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LIMIT THEOREMS FOR DELAYED SUMS[1]

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In this paper, we study analogues of the law of the iterated logarithm for delayed sums of independent random variables. In the i.i.d. case, necessary and sufficient conditions for such analogues are obtained. We apply our results to find convergence rates for expressions of the form $P[|S_n| > b_n]$ and $P[\sup_{k \geqq n} |S_k/b_k| > \varepsilon]$ for certain upper-class sequences (b_n). In this connection, certain theorems of Erdös, Baum and Katz are also generalized.

1. Introduction. Let (a_n) be a sequence of real numbers and let (k_n) be a sequence of positive integers. The numbers

$$\rho_{n,k_n} = \{\sum_{j=1}^{k_n} a_{n+j-1}\}/k_n$$

are called the (*forward*) *delayed first arithmetic means* (cf. [13] page 80). Such delayed averages have been studied in connection with summability methods (cf. [1], [9], [13]). Recently, using the limiting behavior of delayed averages, Chow [3] has found necessary and sufficient conditions for the Borel summability of i.i.d. random variables. Making use of delayed averages, Chow [3] has also obtained very simple proofs of a number of well-known results such as the Hsu–Robbins–Spitzer–Katz theorem which states that if $X_1, X_2, \cdots$ are i.i.d. with $EX_1 = 0$, $E|X_1|^p < \infty$ $(p \geqq 1)$, then $\sum_1^\infty n^{p-2}P[S_n^* > n] < \infty$, where $S_n = X_1 + \cdots + X_n$ and $S_n^* = \max_{1 \leqq j \leqq n} |S_j|$.

Suppose $X_1, X_2, \cdots$ are i.i.d. random variables with $EX_1 = 0$ and $E|X_1|^p < \infty$ for some $p \geqq 1$. Let $S_{n,j} = X_n + \cdots + X_{n+j-1}$, $S_{n,t}^* = \max_{1 \leqq j \leqq t} |S_{n,j}|$. Chow's theorem states that

$$(1) \qquad \lim_{n\to\infty} n^{-1/p} S_{n,n^\alpha}^* = 0 \quad \text{a.e.} \quad \text{for every} \quad 0 < \alpha < \min(2/p, 1)\,.$$

$$(2) \qquad \lim_{n\to\infty} n^{-1/p} S_{n,n}^* = 0 \quad \text{a.e.} \quad \text{if} \quad 2 > p \geqq 1\,.$$

In this paper, we shall consider the rate at which the convergence in (1) or (2) takes place. In other words, we want to find analogues of the law of the iterated logarithm for the forward delayed sums S_{n,k_n}. Obviously, we have corresponding results for the backward delayed sums of the form $X_{n-k_n+1} + \cdots + X_n$. Theorem 1 below sheds light on analogues of the law of the iterated logarithm for certain summability methods for independent random variables. This and related problems will be treated in [7].

An interesting choice of the sequence (k_n) is $k_n = k$ for all n. In this case,

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the backward delayed average $(X_{n-k+1} + \cdots + X_n)/k$ is the usual moving average used in time series analysis and other statistical applications. Suppose $X_1, X_2, \cdots$ are i.i.d. $N(0, \sigma^2)$ random variables. Then

$$\text{(3)} \qquad \limsup_{n\to\infty} |X_{n-k+1} + \cdots + X_n|/(2k\sigma^2 \log n)^{\frac{1}{2}} = \limsup_{n\to\infty} |X_n + \cdots + X_{n+k-1}|/(2k\sigma^2 \log n)^{\frac{1}{2}} = 1 \quad \text{a.e.}$$

On the other hand, if $|X_n| \leqq C$ for all n, then obviously $|X_n + \cdots + X_{n+k-1}| \leqq kC$ for all n. In general, if $X_1, X_2, \cdots$ are i.i.d. random variables and f is a strictly increasing function on $[0, \infty)$ with $\lim_{t\to\infty} f(t) = \infty$, then

$$\text{(4)} \qquad \begin{aligned} &\limsup_{n\to\infty} |X_{n-k+1} + \cdots + X_n|/f(n) = 1 \quad \text{a.e.} \\ &\Leftrightarrow \limsup_{n\to\infty} |X_n + \cdots + X_{n+k-1}|/f(n) = 1 \quad \text{a.e.} \\ &\Leftrightarrow Ef^{-1}(t|X_1 + \cdots + X_k|) < \infty \quad \text{for} \quad t < 1 \quad \text{and} \\ &\quad\; Ef^{-1}(t|X_1 + \cdots + X_k|) = \infty \quad \text{for} \quad t > 1 \,. \end{aligned}$$

