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ON QUASI-COMPACT MARKOV OPERATORS[1]

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Let P be a conservative Markov operator on $L_\infty(X, \Sigma, m)$. The following conditions are proved to be equivalent: (i) P is ergodic and quasi-compact. (ii) P is ergodic and $(I - P)L_\infty$ is closed. (iii) For every $u \in L_1$ with $\int u\,dm = 0$ the sequence $\{\sum_{n=0}^N uP^n\}$ is weakly sequentially compact in L_1. (iv) There exists an invariant probability measure $\lambda \sim m$, and for every $u \in L_1$ with $\int u\,dm = 0$, $\sup_{N\geqq 0} ||\sum_{n=0}^N uP^n||_1 < \infty$. These conditions are used to study the quasi-compactness of the induced operators $P_f = \sum_{n=0}^\infty (PT_{1-f})^n PT_f$ in the case that P is Harris-recurrent: The following conditions are equivalent: (i) P_f is quasi-compact. (ii) f is "special" in the (modified) sense of Neveu. (iii) There exists a σ-finite measure λ, equivalent to m on $A = \{f > 0\}$, with $\int f\,d\lambda < \infty$, and $\int fg\,d\lambda = 0$ implies $\sup_{N\geqq 0} ||\sum_{n=0}^N P^n(fg)||_\infty < \infty$. Using this characterization certain limit theorems for P are obtained.

1. Introduction. Let (X, Σ, m) be a σ-finite measure space and let P be a positive linear contraction of $L_1(X, \Sigma, m)$, denoted by $u \to uP$. The adjoint of P, operating on $L_\infty(X, \Sigma, m)$, will be denoted by P and written to the left of its variable so that $\langle uP, f\rangle = \langle u, Pf\rangle = \int uPf\,dm$ for $u \in L_1$ and $f \in L_\infty$. P is called a *Markov operator*.

P is called *conservative* if $Pf \leqq f$ for $f \in L_\infty$ implies $Pf = f$. It is *conservative and ergodic* if $Pf \leqq f$ for $f \in L_\infty$ implies that f is constant a.e. (all inequalities are assumed a.e.).

A *Harris operator* is a conservative and ergodic Markov operator P such that for some $n > 0$ there is a function $k(x, y) \geqq 0$, measurable in (x, y), satisfying $P^n f(x) \geqq \int k(x, y)f(y)m(dy)$ (and $k \not\equiv 0$). A bounded linear operator T in a Banach space is *quasi-compact* if $||T^n - K|| < 1$ for some $n > 0$ and some compact linear operator $K \neq 0$. Quasi-compactness of a transition probability operator is discussed in Neveu [14] Section V. 3. It is equivalent to Doeblin's condition, which is treated in Doob [5].

Horowitz [10] studied the quasi-compactness of an ergodic and conservative Markov operator P and of certain induced Markov operators. Some additional information on quasi-compact Markov operators is obtained in Section 2, while Section 3 deals with the quasi-compactness of the induced operators

$$P_f = \sum_{n=0}^\infty (PT_{f'})^n T_f \qquad (\text{where } T_f g = fg, f' = 1 - f)$$

for $0 \leqq f \leqq 1$, which is shown to be equivalent to f being "*special*" in the sense

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of Neveu [15], although in a somewhat different context. (Brunel and Revuz [2] have obtained independently and announced this result in the exact context of [15]. Their remark about "changing time" seems to indicate a different proof). The quasi-compactness of P_f is given a characterization in terms of P, and is used to obtain certain limit theorems for Harris operators (only for such Markov operators special functions exist), yielding as corollaries some results of Chung [4] about Markov chains.

2. Quasi-compact conservative Markov operators. In this section we obtain some necessary and sufficient conditions for a conservative and ergodic Markov operator to be quasi-compact.

By Theorems VI. 6.2 and VI. 6.4 of [6] we have that $L_1(I - P)$ is closed if and only if $(I - P)L_\infty$ is closed, and in that case $(I - P)L_\infty = \{f \in L_\infty : uP = u \Rightarrow \langle u, f\rangle = 0\}$. We now obtain a general result.

THEOREM 2.1. *Let P be a conservative Markov operator. If $(I - P)L_\infty$ is closed, then P has a finite invariant measure equivalent to m, and $N^{-1}\sum_{n=0}^{N} P^n f$ converges in L_∞-norm for every $f \in L_\infty$.*

PROOF. By Theorem IV. E in [7], $X = C_0 + C_1$ where on C_1 there is a finite invariant measure $\lambda(\sim m_{|C_1})$, and every finite invariant measure vanishes on C_0. If $uP = u$ then $u^\pm P = u^\pm$ and therefore $\langle u, 1_{C_0}\rangle = 0$. Hence $-1_{C_0} \in (I - P)L_\infty$, so that for some $f \in L_\infty$, $-1_{C_0} = (I - P)f$ or $Pf = f + 1_{C_0}$. Since P is conservative, $m(C_0) = 0$ and there is a finite measure $\lambda \sim m$, $\lambda P = \lambda$. Hence $f \to Pf$ is a contraction of $L_1(\lambda)$, with the adjoint $f \to P^*f$ also a contraction of $L_1(\lambda)$ and $L_\infty(\lambda)$. By the Radon–Nikodym theorem $L_1(\lambda) \sim L_1(m)$, so we assume $m = \lambda$. Then $P^*u = uP = u$ if and only if u is Σ_i-measurable, since P and P^* have the same invariant sets [7]. Denote by $Ef = E_\lambda(f \mid \Sigma_i)$ the conditional expectation projection, and $PEf = Ef$. For $f \in L_\infty$ write $f = Ef + f - Ef$. Then $E(f - Ef) = 0$, and if $uP = u$ then u is Σ_i-measurable and therefore $\langle u, f - Ef\rangle = 0$ and $f - Ef \in (I - P)L_\infty$. Thus we have $L_\infty = (I - P)L_\infty \oplus L_\infty(X, \Sigma_i, \lambda)$ and $\|N^{-1}\sum_{n=0}^{N-1} P^n f - Ef\|_\infty \to 0$ for every $f \in L_\infty$.

