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A NOTE ON THE ASYMPTOTIC INDEPENDENCE OF HIGH LEVEL CROSSINGS FOR DEPENDENT GAUSSIAN PROCESSES

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It is shown that the numbers of high level crossings by p dependent stationary Gaussian processes have asymptotically independent Poisson distributions if the observation interval and the height of the level increase in a coordinated way. The processes may be highly correlated, even linearly dependent.

Let $\{X(t),\ t \geqq 0\}$ be a separable, stationary, zero mean, Gaussian process with unit variance, and with a covariance function r which satisfies

$$1 - r(t) \sim \lambda_2 t^2/2 \qquad \text{as } t \to 0 .$$

If

$$M(T) = \text{the number of upcrossings of the level } u \text{ by } X(t) \text{ for } 0 \leqq t \leqq T ,$$

then

$$E(M(T)) = \frac{T}{2\pi} \lambda_2^{\frac{1}{2}} \exp(-u^2/2) .$$

It is well known that, if $u \to \infty$ and $T \to \infty$ in such a way that $E(M(T))$ remains fixed and equal to θ, then $M(T)$ has an asymptotic Poisson distribution with mean θ. This has been shown by several authors, most recently by Berman (1971), who shows that either one of the following two conditions is sufficient for the Poisson result to hold:

(a) $r(t) \log t \to 0$ as $t \to \infty$
(b) $\int_0^\infty r^2(t)\, dt < \infty$.

Berman also shows that $M(T)$ is asymptotically independent of $N(T) =$ the number of downcrossings of the level $-u$.

The purpose of this note is to show how Berman's methods can be generalized to deal with the joint asymptotic distribution of upcrossings by two or more dependent stationary Gaussian processes, $\{X_1(t), \cdots, X_p(t),\ t \geqq 0\}$. Let the cross-covariance functions be r_{kl},

$$r_{kl}(t) = \operatorname{Cov}(X_k(s), X_l(s+t)) ,$$

and denote the auto-covariance functions by r_k,

$$r_k(t) = r_{kk}(t) .$$

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Suppose

(1) $$1 - r_k(t) \sim \lambda_{2k} t^2/2 \qquad \text{as } t \to 0 \ (k = 1, \cdots, p)$$

(2) $$r_{kl}(t) \log t \to 0 \qquad \text{as } t \to \infty \ (k, l = 1, \cdots, p) .$$

To exclude the possibility of two processes being identical, we also assume that

(3) $$\max_{k \neq l} \sup_t |r_{kl}(t)| < 1 .$$

Let $M_k(T)$ be the number of upcrossings of the level u by the process $X_k(t)$ for $0 \leqq t \leqq T$ $(k = 1, \cdots, p)$.

THEOREM. *If $u \to \infty$ and $T \to \infty$ so that*

$$E(M_k(T)) = \theta_k ,$$

then the conditions (1), (2), *and* (3) *imply that the joint asymptotic distribution of $M_1(T), \cdots, M_p(T)$ consists of p independent Poisson distributions with means $\theta_1, \cdots, \theta_p$. In particular*

$$\lim_{T\to\infty} P(\max_{1 \leqq k \leqq p} \sup_{0 \leqq t \leqq T} X_k(t) \leqq u) = \prod_{k=1}^{p} \lim_{T\to\infty} P(\sup_{0 \leqq t \leqq T} X_k(t) \leqq u)$$
$$= \prod_{k=1}^{p} \exp(-\theta_k) .$$

REMARK 1. The condition (3) does not mean that the processes are necessarily linearly independent. For example, if $Y(t)$ and $Z(t)$ are two independent processes whose covariance functions fulfill (1) and (2), then, for any finite set of constants $\{c_k; |c_k| < 1, k = 1, \cdots, p\}$, the processes

$$X_k(t) = c_k Y(t) + (1 - c_k^2)^{\frac{1}{2}} Z(t)$$

fulfill the condition (3). Therefore, although the processes X_k are completely linearly dependent, the numbers of upcrossings of u in $[0, T]$ are asymptotically independent. This surprising fact illustrates the quite extraordinary character of crossings of high levels.

REMARK 2. Condition (3) can be weakened to

(3′) $$\max_{k \neq l} \sup_t r_{kl}(t) < 1 ,$$

which means that we allow $\inf_t r_{kl}(t) = -1$ for some $k \neq l$. If, say, $\inf_t r_{12}(t) = -1$, then there actually is a t_0 such that $r_{12}(t_0) = -1$. This in turn implies that the process X_2 is the process $-X_1$ translated the distance t_0:

$$X_2(t) = -X_1(t - t_0) .$$

An upcrossing of u by X_2 is therefore a downcrossing of $-u$ by X_1, and, by Berman's results, upcrossings of u and downcrossings of $-u$ by X_1 are asymptotically independent.