A completely different limiting behavior occurs when we take $k_n = [\alpha n]$, $\alpha > 0$. As in the usual law of the iterated logarithm for S_n, it can be shown that if $X_1, X_2, \cdots$ are i.i.d. random variables, then

$$\text{(5)} \qquad \begin{aligned} &\limsup_{n\to\infty} |S^*_{n,\alpha n}|/\{2\alpha n \log\log n\}^{\frac{1}{2}} = \sigma \quad \text{a.e.} \\ &\Leftrightarrow \limsup_{n\to\infty} |X_n + \cdots + X_{n+[\alpha n]-1}|/\{2\alpha n \log\log n\}^{\frac{1}{2}} = \sigma \quad \text{a.e.} \\ &\Leftrightarrow EX_1 = 0 \,, \qquad EX_1^2 = \sigma^2 \,. \end{aligned}$$

When $\alpha = 1$, (5) gives the rate at which the convergence in (2) takes place. To see the second equivalence in (5), we assume $EX_1 = 0$, $EX_1^2 = 1$ and make use of a theorem of Strassen [10]: With probability one, the sequence $(\zeta_n, n \geqq 3)$ is relatively compact in $C[0, 1 + \alpha]$ and its set of limit points in $C[0, 1 + \alpha]$ coincides with the set K, where

$$\begin{aligned} \zeta_n(t) &= (2n \log\log n)^{-\frac{1}{2}}\{([nt] + 1 - nt)S_{[nt]} + (nt - [nt])S_{[nt]+1}\} \,; \\ K &= \{h \in C[0, 1 + \alpha] : h \text{ is absolutely continuous, } h(0) = 0 \\ &\qquad \text{and } \textstyle\int_0^{1+\alpha} (dh/dt)^2\, dt \leqq 1\} \,. \end{aligned}$$

Now $\sup_{h \in K} |h(1 + \alpha) - h(1)| = \alpha^{\frac{1}{2}}$, where the supremum is attained by the function h_0 such that $h_0(t) = 0$, $0 \leqq t \leqq 1$ and $h_0(t) = \alpha^{-\frac{1}{2}}(t - 1)$, $1 \leqq t \leqq 1 + \alpha$. From this, it then follows that

$$\limsup_{n\to\infty} |X_n + \cdots + X_{n+[\alpha n]-1}|/\{2\alpha n \log\log n\}^{\frac{1}{2}} = 1 \quad \text{a.e.}$$

Conversely, if the above limiting relation holds almost everywhere, then by using an argument due to Feller [5], we can prove that $EX_1 = 0$ and $EX_1^2 = 1$.

In the case $X_1, X_2, \cdots$ are i.i.d. $N(0, \sigma^2)$ random variables. (3) states that $\limsup_{n\to\infty} |X_n + \cdots + X_{n+k_n-1}|/(2k_n \log n)^{\frac{1}{2}} = \sigma$ a.e. if $k_n = k$, while (5) states that $\limsup_{n\to\infty} |X_n + \cdots + X_{n+k_n-1}|/(2k_n \log\log n)^{\frac{1}{2}} = \sigma$ a.e. if $k_n = [\alpha n]$. Between these two extremes, there is a whole hierarchy of limiting relations characterized by

$$\text{(6)} \qquad \limsup_{n\to\infty} |X_n + \cdots + X_{n+k_n-1}|/\{2(1 - \alpha)k_n \log n\}^{\frac{1}{2}} = \sigma \quad \text{a.e.} \,,$$

corresponding to $k_n = [n^\alpha]$, $0 < \alpha < 1$ (cf. [6]). In the case where $X_1, X_2, \cdots$ are i.i.d. (but not necessarily normal), we shall find in Section 2 necessary and sufficient conditions for (6) to hold. In Section 3, we shall use our results to obtain convergence rates for expressions of the form $P[|S_n| > b_n]$ for some upper-class sequences (b_n). Certain results of Erdös [4], Baum and Katz [2] are also generalized.

