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A NOTE ON THE SUPERCRITICAL BRANCHING PROCESSES WITH RANDOM ENVIRONMENTS[1]

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Some further results in the theory of Galton Watson processes are extended to the more general set up of a branching process with random environments. The random distribution function of the limit random variable in the supercritical case (Athreya and Karlin, *Ann. Math. Statist.*, **40** (1969) 743-763) is investigated, and a zero-one law is established. It is shown that this random distribution function is w.p. 1. either absolutely continuous on $(0, \infty)$ with only a jump at the origin or w.p. 1. it is singular. A set of conditions is given under which the former case holds.

Let $\{Z_n(\bar{\zeta})\}_{n\geq 0}$ be a branching process with random environments (B.P.R.E.). The process can be described as follows. Assume, given a stationary ergodic sequence, $\{\zeta_n\}_{n\geq 0}$ of "environmental" random variables. For a.e. realization of this process there is associated a sequence $\{\varphi_{\zeta_n}(s)\}_{n\geq 0}$ of probability generating functions, (pgf). When conditioned on the $\{\zeta_n\}_{n\geq 0}$ process, the $\{Z_n\}$ behave as a temporally nonhomogeneous branching process where the number of offspring produced by an individual in the nth generation is governed by $\varphi_{\zeta_n}(s)$. Let $F(\bar{\zeta})$ denote the σ-field generated by the $\{\zeta_n\}_{n\geq 0}$. It follows that,

$$E\{s^{Z_{n+1}} \mid Z_0, Z_1, \cdots, Z_n, F(\bar{\zeta})\} = [\varphi_{\zeta_n}(s)]^{Z_n} \qquad \text{w.p. 1.} \tag{1.1}$$

For a more detailed discussion of a B.P.R.E., the reader should consult the papers of Athreya and Karlin [2], [3]. The notation of those papers will be adhered to as much as possible.

Put

$$q(\bar{\zeta}) = P\{\lim_{n\to\infty} Z_n = 0 \mid Z_0 = 1, F(\bar{\zeta})\}.$$

Using (1.1), it is not difficult to show that w.p. 1.

$$q(\bar{\zeta}) = \lim_{n\to\infty} \varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots \varphi_{\zeta_n}(0))).$$

It is proven in (1) that if $E\{-\log(1 - \varphi_{\zeta_0}(0))\} < \infty$, then

$$E\{\log \varphi'_{\zeta_0}(1)\} > 0 \qquad \text{iff} \quad P\{\bar{\zeta} : q(\bar{\zeta}) < 1\} = 1. \tag{1.2}$$

Henceforth, we assume that $E\{-\log(1 - \varphi_{\zeta_0}(0))\} < \infty$ and $E\{\log \varphi'_{\zeta_0}(1)\} > 0$. This corresponds to the supercritical case for the standard Galton Watson process.

Define the random variables

$$W_n(\bar{\zeta}) = Z_n(\bar{\zeta})/P_n(\bar{\zeta})$$

where

$$P_n(\bar{\zeta}) = \prod_{i=0}^{n-1} \varphi'_{\zeta_i}(1) \qquad n \geq 1.$$

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Using martingale arguments, Athreya and Karlin showed that the following statements hold w.p. 1. for a supercritical B.P.R.E. satisfying $E\{\varphi''_{\zeta_0}(1)\} < \infty$,

(i) $\lim_{n\to\infty} W_n(\bar{\zeta}) = W(\bar{\zeta})$
(ii) $E\{W \mid Z_0 = 1, F(\bar{\zeta})\} = 1$
(iii) $P\{W = 0 \mid Z_0 = 1, F(\bar{\zeta})\} = q(\bar{\zeta})$
(iv) Let $\psi(t, \bar{\zeta}) = E\{e^{itW} \mid Z_0 = 1, F(\bar{\zeta})\}$. Then

$$\psi(t, \bar{\zeta}) = \varphi_{\zeta_0}(\psi(t/\varphi'_{\zeta_0}(1), T\bar{\zeta})) \tag{1.3}$$

(T is the standard shift operator associated with the $\{\zeta_n\}_{n\geq 0}$ process).
The purpose of this note is to examine the conditional distribution of the random variable $W(\bar{\zeta})$ given $Z_0 = 1$ and $F(\bar{\zeta})$. We denote this random distribution function by $G(t, \bar{\zeta})$.

Our first result shows that w.p. 1., $G(t, \bar{\zeta})$ is a pure distribution function.

THEOREM 1. *Assume*:

(a) *there exists a constant* $R > 1$ *such that*

$$P\{\bar{\zeta} : \varphi'_{\zeta_0}(1) < R\} = 1$$

and

(b) $P\{\bar{\zeta} : \varphi_{\zeta_0}(s) = s^{m(\bar{\zeta})}$ *for some integer* $m(\bar{\zeta})$ *depending on* $\bar{\zeta}\} = 0$.