REMARK 3. In the theorem, the constants θ_k are not chosen independently of each other. Since

$$E(M_k(T)) = \frac{T}{2\pi} \lambda_{2k}^{\frac{1}{2}} \exp(-u^2/2) ,$$

once one of the θ_k is fixed, the others are determined. An alternative is to take different levels u_k for each one of the processes, which gives us a free choice for the θ_k-values.

PROOF OF THE THEOREM. We follow the main lines in Berman's paper (1971) about the asymptotic independence of the numbers of high and low level crossings, to which the reader is primarily referred. We will here only indicate what changes have to be made in Berman's proof in order to cover the present situation.

Let $I_1, \cdots, I_m$ be m subintervals of the interval $[0, T]$,

$$I_j = [(j-1)T/m, (j-\beta)T/m],$$

separated by $\beta T/m$, where β $(0 < \beta < 1)$ is a number to be chosen later. Also let G be the set of numbers $\{t_i = iT/n, i = 0, 1, \cdots, n\}$. Here m and n are integers, which for the moment are fixed, but which will be permitted to tend to $+\infty$ at a later stage.

Now define

$$U_{jk} = \max\{X_k(t_i), t_i \in I_j \cap G\}, \qquad j = 1, \cdots, m;\ k = 1, \cdots, p.$$

The point in the proof is that all the variables U_{jk} are asymptotically independent. We start with a lemma.

LEMMA 1. *Let $\phi(x, y; \rho)$ be the standardized, bivariate normal density function with the correlation coefficient ρ. If u_{jk} $(j = 1, \cdots, m;\ k = 1, \cdots, p)$ assume only the values u or $+\infty$, then*

$$\begin{aligned}(4)\qquad &|P(U_{jk} \leqq u_{jk}, j = 1, \cdots, m; k = 1, \cdots, p) - \textstyle\prod_{k=1}^{p}\prod_{j=1}^{m} P(U_{jk} \leqq u_{jk})| \\ &\quad\leqq n \textstyle\sum_{k=1}^{p}\sum_{i=[\beta n/m]}^{n} |r_k(t_i)| \int_0^1 \phi(u, u; \lambda r_k(t_i))\, d\lambda \\ &\qquad + 2n \textstyle\sum_{1\leqq k<l\leqq p}\sum_{i=0}^{n} |r_{kl}(t_i)| \int_0^1 \phi(u, u; \lambda r_{kl}(t_i))\, d\lambda .\end{aligned}$$

PROOF OF THE LEMMA. The lemma is a variant of a well-known inequality, frequently used in this context. It is based on the following inequality (5).

Let $Y_1, \cdots, Y_q$ be q normally distributed random variables with mean zero and unit variance, and let $A = (a_{\nu\mu})$ and $B = (b_{\nu\mu})$ be two alternatives for the covariance matrix of $Y_1, \cdots, Y_q$. If ϕ_A and ϕ_B are the corresponding q-variate normal densities, then

$$\begin{aligned}(5)\qquad &|\textstyle\int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_q} \phi_A(x_1, \cdots, x_q) - \phi_B(x_1, \cdots, x_q)\, dx_1 \cdots dx_q| \\ &\quad\leqq \textstyle\sum_{\nu<\mu} |a_{\nu\mu} - b_{\nu\mu}| \int_0^1 \phi(y_\nu, y_\mu; \lambda a_{\nu\mu} + (1-\lambda) b_{\nu\mu})\, d\lambda .\end{aligned}$$

The proof of (5) can be found in a variety of places, e.g., in Berman's paper (1971).

Now let the variables $X_k(t_i)$, $k = 1, \cdots, p$; $t_i \in G \cap \bigcup_j I_j$, play the role of the q variables $Y_1, \cdots, Y_q$. (Since the I-intervals take the proportion $1 - \beta$ of the interval $[0, T]$, q is approximately $pn(1-\beta)$.) The covariance matrix A is defined as the actual covariance matrix of the q variables, computed by means of the auto- and cross-covariance functions r_k and r_{kl}, while B is the covariance matrix they would have had, provided $X_k(t_{i'})$ and $X_l(t_{i''})$ had been independent as soon as $k \neq l$ or $t_{i'}$ and $t_{i''}$ belong to different I-intervals. This means that

$$b_{\nu\mu} = 0 \quad \text{if } Y_\nu \text{ and } Y_\mu \text{ are obtained from different } X\text{-processes or from different } I\text{-intervals}$$
$$= a_{\nu\mu} \quad \text{otherwise.}$$