2. Limit theorems for delayed sums. In this section, we first prove (6) for independent random variables whose moment generating functions satisfy certain exponential inequalities. Then by a truncation argument, we obtain (6) for i.i.d. random variables satisfying certain moment conditions which we prove to be both necessary and sufficient.

THEOREM 1. *Let $X_1, X_2, \cdots$ be independent random variables such that $EX_n = 0$, $EX_n^2 = \sigma_n^2$ and $\lim_{n\to\infty} \sigma_n^2 = \sigma^2 > 0$. Let $0 < \alpha < 1$. Suppose for $j \geqq j_0$, there exists $\gamma_j \geqq 0$ such that $\gamma_j = o(j^{\alpha/2}(\log j)^{-\frac{1}{2}})$ and*

$$(7) \qquad \exp\{t^2\sigma_j^2(1 - |t|\gamma_j)/2\} \leqq E\exp(tX_j) \leqq \exp\left\{t^2\sigma_j^2\left(1 + \frac{|t|}{2}\gamma_j\right)\Big/2\right\}$$

whenever $|t|\gamma_j \leqq 1$. Then

$$(8) \qquad \limsup_{n\to\infty} (X_n + \cdots + X_{n+[n^\alpha]})/\{2(1-\alpha)n^\alpha \log n\}^{\frac{1}{2}} = \sigma \quad \text{a.e.}$$

$$(9) \qquad \liminf_{n\to\infty} (X_n + \cdots + X_{n+[n^\alpha]})/\{2(1-\alpha)n^\alpha \log n\}^{\frac{1}{2}} = -\sigma \quad \text{a.e.}$$

$$(10) \qquad \limsup_{n\to\infty} \max_{0\leqq j\leqq n^\alpha} |X_n + \cdots + X_{n+j}|/\{2(1-\alpha)n^\alpha \log n\}^{\frac{1}{2}} = \sigma \quad \text{a.e.}$$

PROOF. We first prove that the lim sup in (8) is $\geqq \sigma$ a.e. To do this, we shall make use of Kolmogorov's (lower) exponential bounds (cf. [8], [12]): Given any $\delta > 0$, $\exists \varepsilon_0, \varepsilon_1$ (depending only on δ) such that if $\varepsilon \geqq \varepsilon_0$ and $\varepsilon c \leqq \varepsilon_1$, then

$$(11) \qquad P[S > \varepsilon] \geqq \exp\{-\varepsilon^2(1+\delta)/2\}$$

for any $c > 0$ and any random variable S satisfying

$$(12) \qquad \exp\{t^2(1 - tc)/2\} \leqq Ee^{tS} \leqq \exp\left\{t^2\left(1 + \frac{t}{2}c\right)\Big/2\right\}$$

for all $t > 0$ with $tc \leqq 1$.

Choose $\lambda > 0$ such that $\lambda^\alpha < \lambda/(1-\alpha)$, and set $m_k = [\lambda k^{1/(1-\alpha)}]$. Let $0 < \delta < \delta' < 1$ such that $(1-\delta')^2(1+\delta) < 1$. For the given δ, there exist ε_0 and ε_1 given by the Kolmogorov exponential bounds. Setting $c_k = (\varepsilon_1/(1-\delta'))(2\log k)^{-\frac{1}{2}}$, $s_k^2 = \sum_{j=0}^{[m_k^\alpha]} \sigma_{m_k+j}^2$ and $Z_k = (1/s_k)(X_{m_k} + \cdots + X_{m_k+[m_k^\alpha]})$, we obtain for k sufficiently large and $|t|c_k \leqq 1$,

$$(13) \qquad \exp\{t^2(1 - |t|c_k)/2\} \leqq E\exp(tZ_k) \leqq \exp\left\{t^2\left(1 + \frac{|t|}{2}c_k\right)\Big/2\right\}.$$

Letting $\varepsilon = (1-\delta')(2\log k)^{\frac{1}{2}}$ in (11), we have for all large k,

$$(14) \qquad \begin{aligned} &P[X_{m_k} + \cdots + X_{m_k+[m_k^\alpha]} \geqq (1-\delta)\sigma\{2(1-\alpha)m_k^\alpha \log m_k\}^{\frac{1}{2}}] \\ &\geqq P[Z_k \geqq (1-\delta')(2\log k)^{\frac{1}{2}}] = P[Z_k \geqq \varepsilon] \\ &\geqq \exp\{-\varepsilon^2(1+\delta)/2\} = \exp\{-(1+\delta)(1-\delta')^2 \log k\}. \end{aligned}$$