REMARKS. 1. If we assume only $P1 = 1$, we obtain (by restriction to the conservative part) an invariant measure supported on the conservative part.

2. P need not be ergodic ($P = I$ satisfies our conditions).

Theorem 2.1 yields the following improved version of Horowitz' result [10].

THEOREM 2.2. *Let P be conservative and ergodic. Then the following conditions are equivalent:*

(i) *$(I - P)L_\infty$ is closed (in L_∞-norm);*
(ii) *$L_1(I - P)$ is closed (in L_1-norm);*
(iii) *P is quasi-compact.*

REMARK. Without restricting the size of the invariant field Σ_i the theorem is not true. Instead of "ergodic" we put "Σ_i finite" (and apply the theorem to each of its atoms). Since if P is quasicompact $N^{-1}\sum_{n=1}^{N} P^n$ converges to a finite dimensional projection ([6] pages 709–711), Σ_i must be finite if (iii) holds.

THEOREM 2.3. *Let P be conservative. Then the following conditions are equivalent*:

(i) *For every $u \in L_1$ with $\int u\,dm = 0$ the sequence $\{\sum_{n=0}^{N} uP^n\}$ is weakly sequentially compact in L_1.*

(ii) *There exists an invariant probability measure $\lambda \sim m$, and for every $u \in L_1$ with $\int u\,dm = 0$ we have $\sup_{N\geqq 0} \|\sum_{n=0}^{N} uP^n\|_1 < \infty$.*

(iii) *P is quasi-compact and ergodic.*

PROOF. (i) $\Rightarrow$ (ii): Fix $u \in L_1$ with $\int u\,dm = 0$. Clearly $\sup_{N\geqq 0} \|\sum_{n=0}^{N} uP^n\|_1 = K < \infty$, and we have to show the existence of a finite invariant measure. Define $v_n = \sum_{i=0}^{n-1} uP^i$. Then $v_n(I - P) = u - uP^n$. It follows easily from Theorem IV. 8.9, in [6] that also the sequence of averages $w_N = \sum_{n=1}^{N} v_n/N$ is weakly sequentially compact. If v is a weak limit of $\{w_N\}$, say $w_{N_j} \to_w v$, then for $f \in L_\infty$

$$\begin{aligned}\langle v(I-P), f\rangle &= \langle v, (I-P)f\rangle = \lim \langle w_{N_j}(I-P), f\rangle \\ &= \lim \langle u - N_j^{-1} \textstyle\sum_{n=1}^{N_j} uP^n, f\rangle = \langle u, f\rangle\end{aligned}$$

and $v(I - P) = u$. By (i) P is ergodic (If $1 \not\equiv P1_A = 1_A \not\equiv 0$, we take $u_1 \geqq 0$ supported in A and $u_2 \geqq 0$ supported in $X - A$ with $\int u_2 = \int u_1$ and $u = u_1 - u_2$ will satisfy $\int u\,dm = 0$, but $\sum_{n=0}^{N} \langle uP^n, 1_A\rangle = N\langle u_1, 1_A\rangle \to \infty$–a contradiction), so that

$$\{u \in L_1 \colon \textstyle\int u\,dm = 0\} \subset L_1(I-P) \subset clmL_1(I-P) = \{u \in L_1 \colon \textstyle\int u\,dm = 0\}$$

and $L_1(I - P)$ is closed. Now Theorem 2.1 yields the existence of a finite invariant measure.

(ii) $\Rightarrow$ (iii): We want to show first that if $g \in L_\infty$ satisfies $\int g\,d\lambda = 0$, then $\sup_{N\geqq 0} \|\sum_{n=0}^{N} P^n g\|_\infty < \infty$. Denote $g_N = \sum_{n=0}^{N} P^n g$. For $v \in L_1$ define $u = v - (\int v\,dm)u_0$, where $u_0 = d\lambda/dm$. Then $\int u\,dm = 0$, and by our assumption $\sup \|\sum_{n=0}^{N} uP^n\|_1 = K_v < \infty$. But

$$\langle vP^n, g\rangle = \langle uP^n, g\rangle + \textstyle\int v\,dm \langle u_0, g\rangle = \langle uP^n, g\rangle\,, \qquad \text{yielding}$$

$$|\langle v, g_N\rangle| = |\textstyle\sum_{n=0}^{N} \langle vP^n, g\rangle| = |\textstyle\sum_{n=0}^{N} \langle uP^n, g\rangle| \leqq K_v \|g\|_\infty\,.$$

By the Banach–Steinhaus theorem $\sup_N \|g_N\|_\infty < \infty$.

Now for $B \in \Sigma$ with $\lambda(B) > 0$ define $g = \lambda(B)1 - 1_B$. Then $\|\sum_{n=0}^{N} P^n g\|_\infty \leqq K$ for $N = 0, 1, \cdots$. If $R \geqq (K+1)/\lambda(B)$, then $\sum_{n=0}^{R} P^n 1_B \geqq \lambda(B) \sum_{n=0}^{R} P^n 1 - K \geqq 1$. By Horowitz [10] this condition implies that P is quasi-compact. (P is ergodic, as proved before).

