tau < \infty . \end{aligned}$$

Then $y_p \to x$ in D. Moreover,

$$T_p^c(x_p) \leqq T^c(y_p) + O(1/p)$$

for all p; this and the uppersemicontinuity of T^c imply the right half of (8.23).

To get the left half of (8.23) choose a $\beta \in (b, c)$ and define $S^\beta: D \to [0, 1]$ by $S^\beta(z) = \mu\{t: z(t) < \beta/t\}$. For large p, we have

$$S^\beta(y_p) - O(1/p) \leqq T_p^c(x_p) .$$

Since S^β is lowersemicontinuous and $S^\beta(x) \geqq T^b(x)$, the left half of (8.23) holds. □

Note that for $k \leqq n \leqq p$, $m_n^{(k)}/(\log_2 n/n) \leqq c$ iff $\mathscr{H}_p(n/p) \leqq c/(n/p)$, where $\mathscr{H}_p = (\mathscr{H}_p(t))_{0<t<\infty}$ is defined by

$$\mathscr{H}_p(t) = (\log_2 p)H_p(t)/(\log_2 pt)$$

(cf. (2.17)). Thus $f_{p,c} = T_p^c(\mathscr{H}_p)$. Theorem 1A implies that wp 1 $(\mathscr{H}_p)$ is relatively compact in D and has K as its set of limit points. Corollary 2.2 follows easily from this, Lemma 8.5, Lemma 8.6, and the continuity of the mapping $c \to f(c)$.

(c) Proof of Corollary 2.3. We can write

$$H_n(t) = Y(nt)/a_n ,$$

where

$$Y(u) = m^{(k)}_{\max(k,[u])} \qquad \text{and} \qquad a_n = \log_2 n/n$$

Y takes values in $\mathscr{D} = \{x \in D_-: x(t) \uparrow 0 \text{ as } t \uparrow \infty\}$. Consider the mapping $\mathscr{T}$ which sends $x \in \mathscr{D}$ into $\mathscr{T}x \in -D_-$, defined by

$$(\mathscr{T}x)(s) = \inf\{t: x(t) \geqq -s\}$$

$(0 < s < \infty)$. This mapping is continuous with respect to the topology of weak convergence (cf. Breiman (1968) page 294), and

$$(\mathscr{T}H_n)(s) = \inf\{t\colon Y(nt) \geqq -sa_n\} = n^{-1}(\mathscr{T}Y)(sa_n) = n^{-1}z_{sa_n'} = Z_t(s)\,,$$

where $t = a_n$ (note that $n \sim t^{-1}\log_2 t^{-1}$ as $n \to \infty$). By Theorem 1A and Lemma 2.1, it follows that wp 1 the net (Z_t) is relatively compact as $t \downarrow 0$ with limit points $\mathscr{T}(K) = -K$.

(d) Proof of Corollary 2.4. Suppose y_n, $n \geqq 1$, and y are points in Δ, with $y(0+) = 0 = y(\infty-)$. A sufficient (and necessary) condition for y_n to converge to y in the M_1-topology is the following (cf. Skorohod (1956), Section 2.4):

(i) $\lim_n y_n(t) = y(t)$, for a set of t-values dense in $(0, \infty)$,

(ii) $\lim_{\delta\downarrow 0} \limsup_n \sup_{t\leqq\delta} |y_n(t)| = 0 = \lim_{\delta\uparrow\infty} \limsup_n \sup_{t\geqq\delta} |y_n(t)|$, and

(iii) $\lim_{\delta\downarrow 0} \sup\{\theta(y_n; s, t, u)\colon c \leqq s < t < u \leqq 1/c,\ u - s \leqq \delta\} = 0$ for a set of c-values dense at 0; here $\theta(y_n; s, t, u)$ is the distance from $y_n(t)$ to the line segment having $y_n(s)$ and $y_n(u)$ as endpoints.

Because the function w of (2.29) is continuous, (i) and (iii) automatically hold when y_n and y are respectively of the form wx_n and wx, with $x_n \to x$ in D. Also, for each x in K, one has $(wx)(0+) = 0 = (wx)(\infty-)$. Combining these observations with (8.13) and (8.15), we get Corollary 2.4 as a consequence of Theorem 1A and Lemma 2.1 (with T_n and T taken to be the identity map from D_w, under the topology of weak convergence, to D_w, under the d_w-topology).