Then either:

(i) $P\{\bar{\zeta} : G(t, \bar{\zeta})$ *is absolutely continuous on* $(0, \infty)$ *with a jump at the origin of size* $q(\bar{\zeta})\} = 1$, *or*
(ii) $P\{\bar{\zeta} : G(t, \bar{\zeta})$ *is singular*$\} = 1$.

It has been pointed out by the referee that the set

$$A = \{\bar{\zeta} : G(t, \bar{\zeta}) \text{ is absolutely continuous on } (0, \infty)\}$$

is invariant under T and hence has probability 0 or 1. The same is true of the set

$$B = \{\bar{\zeta} : G(t, \bar{\zeta}) \text{ is singular}\}.$$

These results are of a weaker nature than Theorem 1 since they do not rule out the possibility that $G(t, \bar{\zeta})$ could be a mixed distribution with positive probability.

PROOF OF THEOREM 1. Using arguments in Doob [4], it is not hard to establish that all sets in question are measurable, and therefore no further discussion of this technical problem will be given. The idea of the proof is to show that if (ii) does not hold, then $\int_{-\infty}^{\infty} |(d/dt)\psi(y, \bar{\zeta})|\, dy < \infty$ w.p. 1. This together with Lemma 3 of [1] implies (i). The details of the argument will be carried out in a series of lemmas.

LEMMA 1. $P\{\bar{\zeta} : G(t, \bar{\zeta})$ *is not degenerate*$\} = 1$.

PROOF. Let $D = \{\bar{\zeta} : G(t, \bar{\zeta}) \text{ is degenerate}\}$. It is a consequence of (1.3) that $T^{-1}D \subset D$. Since T is measure preserving and ergodic, $P(D) = 0$ or 1. Assume $P(D) = 1$. Then necessarily, $\psi(it, \bar{\zeta}) = e^{-t}$ w.p. 1. and substituting in (1.3) we obtain

$$e^{-t} = \varphi_{\zeta_0}(\exp[-t/\varphi'_{\zeta_0}(1)]) .$$

However, Jensen's inequality implies $e^{-t} < \varphi_{\zeta_0}(\exp[-t/\varphi'_{\zeta_0}(1)])$ unless $\varphi_{\zeta_0}(s) = s^j$ for some j which is ruled out by our assumptions. □

LEMMA 2. $P\{\bar{\zeta}: |\psi(t, \bar{\zeta})| < 1, |t| \neq 0\} = 1$.

PROOF. Define

$$\eta(\bar{\zeta}) = \sup \{\eta: |\psi(t, \bar{\zeta})| < 1, 0 < |t| < \eta\} .$$

By Lemma 1, $P\{\bar{\zeta}: \eta(\bar{\zeta}) > 0\} = 1$. If follows from (1.3) that $\eta(\bar{\zeta}) \geqq \varphi'_{\zeta_0}(1)\eta(T\bar{\zeta})$ w.p. 1. Iterating this relationship yields,

$$\eta(\bar{\zeta}) > P_n(\bar{\zeta})\eta(T^n\bar{\zeta}) . \qquad \text{w.p. 1.}$$

Since the process is supercritical, $\eta(\bar{\zeta}) = \infty$ w.p. 1. □

LEMMA 3. *If* (ii) *does not hold, then*

$$P\{\bar{\zeta}: \lim_{|t|\to\infty} |\psi(t, \bar{\zeta}) - q(\bar{\zeta})| = 0\} = 1 .$$

PROOF. If (ii) does not hold, then $G(t, \bar{\zeta})$ has with positive probability an absolutely continuous part and hence for ε sufficiently small $\alpha(\bar{\zeta}) = \limsup_{|t|\to\infty} |\psi(t, \bar{\zeta})| < 1 - \varepsilon$ with positive probability. It follows then from the Ergodic Theorem, that there exists w.p. 1. a sequence of integers $n_k \to \infty$, depending on the sample path, such that $\alpha(T^{n_k}\bar{\zeta}) < 1 - \varepsilon$. Thus, for each n_k

$$\begin{aligned}\limsup_{|t|\to\infty} |\psi(t, \bar{\zeta}) - q(\bar{\zeta})| &\leqq \limsup_{|t|\to\infty} |\varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots \varphi_{\zeta_{n_k-1}}(\psi(t, T^{n_k}\bar{\zeta})))) - q(\bar{\zeta})| \\ &\leqq |\varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots \varphi_{\zeta_{n_k-1}}(1 - \varepsilon))) - q(\bar{\zeta})| .\end{aligned}$$

It is proven in (1) that

$$\lim_{n\to\infty} \varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots \varphi_{\zeta_n}(1 - \varepsilon))) = q(\bar{\zeta}) . \qquad \square$$