(iii) $\Rightarrow$ (i): By Theorem 2.2, $L_1(I - P)$ is closed. Thus, $L_1(I - P) = \{u \in L_1 \colon \int u\,dm = 0\}$. Hence $\int u\,dm = 0 \Rightarrow u = v(I - P)$ and $\sum_{n=0}^{N} uP^n = v - vP^{N+1}$. Since P is necessarily Harris, it has a period k, and necessarily vP^{nk+d}

converges weakly as $n \to \infty$, for fixed $0 \leqq d \leqq k$ (see [7] page 91. In fact, L_1-norm convergence holds). Thus $\{vP^N\}$ is weakly sequentially-compact.

COROLLARY 2.1. *Let P be conservative. Then P is quasi-compact aperiodic if and only if for every* $u \in L_1$ *with* $\int u\,dm = 0$ *we have* $\sum_{n=0}^{\infty} ||uP^n||_1 < \infty$.

PROOF. Assume that P is quasi-compact aperiodic. Then, in L_∞, $||P^n - E|| \to 0$, where $Ef = \int f\,d\lambda$ (see Neveu [14]. λ is the invariant probability). But $uE = (\int u\,dm)u_0(u_0 = d\lambda/dm)$, so that $||P^n - E|| \to 0$ as L_1 operators, and $||P^n - E|| \leqq Kr^n$ with $0 < r < 1$. Hence, if $\int u\,dm = 0$, $||uP^n|| \leqq Kr^n||u||$, and $\sum_{n=0}^{\infty} ||uP^n||_1 < \infty$.

For the converse, note that P is quasi-compact by Theorem 2.3, since (i) is clearly satisfied. $\sum_{n=0}^{\infty} ||uP^n||_1 < \infty$ implies $||uP^n||_1 \to 0$ whenever $\int u\,dm = 0$, hence all the iterates P^k are ergodic.

COROLLARY 2.2. *If P is conservative, quasi-compact and aperiodic, then for every* $u \in L_1$ *we have* $\lim_{n\to\infty} uP^n(x) = (\int u\,dm)u_0$, (*where* $u_0P = u_0$ *and* $\int u_0\,dm = 1$).

PROOF. $\sum_{n=0}^{\infty} |uP^n|(x)$ converges a.e. for $u \in L_1$ with $\int u\,dm = 0$. If $v \in L_1$ put $u = v - (\int v\,dm)u_0$ and $uP^n(x) \to 0$ a.e.

REMARK. If P is quasi-compact, P^* may fail to be quasi-compact. Take on $X = \{0, 1, 2, \cdots\}$ $P_{i0} = \frac{1}{2}$, $P_{i,i+1} = \frac{1}{2}$, $P_{ij} = 0$ otherwise. Then $\sum_{n=0}^{j+1} P^n 1_{\{j\}}(i) \geqq 2^{-j-1}$ for every i and P is quasi-compact, with invariant measure $\lambda\{i\} = 2^{-1-i}$ for $i \geqq 0$. P^* is given by $P^*_{0j} = 2^{-j-1}$, $P^*_{i,i-1} = 1$ for $i \geqq 1$, $P^*_{ij} = 0$ otherwise, and is not quasi-compact.

3. Application to Harris operators. In this section we obtain some results for Harris operators (for their properties see Foguel [7]), using the quasi-compactness of certain induced Markov operators.

We start by generalizing a result of Butzer and Westphal [3] to nonreflexive spaces (with a shorter proof).

THEOREM 3.1. *Let T be a linear operator on a Banach space Y satisfying* $\sup_{n\geqq 0} ||T^n|| = K < \infty$, *and let* $X = Y^*$. *Then for* $x \in X$, $\sup_{N\geqq 0} ||\sum_{n=0}^{N} T^{*n}x|| < \infty$ *if and only if* $x \in (I^* - T^*)X$.

PROOF. Let $x \in X$ satisfy $\sup_{N\geqq 0} ||\sum_{n=0}^{N} T^{*n}x|| = c < \infty$. Then

$$||N^{-1} \textstyle\sum_{n=0}^{N-1} T^{*n}x|| \to 0\,.$$

Let $u_n = \sum_{i=0}^{n-1} T^{*i}x$ and let $z \in X$ be a weak-* limit point of $v_N = N^{-1} \sum_{n=1}^{N} u_n$. For $y \in Y$ there is a sequence N_j such that $\langle z, (I - T)y\rangle = \lim \langle v_{N_j}, (I - T)y\rangle$. But

$$\begin{aligned}\langle (I - T^*)z, y\rangle &= (z, (I - T)y\rangle = \lim \langle (I - T^*)v_{N_j}, y\rangle \\ &= \lim \langle x - N_j^{-1} \textstyle\sum_{n=1}^{N_j} T^{*n}x, y\rangle = \langle x, y\rangle\,.\end{aligned}$$

Thus $x = (I^* - T^*)\,z \in (I^* - T^*)X$. The converse is obvious.

COROLLARY 3.1. [3]. *Let X be a reflexive Banach space and T a linear operator with* $\sup_{n\geqq 0} ||T^n|| < \infty$. *Then* $x \in X$ *satisfies* $\sup_N ||\sum_{n=0}^{N} T^n x|| < \infty$ *if and only* $x \in (I - T)X$.

In the remainder of this section P is a conservative and ergodic Markov operator. Let $0 \leqq f \leqq 1$ be in L_∞ with $f \not\equiv 0$, denote $f' = 1 - f$ and let T_f be the operator of multiplication by f. If $f = 1_A$ we put I_A instead of T_f.

LEMMA 3.1. (i) *The operator* $P_f = \sum_{n=0}^{\infty} (PT_{f'})^n PT_f$ *is a Markov operator with* $P_f 1 = 1$.

(ii) $\sum_{n=1}^{N} P^n(T_f - T_f P_f) = (I - P^N)P_f$.

(iii) *The conservative part of* P_f *is* $A = \{x: f(x) > 0\}$ *and* $I_A P_f$ *is conservative and ergodic.*

(iv) *If* P_f *has a finite invariant measure* μ, *then* P *has a* σ*-finite invariant measure* λ *and* $\int f \, d\lambda < \infty$.