From the fact that $\lambda^\alpha < \lambda/(1-\alpha)$, it is easy to see that the σ-fields $\mathscr{F}_k = \mathscr{B}(X_j : m_k \leqq j \leqq m_k + [m_k{}^\alpha])$ are independent for all large k. Hence it follows from (14) and the Borel–Cantelli lemma that

$$(15)\qquad \limsup_{k\to\infty} (X_{m_k} + \cdots + X_{m_k+[m_k{}^\alpha]})/\{2(1-\alpha)m_k{}^\alpha \log m_k\}^{\frac12} \geqq (1-\delta)\sigma \quad \text{a.e.}$$

Since δ is arbitrary, we have

$$(16)\qquad \limsup_{n\to\infty} (X_n + \cdots + X_{n+[n^\alpha]})/\{2(1-\alpha)n^\alpha \log n\}^{\frac12} \geqq \sigma \quad \text{a.e.}$$

Replacing X_n by $-X_n$ in the above argument, we obtain

$$(17)\qquad \liminf_{n\to\infty} (X_n + \cdots + X_{n+[n^\alpha]})/\{2(1-\alpha)n^\alpha \log n\}^{\frac12} \leqq -\sigma \quad \text{a.e.}$$

Given $0 < \delta < 1$, we shall now show that

$$(18)\qquad \limsup_{n\to\infty} (\max_{0\leqq j\leqq n^\alpha} |X_n + \cdots + X_{n+j}|/\{2(1-\alpha)n^\alpha \log n\}^{\frac12}) \leqq (1+\delta)\sigma + \delta \quad \text{a.e.}$$

To prove (18), we shall use Kolmogorov's (upper) exponential bounds (cf. [8], [12]): If S is a random variable such that $Ee^{tS} \leqq \exp\{t^2(1+(t/2)c)/2\}$ for some $c < 0$ and all $0 < tc \leqq 1$, then

$$(19)\qquad P[S > \varepsilon] \leqq \exp\left\{-\varepsilon^2\left(1 - \frac{\varepsilon}{2}c\right)\Big/2\right\} \qquad \text{if} \quad 0 < \varepsilon c \leqq 1\,.$$

Let $n_k = [\eta_k k^{1/(1-\alpha)}]$, where $\eta_k = (\log k)^{-1/(1-\alpha)}$. By the Lévy inequality ([8], page 248),

$$\begin{aligned}
P[\max_{0\leqq j\leqq n_{k+1}^\alpha} &|X_{n_k} + \cdots + X_{n_k+j}| \geqq (1+\delta)\sigma\{2(1-\alpha)n_k{}^\alpha \log n_k\}^{\frac12}]\\
&\leqq 2P[|X_{n_k} + \cdots + X_{n_k+[n_{k+1}^\alpha]}| \geqq (1+\delta)\sigma\{2(1-\alpha)n_k{}^\alpha \log n_k\}^{\frac12}\\
&\qquad - (2\textstyle\sum_{j=0}^{[n_{k+1}^\alpha]} \sigma^2_{n_k+j})^{\frac12}]\\
&\leqq 2P\left[|X_{n_k} + \cdots + X_{n_k+[n_{k+1}^\alpha]}| \geqq \left(1+\frac{\delta}{2}\right)\sigma(2n_k{}^\alpha \log k)^{\frac12}\right]\\
&\hspace{20em}\text{for all large } k\\
&\leqq 2P\left[Y_k \geqq \left(1+\frac{\delta}{4}\right)(2\log k)^{\frac12}\right] + 2P\left[-Y_k \geqq \left(1+\frac{\delta}{4}\right)(2\log k)^{\frac12}\right]\\
&\qquad (\text{where } Y_k = \{\textstyle\sum_{j=0}^{[n_{k+1}^\alpha]} \sigma^2_{n_k+j}\}^{-\frac12}(X_{n_k} + \cdots + X_{n_k+[n_{k+1}^\alpha]}))\\
&\leqq 4\exp\left\{-\left(1+\frac{\delta}{4}\right)^2\left(1-\frac{\delta}{8}\right)\log k\right\},\\
&\hspace{14em}\text{by the exponential bound (19).}
\end{aligned}$$