Acknowledgment. I would like to thank Wim Vervaat and the referee for making a number of helpful suggestions.

REFERENCES

[1] Balkema, A. A. and de Haan, L. (1972). On R. von Mises' condition for the domain of attraction of $\exp(-e^{-x})$. *Ann. Math. Statist.* **43** 1352–1354.

[2] Barndorff-Nielsen, O. (1961). On the rate of growth of the partial maxima of a sequence of independent identically distributed random variables. *Math. Scand.* **9** 383–394.

[3] Barndorff-Nielsen, O. (1963). On the limit behavior of extreme order statistics. *Ann. Math. Statist.* **34** 992–1002.

[4] Billingsley, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

[5] Breiman, L. (1968). *Probability.* Addison-Wesley, Reading.

[6] David, H. A. (1970). *Order Statistics.* Wiley, New York.

[7] Dwass, M. (1964). Extremal Processes. *Ann. Math. Statist.* **35** 1718–1725.

[8] Feller, W. (1971). *An Introduction to Probability Theory and its Application* **2** 2nd ed. Wiley, New York.

[9] Frankel, J. (1972). Iterated logarithm laws for order statistics. Ph. D. dissertation. Columbia Univ.

[10] Gnedenko, B. (1943). Sur la distribution limite du terme maximum d'une série aléatoire. *Ann. Math.* **41** 423–453.

[11] Grenander, U. (1965). A limit theorem for sums of minima of stochastic variables. *Ann. Math. Statist.* **36** 1041–1042.

[12] Gumbel, E. J. (1958). *Statistics of Extremes.* Columbia Univ. Press.

[13] de Haan, L. and Hordijk, A. (1972). The rate of growth of sample maxima. *Ann. Math. Statist.* **43** 1185–1196.

[14] DE HAAN, L. (1970). *On Regular Variation and its Application to the Weak Convergence of Sample Extremes.* Mathematisch Centrum, Amsterdam.

[15] KIEFER, J. (1972). Iterated logarithm analogues for sample quantites when $p_n \downarrow 0$. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* Univ. of California Press **1** 227–244.

[16] MARCUS, M. and PINSKY, M. (1969). On the domain of attraction of $e^{-e^{-x}}$. *J. Math. Anal. Appl.* **28** 440–449.

[17] MISES, R. VON (1936). La distribution de la plus grande de n valeurs. *Rev. Math. Union Interbalkanique.* **1** 141–160.

[18] MOTOO, M. (1959). Proof of the law of iterated logarithm through diffusion equation. *Ann. Inst. Statist. Math.* **10** 21–28.

[19] PICKANDS, J. III (1967). Sample sequences of maxima. *Ann. Math. Statist.* **38** 1570–1574.

[20] PICKANDS, J. III (1971). The two dimensional Poisson process and extremal processes. *J. Appl. Prob.* **8** 745–756.

[21] RESNICK, S. I. and TOMKINS, R. J. (1973). Almost sure stability of maxima. *J. Appl. Prob.* **10**.

[22] ROBBINS, H. and SIEGMUND, D. (1971). On the law of the iterated logarithm for maxima and minima. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* Univ. of California Press **3** 51–70.

[23] SKOROHOD, A. V. (1956). Limit theorems for stochastic processes. *Theor. Probability Appl.* **1** 261–290.

[24] STRASSEN, V. (1964). An invariance principle for the law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **3** 211–226.

[25] VERVAAT, W. (1974). Limit theorems for partial maxima and records.

[26] WICHURA, M. J. (1973a). Some Strassen type laws of the iterated logarithm for multiparameter stochastic processes with independent increments, *Ann. Probability* **1** 272–296.

[27] WICHURA, M. J. (1973b). Boundary crossing probabilities associated with Motoo's law of the iterated logarithm. *Ann. Probability* **1** 437–456.

DEPARTMENT OF STATISTICS
UNIVERSITY OF CHICAGO
1118 EAST 58TH STREET
CHICAGO, ILLINOIS 60637