Since $U_{jk} = \max\{X_k(t_i),\, t_i \in I_j \cap G\}$ we then have

$$P(U_{jk} \leqq u_{jk}, j = 1, \cdots, m; k = 1, \cdots, p) = \int_{-\infty}^{u_{11}} \cdots \int_{-\infty}^{u_{mp}} \phi_A(x_1, \cdots, x_q)\, dx_1 \cdots dx_q .$$

Under the covariance matrix B the variables U_{jk} are all independent, so that

$$\prod_{k=1}^{p} \prod_{j=1}^{m} P(U_{jk} \leqq u_{jk}) = \int_{-\infty}^{u_{11}} \cdots \int_{-\infty}^{u_{mp}} \phi_B(x_1, \cdots, x_q)\, dx_1 \cdots dx_q .$$

Thus the left-hand sides of (4) and (5) coincide.

On the right-hand side of (5) there will be a contribution from those pairs (ν, μ) which correspond to different X-processes or to different I-intervals. Otherwise $a_{\nu\mu} = b_{\nu\mu}$ and the term is 0. Since the variables y_ν $(= u_{jk})$ assume only the values u or ∞ we have

$$\begin{aligned} &\sum_{\nu<\mu} |a_{\nu\mu} - b_{\nu\mu}| \int_0^1 \phi(y_\nu, y_\mu; \lambda a_{\nu\mu} + (1-\lambda) b_{\nu\mu})\, d\lambda \\ (6) \quad &\leqq \sum_{k=1}^{p} \sum_{0 \leqq i < j \leqq n,\, j-i > \beta n/m} |r_k(t_j - t_i)| \int_0^1 \phi(u, u; \lambda r_k(t_j - t_i))\, d\lambda \\ &\quad + \sum_{1 \leqq k < l \leqq p} \sum_{i,j=0}^{n} |r_{kl}(t_j - t_i)| \int_0^1 \phi(u, u; \lambda r_{kl}(t_j - t_i))\, d\lambda . \end{aligned}$$

The first sum in the right-hand side in (6) is extended to include all values of i and j for which t_i and t_j belong to different I-intervals: $|t_j - t_i| > \beta T/m$ implies $|j - i| > \beta n/m$. By stationarity, the sums in (6) are less than or equal to

$$\sum_{k=1}^{p} n \sum_{i=[\beta n/m]}^{n} |r_k(t_i)| \int_0^1 \phi(u, u; \lambda r_k(t_i))\, d\lambda + \sum_{1 \leqq k < l \leqq p} 2n \sum_{i=0}^{n} |r_{kl}(t_i)| \int_0^1 \phi(u, u; \lambda r_{kl}(t_i))\, d\lambda ,$$

which is the content of the lemma.

We can now follow Berman (1971) for the rest of the proof of the theorem. First, we notice that condition (2), i.e., $r_{kl}(t) = o(1/\log t)$, implies that the right-hand side of (5) tends to 0 if $n \to \infty$ sufficiently slowly with u and T, while m and β remain fixed. This follows exactly as in Berman's Lemma 5.1 as far as the first sum in (5) is concerned. The second sum is even simpler, since we always have $\sup_t |r_{kl}(t) < 1$.

Next, define

$$\xi_{jk} = 1 \quad \text{if } \max\{X_k(t_i),\, t_i \in I_j \cap G\} > u$$
$$= 0 \quad \text{otherwise.}$$

Then, our Lemma 1 and its implications show that the ξ-variables are asymptotically independent: if x_{jk} equals 0 or 1, then

$$\lim_{T\to\infty} |P(\xi_{jk} = x_{jk}, j = 1, \cdots, m; k = 1, \cdots, p) - \prod_{k=1}^{p} \prod_{j=1}^{m} P(\xi_{jk} = x_{jk})| = 0 ,$$

and

(7) $\lim_{T\to\infty} P(\sum_{j=1}^{m} \xi_{jk} = z_k,\ k = 1, \cdots, p) = \lim_{T\to\infty} \prod_{k=1}^{p} P(\sum_{j=1}^{m} \xi_{jk} = z_k)$.

As $\beta \to 0$ and $m \to \infty$, the right-hand side in (7) tends to the product $\prod_{k=1}^{p} P(Z_k = z_k)$ where Z_k, $k = 1, \cdots, p$ are Poisson variables with means θ_k respectively. This follows from Berman's proof of the univariate theorem, which also shows that $M_k(T)$ and $\sum_j \xi_{jk}$ are asymptotically equal:

$$\lim_{m\to\infty, \beta\to 0} \liminf_{T\to\infty} P(M_k(T) = \textstyle\sum_{j=1}^{m} \xi_{jk},\ k = 1, \cdots, p) = 1 .$$

This concludes the proof of the theorem.

REFERENCE

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