Define

$$C_n(t, \bar{\zeta}) = \frac{d}{ds} [\varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots \varphi_{\zeta_{n-1}}(s)))]\Big|_{s=\psi(P_n(\bar{\zeta})t, T^n\bar{\zeta})} \qquad n \geqq 1$$

LEMMA 4. *There exists a* $\delta > 0$ *such that*

$$P\{\bar{\zeta}: \limsup_{n\to\infty} [\sup_{1\leqq|t|\leqq R} |P_n(\bar{\zeta})^\delta C_n(t, \bar{\zeta})|] = 0\} = 1$$

The proof of this lemma is not difficult, but tedious and will be eliminated. Finally, we come to

LEMMA 5. $P\{\bar{\zeta}: \int_{-\infty}^{\infty} |(d/dt)\psi(y, \bar{\zeta})| \, dy < \infty\} = 1$.

PROOF. Let $\mu = E\{\log \varphi'_{\zeta_0}(1)\}$. By the Ergodic Theorem, there exists for $0 < \varepsilon < \mu$, an integer $N(\bar{\zeta})$ depending on the sample path such that w.p. 1.

$$e^{n(\mu-\varepsilon)} \leqq P_n(\bar{\zeta}) \leqq e^{n(\mu+\varepsilon)} \qquad n > N(\bar{\zeta}) . \tag{1.4}$$

Observe

$$\int_{-\infty}^{\infty} \left|\frac{d}{dt}\psi(y,\bar{\zeta})\right| dy = \int_{|y|\leq P_{N(\bar{\zeta})}(\bar{\zeta})} + \int_{|y|>P_{N(\bar{\zeta})}(\bar{\zeta})} \left|\frac{d}{dt}\psi(y,\bar{\zeta})\right| dy$$
$$= A(\bar{\zeta}) + B(\bar{\zeta}) .$$

$A(\bar{\zeta}) < \infty$ w.p. 1. It remains to show the same of $B(\bar{\zeta})$.

Note that

$$B(\bar{\zeta}) \leqq \sum_{k=N(\bar{\zeta})}^{\infty} \int_{P_k(\bar{\zeta})\leqq|y|\leqq RP_k(\bar{\zeta})} \left|\frac{d}{dt}\psi(y,\bar{\zeta})\right| dy .$$

Also,

$$\int_{P_k(\bar{\zeta})\leqq|y|\leqq RP_k(\bar{\zeta})} \left|\frac{d}{dt}\psi(y,\bar{\zeta})\right| dy \leqq P_k(\bar{\zeta}) \int_{1\leqq|y|\leqq R} \left|\frac{d}{dt}\psi(P_k(\bar{\zeta})y,\bar{\zeta})\right| dy$$
$$\leqq \frac{2(R-1)}{P_k(\bar{\zeta})^{\delta}} \sup_{1\leqq|t|\leqq R} [P_k(\bar{\zeta})^{\delta} C_k(t,\bar{\zeta})] .$$

The result now follows from Lemma 4 and (1.4). This completes the proof of Theorem 1. □

REMARK. It is possible to show that if the $\{\zeta_n\}_{n\geqq 0}$ are i.i.d., then $P\{\bar{\zeta}: G(t,\bar{\zeta})$ is continuous on $(0,\infty)\} = 1$. Thus, if $G(t,\bar{\zeta})$ is singular on $(0,\infty)$, the singularities cannot be jumps.

We now turn our attention to finding conditions which imply (i) of Theorem 1. In general this seems to be a difficult problem. However, we are able to obtain the following.

THEOREM 2. *Assume the conditions of Theorem* 1. *In addition assume*

(a) $P\{\bar{\zeta}: \varphi_{\zeta_0}(0) > \eta\} = 1$ *for some* $\eta > 0$, *and*

(b) *If* $V(\bar{\zeta}) = E\{W^3 \mid Z_0 = 1, F(\bar{\zeta})\}$, *then* $E\{V(\bar{\zeta})\} < \infty$.

Then (i) *of Theorem* 1 *holds.*

PROOF. Our goal is to show that $\limsup_{|t|\to\infty} |\psi(t,\bar{\zeta})| < 1$ w.p. 1. We first expand $\psi(t,\bar{\zeta})$ in a three term Taylor series expansion,

$$\psi(t,\bar{\zeta}) = 1 + it - \frac{t^2}{2} U(\bar{\zeta}) + \gamma(\bar{\zeta}) \frac{t^3}{6} V(\bar{\zeta})$$