(v) *If* $\lambda \sim m$ *is a* σ*-finite invariant measure for* P *with* $\int f \, d\lambda < \infty$, *then* $\mu = \lambda T_f$ *is a finite invariant measure for* P_f.

PROOF. Except for (iii), the lemma is proved in [8]. Clearly $L_1(X)P_f \subset L_1(A)$, so that $X - A$ must be in the dissipative part of P_f, and since $L_1(A)P_f = L_1(A)I_A P_f \subset L_1(A)$, we have to show that $I_A P_f$ is conservative (and ergodic).

Let $0 \leqq g \in L_\infty(A)$ satisfy $I_A P_f g \leqq g$. Then $0 \leqq T_f g - T_f I_A P_f g = (T_f - T_f P_f)g$ and by (ii) $0 \leqq P(T_f - T_f P_f)g = (I - P)P_f g$. Since P is conservative and ergodic, we have $P_f g = c$ (a constant) a.e. Hence $g \geqq I_A P_f g \geqq c 1_A$, and

$$\begin{aligned} fg - cf \geqq 0 = c - cP_f 1 = c - cP_f 1_A = P_f(g - c1_A) \\ \geqq PT_f(g - c1_A) \doteq P(fg - cf) \, . \end{aligned}$$

Thus $P(fg - cf) = 0 = (fg - cf)$ and $g = c$ on A. Therefore $I_A P_f$ is ergodic and conservative.

REMARK. For the case $f = 1_A$ the lemma is well known.

We now investigate the quasi-compactness of $I_A P_f$. It is equivalent to the quasi-compactness of P_f, because $P_f^n = P_f(I_A P_f)^{n-1}$.

THEOREM 3.2. *Let* P *be conservative and ergodic and let* $0 \leqq f \leqq 1$, *with support* $A = \{x: f(x) > 0\}$. *Then the following conditions are equivalent*:

(i) *The operator* $I_A P_f = I_A \sum_{n=0}^{\infty} (PT_{f'})^n PT_f$ *is quasi-compact (on* $L_\infty(A)$*).*

(ii) *There exists a* σ*-finite measure* λ, $(\lambda I_A \sim m I_A)$ *with* $\int f \, d\lambda < \infty$, *such that for every* $g \in L_\infty(A)$ *with* $\int gf \, d\lambda = 0$ *we have* $\sup_{N \geqq 0} \|\sum_{n=0}^{N} P^n(fg)\|_\infty < \infty$.

PROOF. (i) $\Rightarrow$ (ii): $I_A P_f$ is conservative, ergodic and quasi-compact, so it has a finite invariant measure μ, by Theorems 2.1 and 2.2. Thus P has a σ-finite invariant measure λ (see [8]), with $\int f \, d\lambda < \infty$, and $\mu = \lambda T_f$. Thus $0 = \int fg \, d\lambda = \int g \, d\mu$. By [10] $g \in (I_A - I_A P_f)L_\infty(A)$ i.e., $g = (I_A - I_A P_f)h$ with $h \in L_\infty(A)$. Then by Lemma 3.1 $P(fg) = P(T_f - T_f P_f)h = (I - P)P_f h$. This shows that $fg = (I - P)(P_f h - fg)$ and $\|\sum_{n=0}^{N} P^n(fg)\|_\infty \leqq 2\|P_f h - fg\|_\infty$ for every $N \geqq 0$.

(ii) $\Rightarrow$ (i). Take $g \in L_\infty(A)$ with $\int fg \, d\lambda = 0$. By (ii) and Theorem 3.1 there is an $h \in L_\infty(X)$ satisfying $(I - P)h = fg$. Now $(I_A - I_A \sum_{n=0}^{N} (PT_{f'})^n PT_f)P = I_A P - I_A \sum_{n=0}^{N} (PT_{f'})^n P^2 + I_A \sum_{n=0}^{N} (PT_{f'})^n PT_{f'} P = I_A \sum_{n=0}^{N} (PT_{f'})^n P(I - P) + I_A(PT_{f'})^{N+1}P$. But $(I - P)h = fg$, and $(PT_{f'})^n 1 \to 0$ by Lemma 2.1 of [8], so that $(I_A - I_A P_f)I_A Ph = (I_A - I_A P_f)Ph = \lim_{N\to\infty} I_A \sum_{n=0}^N (PT_{f'})^n P(I-P)h + \lim_{N\to\infty} I_A(PT_{f'})^{N+1}Ph = \lim_N I_A \sum_{n=0}^N (PT_{f'})^n P(fg) = I_A P_f g$. Thus $I_A P_f g = (I_A - I_A P_f)I_A Ph$ and $(I_A - I_A P_f)(I_A Ph + g) = I_A g = g$. Since $\{g \in L_\infty(A): \int fg\, d\lambda = 0\}$ is a closed subspace of $L_\infty(A)$, $(I_A - I_A P_f)L_\infty(A)$ has closed range, and by Theorem 2.2 $I_A P_f$ is quasi-compact, being conservative and ergodic by Lemma 3.1.

REMARKS. 1. (i) $\Rightarrow$ (ii). For the case $f = 1_A$ was similarly proved first by Brunel [1] (using one of the equivalent conditions of quasi-compactness).

2. Horowitz [10] showed only that if (ii) holds for $f = 1_A$, then for certain *subsets* B of A, $I_B P_{1_B}$ is quasi-compact.

3. Following the method in [10] it can be shown that if f is "*small*" in the sense of Ornstein and Sucheston [17] (and $0 \leqq f \leqq 1$), then $I_A P_f$ is quasi-compact.