$(U(\bar{\zeta}) = E\{W^2 \mid Z_0 = 1, F(\bar{\zeta})\}$ and $|\gamma(\bar{\zeta})| \leqq 1)$. Thus,

$$|\psi(t,\bar{\zeta})| \leqq [1 - (U(\bar{\zeta}) - 1)t^2 + U^2(\bar{\zeta})t^4]^{\frac{1}{2}} + \frac{t^3}{6} V(\bar{\zeta})$$

providing $|t|$ is sufficiently small. Since $E(U(\bar{\zeta}))$ and $E(V(\bar{\zeta}))$ are finite, it can be shown using stationarity and the Borel Cantelli Lemma that

$$P\{\bar{\zeta}: U(T^n\bar{\zeta}) > P_n(\bar{\zeta})^{\frac{1}{8}} \text{ i.o.}\} = 0$$

and

$$P\{\bar{\zeta}: V(T^n\bar{\zeta}) > P_n(\bar{\zeta})^{\frac{1}{8}} \text{ i.o.}\} = 0 .$$

Furthermore,

$$U(\bar{\zeta}) - 1 \geqq \frac{q(\bar{\zeta})}{1-q(\bar{\zeta})} \geqq \frac{\eta}{1-\eta} \qquad \text{w.p. 1.}$$

Hence, for n sufficiently large

$$|\psi(t, T^n\bar{\zeta})| \leqq 1 - \frac{\eta}{4} t^2 + t^3 P_n(\bar{\zeta})^{\frac{1}{2}}$$

providing $|t| < 1/P_n(\bar{\zeta})^{\frac{1}{4}}$. It is not difficult to show that

$$\frac{\theta}{P_n(\bar{\zeta})} \leqq \tfrac{1}{4}\eta t^2 - t^3 P_n(\bar{\zeta})^{\frac{1}{2}} \leqq 1$$

providing

$$\frac{\varepsilon\eta}{6P_n(\bar{\zeta})^{\frac{1}{2}}} \leqq |t| \leqq \frac{\eta}{6P_n(\bar{\zeta})^{\frac{1}{2}}} .$$

The constant θ depends only on ε and not on $\bar{\zeta}$. Let

$$\alpha_n = \frac{\varepsilon\eta}{6P_n(\bar{\zeta})^{\frac{1}{2}}} \quad \text{and} \quad \beta_n = \frac{\eta}{6P_n(\bar{\zeta})^{\frac{1}{2}}} .$$

It follows that

$$|\psi(t, T^n\bar{\zeta})| \leqq 1 - \frac{\theta}{P_n(\bar{\zeta})} \leqq e^{-\theta/P_n(\bar{\zeta})}$$

providing $\alpha_n \leqq |t| \leqq \beta_n$.

Also observe that for ε sufficiently small,

$$\alpha_{n+1} P_{n+1}(\bar{\zeta}) \leqq \beta_n P_n(\bar{\zeta}) \qquad n \geqq 1 .$$

It now follows that

$$\begin{aligned} \sup_{|t|>\alpha_1} |\psi(t, \bar{\zeta})| &\leqq \sup_{n\geqq 1} [\sup_{\alpha_n P_n(\bar{\zeta}) \leqq |t| \leqq \beta_n P_n(\bar{\zeta})} |\psi(t, \bar{\zeta})|] \\ &\leqq \sup_{n\geqq 1} [\varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots \varphi_{\zeta_{n-1}}(\sup_{\alpha_n \leqq |t| \leqq \beta_n} |\psi(t, T^n\bar{\zeta})|)))] \\ &\leqq \sup_{n\geqq 1} [\varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots \varphi_{\zeta_{n-1}}(e^{-\theta/P_n(\bar{\zeta})})))] < 1 . \end{aligned}$$

The last inequality holds since

$$\lim_{n\to\infty} \varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots \varphi_{\zeta_{n-1}}(e^{-\theta/P_n(\bar{\zeta})}))) = \psi(i\theta, \bar{\zeta}) < 1 . \qquad \square$$

It remains to give sufficient condition for when $E\{V(\bar{\zeta})\} < \infty$. The following are two such sets of conditions. The details of the proofs are omitted.

CONDITION I. Assume there exist constants $1 < C_1 < C_2 < \infty$ such that

(a) $P\{\bar{\zeta} : C_1 \leqq \varphi'_{\zeta_0}(1) \leqq C_2\} = 1$, and
(b) $P\{\bar{\zeta} : \varphi'''_{\zeta_0}(1) < C_2\} = 1$, and

CONDITION II. Assume the $\{\zeta_n\}_{n\geqq 0}$ to be i.i.d., $E\{(1/\varphi'_{\zeta_0}(1))^2\} < 1$ and $P\{\zeta : \varphi'''_{\zeta_0}(1) < C_2\} = 1$ for some C_2. Finally we obtain

THEOREM 3. *Assume the conditions of Theorem* 1. *In addition assume*:

(a) $P\{\bar{\zeta} : \varphi_{\zeta_0}(0) > \eta\} = 1$ *for some* $\eta > 0$, *and*
(b) *Either Condition* I *or Condition* II *holds.*

Then (i) *of Theorem* 1 *holds.*

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