4. Let f satsify (ii) in the theorem. If $E = \{x: f(x) \geqq a\}$, then $\lambda(E) < \infty$, and $\lambda(E) > 0$ for a small enough. For $B \subset E$ define (on E) $g = f^{-1}(1_B - \lambda(B)/\lambda(E)1_E)$. Then $g \in L_\infty(A)$ and $\int fg\, d\lambda = 0$. Thus $\sup_{N\geqq 0} \|\sum_{n=0}^N P^n(1_B - \lambda(B)/\lambda(E)1_E)\|_\infty < \infty$ for $B \subset E$ and by Horowitz [10] P is Harris. (By Theorem 3.2 $I_E P_{1_E}$ is quasi-compact). Thus Theorem 3.2 implicitly assumes that P is Harris.

DEFINITION. Let P be a conservative and ergodic Markov operator. A function $0 \leqq f$ is called *special* if for every $0 \leqq h \leqq 1$ in L_∞, $h \not\equiv 0$, we have $\sum_{n=0}^\infty (PT_{h'})^n Pf \in L_\infty$. If 1_A is special then A is a *special set*.

Special functions were introduced in Neveu's remarkable paper [15], while special sets were treated by Brunel [1], who proved that A is special if and only if $L_1(I_A - I_A P_{1_A})$ is closed ($I_A P_{1_A}$ is quasi-compact). We extend this result to special functions, with a different proof, once we assume that P is Harris. Brunel's result is needed, however, to establish the fact that special functions exist only for Harris operators, which is done in the next lemma.

LEMMA 3.2. *Let P be ergodic and conservative.*

(i) *If there exists a special function $f \not\equiv 0$, then P is Harris.*

(ii) *If P is Harris, with σ-finite invariant measure λ, then $\int f\, d\lambda < \infty$ for every special function f.*

PROOF. (i) Let $f \geqq 0$ be special. Then Pf is bounded, and $Pf \not\equiv 0$. For $h \not\equiv 0$ with $0 \leqq h \leqq 1$ we have

$$\sum_{n=0}^\infty (PT_{h'})^n P(Pf) = \sum_{n=0}^\infty (PT_{h'})^n PT_{h'} Pf + \sum_{n=0}^\infty (PT_{h'})^n PT_h Pf$$
$$\leqq \sum_{n=0}^\infty (PT_{h'})^n Pf + \|f\|_\infty P_h 1 .$$

Thus $Pf \not\equiv 0$ is special and bounded. Clearly $A = \{x: Pf(x) \geqq \alpha\}$ is a special set. By Brunel [1] $I_A P_{1_A}$ is quasi-compact and thus P is Harris by Horowitz [10].

(ii) Let $0 \leqq h \leqq 1$ satisfy $\int h\, d\lambda < \infty$. Then

$$||h||_1 ||\sum_{n=0}^{\infty} (PT_{h'})^n Pf||_\infty \geqq \int h \cdot \sum_{n=0}^{\infty} (PT_{h'})^n Pf\, d\lambda$$
$$= \int f \cdot \sum_{n=0}^{\infty} (P^* T_{h'})^n P^* T_h 1\, d\lambda = \int f\, d\lambda .$$

(Lemma 3.1 is applied to the adjoint process P^*).
Some of Neveu's results [15] concerning special functions are collected in the following proposition.

PROPOSITION 3.1. *Let P be a Harris operator.*

(i) *There exists a function* $0 < h \in L_\infty$ *with* $0 < h \leqq 1$ a.e. *and a measure* $m_0 \sim m$, *such that* $\sum_{n=0}^{\infty} (PT_{h'})^n P1_A \geqq m_0(A)$ a.e. *for every* $A \in \Sigma$.

(ii) *A function* $f \geqq 0$ *is special if and only if (for the function h in* (i)) $\sum_{n=0}^{\infty} (PT_{h'})^n Pf \in L_\infty$.

(iii) *If* $\int g\, d\lambda = 0$ *and* $|g| \in L_\infty$ *is special, then* $g \in (I - P)L_\infty$.

REMARK. Neveu's results are obtained for the operators on the space $B(X, \Sigma)$ of bounded measurable functions, induced by transition probabilities, with the probabilistic Harris condition: if $m(A) > 0$ then $\sum_{n=0}^{\infty} (PI_{A'})^n P1_A(x) = 1$ for every x, and with the assumption of separability of Σ. It does not seem that Doob's method of admissible sub-σ-algebras ([5] page 209) can be used to obtain the proof in the general case. We therefore give the reduction of Proposition 3.1 to Neveu's results.

PROOF. We may and do assume $m(X) = 1$. Let X' be the compact Hausdorff space such that $C(X')$ is isometrically isomorphic to $L_\infty(X, \Sigma, m)$, and it is known that order will also be preserved (see Horowitz [10], whose results we are now using). P is then represented by an operator P' on $C(X')$ which is induced by a transition probability $P'(x', A)$, $x' \in X'$, $A \in \Sigma' =$ Baire σ-algebra of X', and m is represented by a measure m' on (X', Σ'), such that $L_\infty(x, \Sigma, m)$ and $L_\infty(x', \Sigma', m')$ are isometrically and order-preserving isomorphic, and pointwise convergence is also preserved under this isomorphism.

Since P is Harris, by Theorem 3.2 of [10] there exists a set $N \in \Sigma'$ with $m'(N) = 0$ such that if $m'(A) > 0$ then $\sum_{n=0}^{\infty} (P'I_A)^n P'1_A(x') = 1$ for $x' \notin N$. From theorem 6 of Moy [13] it follows that P' is a Harris operator on (X', Σ', m') (separability is not necessary to obtain the existence of a kernel). Let $q_n'(x', y')$ be the maximal kernel of P'^n, with corresponding integral operator Q_n'. Then $R_n' = P'^n - Q_n'$ is given by a transitive probability, and for a.e. x' $R_n'(x', \cdot)$ is singular with respect to m' [7].

Define the measures $U_\theta(x'A) = \sum_{n=1}^{\infty} (1 - \theta)^{n-1} P'^n(x', A)$ on (X', Σ'). Then

$$U_\theta(x', A) = \sum_{n=1}^{\infty} (1 - \theta)^{n-1} Q_n'(x', A) + \sum_{n=1}^{\infty} (1 - \theta)^{n-1} R_n'(x', A) .$$

Also $R_n'(x', X') \to 0$ m'-a.e. which shows that $Q_n'(x', X') \to 1$ a.e. and if

$$p_\theta(x', y') = \sum_{n=1}^{\infty} (1 - \theta)^{n-1} q_n'(x', y') ,$$

then for a.e. x', $p_\theta(x', y') > 0$ m'-a.e.

Let $N_0 = \{x' : m'\{y' : p_\theta(x', y') = 0\} > 0\}$, and let $N_1 = N_0 \cup N$. Define $N_2 = \{x' : \sum_{n=0}^{\infty} P'^n(x', N_1) > 0\}$. Since $m'(N_0) = m'(N) = 0$, also $m'(N_1) = m'(N_2) = 0$ (P' is a Markov operator on $L_\infty(X', \Sigma', m')$). Let $E = X' - N_2$. Then $m'(E) = 1$ and also $P'(y', E) = 1$ for $y' \in E$. On $(E, \Sigma' \cap E)$ we have a transition probability P'' satisfying the Harris condition and also $\sum_{n=1}^{\infty} (1-\theta)^{n-1} P''^n(x', A) \geqq \int_A p_\theta(x', y') m'(dy')$ for every $x' \in E$, with $p_\theta(x', y') > 0$ on $E \times E$. These are exactly the conditions in [15] to obtain all the results stated in Proposition 3.1 (with equalities everywhere). Going back to P' on (X', Σ') we have that Proposition 3.1 is true for P' on $L_\infty(X', \Sigma', m')$ and therefore for P on $L_\infty(X, \Sigma, m)$.

THEOREM 3.3. *Let P be an ergodic and conservative Markov operator with σ-finite invariant measure $\lambda \sim m$. Let $0 \leqq f \leqq 1$ have support $A = \{x : f(x) > 0\}$. Then f is special if and only if $I_A P_f$ is quasi-compact.*

PROOF. If $\lambda(A) = 0$ both conditions are trivailly satisfied, so we assume $\lambda(A) > 0$.

Let $I_A P_f$ be quasi-compact. Without loss of generality we may assume $\int f\, d\lambda = 1$, by Lemma 3.1. Let $0 \leqq h \leqq 1$ and $0 \not\equiv h \in L_\infty(X)$. Then $g = h1_A - (\int hf\, d\lambda) 1_A \in L_\infty(A)$ and $\int gf\, d\lambda = 0$. By Theorem 3.2 there exists an $h_1 \in L_\infty$ with $(I_A - I_A P_f) h_1 = g$ (since $d\mu = f\, d\lambda$ is the invariant measure of $I_A P_f$). Thus $PT_f g = P(T_f - T_f P_f) h_1 = (I - P) P_f h_1$ by Lemma 3.1 and

$$\begin{aligned} hf - (\textstyle\int hf\, d\lambda) f &= fg = T_f g = (I - P)(P_f h_1 + fg) \\ &= (I - T_h P - T_{h'} P)(P_f h_1 + fg)\,. \end{aligned}$$

Denote $h_2 = P_f h_1 + fg$. Then $(\int hf\, d\lambda) f = T_h(f + Ph_2) + T_{h'} Ph_2 - h_2$.

$$\int hf\, d\lambda \sum_{n=0}^{\infty} (PT_{h'})^n Pf = P_h(f + Ph_2) + \lim_{N\to\infty} (PT_{h'})^N Ph_2 - Ph_2\,.$$

Since $(PT_{h'})^N 1 \downarrow 0$ a.e. we have that if $\int hf\, d\lambda \not\equiv 0$, then $\sum_{n=0}^{\infty} (PT_{h'})^n Pf \in L_\infty$. Take $0 < h \leqq 1$ which is defined in the proposition, and by (ii) of that proposition f is special.

For the converse, assume $0 \leqq f \leqq 1$ special. The invariant measure λ satisfies $\int f\, d\lambda < \infty$. Let $g \in L_\infty(A)$ satisfy $\int gf\, d\lambda = 0$. Then $|gf|$ is special, and by part (iii) of Proposition 3.1 $gf \in (I-P)L_\infty$. Thus $\sup_{N \geqq 0} \|\sum_{n=0}^{N} P^n(fg)\|_\infty$. By Theorem 3.2 $I_A P_f$ is quasi-compact.

COROLLARY 3.2. *Let $0 \leqq f \leqq 1$. If there exists a measure m_0 with $m_0 I_A \sim m I_A$ such that for every $0 \leqq g \in L_\infty(A)$ we have $\sum_{n=0}^{\infty} (PT_{f'})^n Pg \geqq \int g\, dm_0$, then f is special.*

PROOF. For $0 \leqq h \in L_\infty(A)$ we have

$$I_A P_f h = I_A \sum_{n=0}^{\infty} (PT_{f'})^n P(fh) \geqq I_A \int fh\, dm_0 = cI_A (c > 0, \text{ if } h \not\equiv 0)$$

and by Horowitz [10] $I_A P_f$ is quasi-compact, and by Theorem 3.3, f if special.

REMARKS. 1. Neveu's Corollary 4.8 [15] is the particular case when $A = X$.

2. If $0 \leqq f$ is special with $\|f\|_\infty < 1$, then an m_0 as in the corollary must exist [15].

THEOREM 3.4. *Let P be a Harris process. Then for every two probability measures* μ, $\nu \ll \lambda$ *and for every two bounded special functions f and g, we have*

$$\lim_{N\to\infty} \sum_{n=0}^{N} \langle \mu P^n, f\rangle / \sum_{n=0}^{N} \langle \nu P^n, g\rangle = \int f\,d\lambda / \int f\,d\lambda\,.$$

(f, $g \in L_1(\lambda)$ by Lemma 3.2).

PROOF. Define $h = f - (\int f\,d\lambda/\int g\,d\lambda)g$. Then h is a special function with $\int h\,d\lambda = 0$. By Proposition 3.1 $h \in (I-P)L_\infty$. Thus for every probability measure $\mu \ll \lambda$,

$$\sup_{N\geq 0} \left|\sum_{n=0}^{N} \langle \mu P^n, f\rangle - (\int f\,d\lambda/\int g\,d\lambda) \sum_{n=0}^{N} \langle \mu P^n, g\rangle\right| \leqq \sup_N \left\|\sum_{n=0}^{N} P^n h\right\|_\infty < \infty\,.$$

Thus $\sum_{n=0}^{N} \langle \mu P^n, f\rangle / \sum_{n=0}^{N} \langle \mu P^n, g\rangle \to \int f\,d\lambda / \int g\,d\lambda$. By Lin [11] there exists a special set A such that $\sum_{n=0}^{N} \mu P^n(A)/\sum_{n=0}^{N} \nu P^n(A) \to 1$. The theorem follows by applying our result first to μ, f and 1_A and then to ν, 1_A and g.

REMARK. When P is given by a transition probability with the Harris condition, the theorem was proved (probabilistically) by Metivier [12]. However, using the previously known ratio limit theorem his result can be obtained as in our proof. (Note that his proof is based on Neveu's work [15], so separability of $(X, \sum)$ was implicitly assumed.)

COROLLARY 3.3. *Let P be Harris, with σ-finite invariant measure λ. Then there exists a probability measure* $\mu \sim \lambda$, *such that* $\lambda(A) < \infty$ *if and only if for every* $B \subset A$; $\sum_{n=0}^{N} \mu P^n(B)/\sum_{n=0}^{N} \mu P^n(A)$ *converges.*

PROOF. Since P is Harris, so is P^* (see [7]). By Proposition 3.1 there exists a bounded function $h > 0$ a.e. which is special for P^*. Let $d\mu/d\lambda = h$ (we may and do assume $\int h\,d\lambda = 1$). Then if $\lambda(A) < \infty$, we have for $B \subset A$ that

$$\lim_{N\to\infty} \sum_{n=0}^{N} \mu P^n(B)/\sum_{n=0}^{N} \mu P^n(A) = \lim_{N\to\infty} \sum_{n=0}^{N} \langle 1_B, P^{*n}h\rangle / \sum_{n=0}^{N} \langle 1_A, P^{*n}h\rangle = \lambda(B)/\lambda(A)$$

by applying the previous theorem to P^*.

For the converse, it can be proved that

$$\nu(B) = \lim_{N\to\infty} \sum_{n=0}^{N} \mu P^n(B)/\sum_{n=0}^{N} \mu P^n(A)$$

(which is a measure by the Vitali–Hahn–Saks theorem) is invariant for $I_A P_A$ (similar to the proof in [9]).

REMARK. In [8] it is indicated that not every $\mu \sim \lambda$ will necessarily satisfy the conclusion of the corollary (when $\lambda(X) = \infty$).

The remainder of this section deals with some results which use the special functions.

THEOREM 3.5. *Let P be Harris with invariant measure λ, and let* $f \in L_\infty$ *satisfy* $\int f\,d\lambda = 0$ *and* $|f|$ *special.*

(i) *If* $\lambda(X) = 1$ *and P has period d, then* $\lim_{N\to\infty} \sum_{n=0}^{Nd+r} P^n f(x)$ *exists* a.e., *for fixed* $0 \leqq r < d$.

(ii) *If* $\lambda(x) = \infty$, *then for every* $v \in L_1(m)$ *with* $\int v\,dm = 0$ *the limit* $\lim_{N\to\infty} \sum_{n=0}^{N} \langle v, P^n f\rangle$ *exists.*

PROOF. (i) By Proposition 3.1 $f=(I-P)g$ for some $g \in L_\infty$, hence $\sum_{n=0}^{Nd+r} P^n f = g - P^{Nd+r}g$. Since $g \in L_\infty$, we have that $P^{Nd+r}g$ converges a.e. [7].

(ii) Again $f = (I - P)g$ and $\|\sum_{n=0}^{N} P^n f\|_\infty \leqq 2\|g\|_\infty$ for every N. Fix $v \in L_1(m)$ with $\int v\,dm = 0$. Let N_i be a sub-sequence such that $\sum_{n=0}^{N_i} \langle v, P^n f\rangle$ converges. Define a linear functional on $L_1(m)$ by a Banach limit (and represent it by $h \in L_\infty$): $\langle u, h\rangle = \mathrm{LIM}\{\sum_{n=0}^{N_i} \langle u, P^n f\rangle\}$. Then

$$\begin{aligned}\langle u, (I-P)h\rangle &= \langle u(I-P), h\rangle = \mathrm{LIM}\{\textstyle\sum_{n=0}^{N_i} \langle u(I-P), P^n f\rangle\} \\ &= \mathrm{LIM}\{\langle u, f\rangle - \langle u, P^{N_i+1}f\rangle\} = \langle u, f\rangle .\end{aligned}$$

Since $P^n f(x) \to 0$ a.e. (by [7] we have $P^n 1_A(x) \to 0$ a.e. when $\lambda(A) < \infty$. But $|f| \in L_\infty \cap L_1(\lambda)$ so that for every $\varepsilon > 0$, $A = \{x\colon |f|(x) \geqq \varepsilon\}$ has finite λ measure, and $P^n|f| \leqq \varepsilon + \|f\|_\infty P^n 1_A \to \varepsilon$. Hence $|P^n f| \leqq P^n|f| \to 0$ a.e.).

Thus $(I - P)h = f$ and $h = g +$ const. Since $\int v\,dm = 0$, we have

$$\lim_{N_i} \sum_{n=0}^{N_i} \langle v, P^n f\rangle = \langle v, h\rangle = \langle v, g\rangle,$$

so that the limit does not depend on $\{N_i\}$, and the convergence is proved.

COROLLARY 3.4. *Let P be Harris, and* $\lambda(X) = \infty$. *Let* $f \in L_\infty$ *with* $|f|$ *special satisfy* $\int f\,d\lambda = 0$. *Then* $\sum_{n=0}^{N} P^n f(x)$ *converges* a.e. *if and only if for some* $0 \leqq u \in L_1(m)$ (*with* $\int u\,dm > 0$) $\sum_{n=0}^{N} \langle u, P^n f\rangle$ *converges.*

PROOF. (i) Assume $\sum_{n=0}^{N} P^n f(x)$ converges a.e. Since $\|\sum_{n=0}^{N} P^n f(x)\|_\infty \leqq K$ for every N, we can apply the bounded convergence theorem.

(ii) Assume $\sum_{n=0}^{N} \langle uP^n, f\rangle$ converges for some $u \in L_1$, $\int u\,dm \neq 0$. Then from the theorem we have that $\sum_{n=0}^{N} \langle uP^n, f\rangle$ converges for *every* $u \in L_1(m)$. The fact that $\sum_{n=0}^{N} P^n f(x)$ is a Cauchy sequence a.e. can now be proved using the (standard) technique of ([11] page 364).

COROLLARY 3.5. *Let* $\{p_{ij}^{(n)}\}$ *be the n-step transition probabilities of an irreducible chain.*

(i) *If the chain is positive-recurrent with period d, and invariant probability* $\{\lambda_i\}$, *then for every four points i, j, k, t the limit* $\lim_{N\to\infty} \sum_{n=0}^{Nd+r} \{\lambda_k p_{ij}^{(n)} - \lambda_j p_{tk}^{(n)}\}$ *exists* $(0 \leqq r < d)$.

(ii) *If the chain is null-recurrent with invariant measure* $\{\lambda_i\}$ *then for every four points i, j, k, t the sequence* $\sum_{n=0}^{N} \{\lambda_k(p_{ij}^{(n)} - p_{tj}^{(n)}) - \lambda_j(p_{ik}^{(n)} - p_{tk}^{(n)})\}$ *converges.*

PROOF. The operator P induced by $p_{ij}^{(1)}$ is Harris, and so is P^*, and finite sets are special. (i) follows from applying Theorem 3.5 both to P and P^* (which have the same period). (ii) follows from Theorem 3.5.

REMARKS. 1. (i) is proved by Orey [16] using generating functions (even absolute convergence is shown). See also Chung ([4] pages 66–69).

2. (ii) is proved by Chung ([4] page 69) only with $t = j$ and with certain restrictions, but there the limit is also identified. Orey [16] shows that even for a chain the conditions of Corollary 3.4 may not be satisfied.

THEOREM 3.6. *Let P be conservative and ergodic with invariant probability measure λ. If the space $\{V \in Ba(X, \Sigma, m): \langle V, Pf\rangle = \langle V, f\rangle$ for $f \in L_\infty\}$ of finitely additive invariant measures (charges) is separable, then P is Harris.*

PROOF. Given a pure charge V, we can find sets A_ε with $V(A_\varepsilon) = 0$ and $\lambda(A_\varepsilon) > 1 - \varepsilon$, for every $\varepsilon > 0$ (see [7]). If V_i is a dense sequence in the space of invariant charges, then $V_i = \alpha_i \lambda + V_{0i}$ with V_{0i} a pure charge. Find A_i such that $V_{0i}(A_i) = 0$ and $\lambda(A_i) \geqq 1 - \varepsilon 2^{-i}$. Let $A = \bigcap_{i=1}^{\infty} A_i$. Then $\lambda(A) \geqq 1 - \varepsilon$ and $V_{0i}(A) = 0$ for every i. For $B \subset A$ we put $f = 1_B - (\lambda(B)/\lambda(A))1_A$. Then $V_i(f) = 0$ for every i and therefore $V(f) = 0$ for every $V \in Ba(X, \Sigma, m)$ with $VP = V$. Thus $\|N^{-1} \sum_{n=0}^{N-1} P^n f\|_\infty \to 0$ (by the Hahn–Banach theorem). By the ergodic theorem $N^{-1} \sum_{n=0}^{N-1} P^n 1_A(x) \to \lambda(A)$ a.e. and by Egorov's theorem there is a set $A_1 \subset A$ such that this convergence is uniform. Let $B \subset A_1 \subset A$. For $x \in A_1$ we have $N^{-1} \sum_{n=0}^{N-1} P^n 1_B(x) \geqq (\lambda(B)/\lambda(A))N^{-1} \sum_{n=0}^{N-1} P^n 1_A(x) - \delta \geqq \lambda(B)(1-\delta) - \delta$ if $N \geqq N_0(\delta)$. If δ is small enough, $c = \lambda(B)(1 - \delta) - \delta > 0$ and $\sum_{n=0}^{N-1} P^n 1_B \geqq Nc1_{A_1}$ for $B \subset A_1$. By Horowitz [10] ($I_{A_1} P_{A_1}$ is quasi-compact and) P is Harris.

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