Since $\delta < 1$, it follows from the Borel–Cantelli lemma that

$$(20)\qquad \limsup_{k\to\infty} (\max_{0\leqq j\leqq n_{k+1}^\alpha} |X_{n_k} + \cdots + X_{n_k+j}|/\{2(1-\alpha)n_k{}^\alpha \log n_k\}^{\frac12}) \leqq (1+\delta)\sigma \quad \text{a.e.}$$

Now pick $M > 1$ such that $M\delta > (1 + \alpha)/(1 - \alpha)$. Setting $t = M(\log k)^{\frac{1}{2}}$, we have for $n_k < n \leqq n_{k+1}$, $n - n_k \leqq j \leqq n_{k+1}^{\alpha}$ and all large k,

$$\begin{aligned}P[(X_n + \cdots + X_{n+j}) - (X_{n_k} + \cdots + X_{n_k+j}) > \delta\{2(1-\alpha)n_k^{\alpha}\log n_k\}^{\frac{1}{2}}] \\ \leqq P[tn_k^{-\alpha/2}(-X_{n_k} - \cdots - X_{n-1} + X_{n_k+j+1} + \cdots + X_{n+j}) > t\delta(\log k)^{\frac{1}{2}}] \\ \leqq \{\exp(-t\delta(\log k)^{\frac{1}{2}})\} \\ \times E\exp\{tn_k^{-\alpha/2}(-X_{n_k} - \cdots - X_{n-1} + X_{n_k+j+1} + \cdots + X_{n+j})\} \\ \leqq k^{-M\delta}\exp\{2\sigma^2 t^2 n_k^{-\alpha}(n_{k+1} - n_k)\}, \quad \text{by (7).}\end{aligned}$$

In the case $0 \leqq j < n - n_k$, we have

$$\begin{aligned}P[(X_n + \cdots + X_{n+j}) - (X_{n_k} + \cdots + X_{n_k+j}) > \delta\{2(1-\alpha)n_k^{\alpha}\log n_k\}^{\frac{1}{2}}] \\ \leqq \{\exp(-t\delta(\log k)^{\frac{1}{2}})\} \\ \times E\exp\{tn_k^{-\alpha/2}(X_n + \cdots + X_{n+j} - X_{n_k} - \cdots - X_{n_k+j})\} \\ \leqq k^{-M\delta}\exp\{2t^2\sigma^2 n_k^{-\alpha}(j+1)\} \\ \leqq k^{-M\delta}\exp\{2t^2\sigma^2 n_k^{-\alpha}(n_{k+1} - n_k)\} = O(k^{-M\delta}).\end{aligned}$$

The last relation above follows from the fact that $n_{k+1} - n_k \sim \eta_k k^{\alpha/(1-\alpha)}$, while $n_k^{\alpha} \sim \eta_k^{\alpha} k^{\alpha/(1-\alpha)}$ and $\eta_k^{1-\alpha}\log k = 1$. In a similar way, we can prove that in either case,

$$\begin{aligned}P[(X_n + \cdots + X_{n+j}) - (X_{n_k} + \cdots + X_{n_k+j}) < -\delta\{2(1-\alpha)n_k^{\alpha}\log n_k\}^{\frac{1}{2}}] \\ = O(k^{-M\delta}).\end{aligned}$$

Therefore for all large k,

$$\begin{aligned}\textstyle\sum_{n=n_k+1}^{n_{k+1}} \sum_{j=0}^{n_{k+1}^{\alpha}} P[|(X_n + \cdots + X_{n+j}) - (X_{n_k} + \cdots + X_{n_k+j})| \\ > \delta\{2(1-\alpha)n_k^{\alpha}\log n_k\}^{\frac{1}{2}}] \\ \leqq Ck^{-M\delta}(n_{k+1} - n_k)n_{k+1}^{\alpha} = O(k^{-M\delta+2\alpha/(1-\alpha)}).\end{aligned}$$

Since $M\delta - 1 > 2\alpha/(1-\alpha)$, we obtain by the Borel–Cantelli lemma that

$$\begin{aligned}\limsup_{k\to\infty}\{2(1-\alpha)n_k^{\alpha}\log n_k\}^{-\frac{1}{2}} \\ \times (\max_{n_k<n\leqq n_{k+1}}\max_{0\leqq j\leqq n_{k+1}^{\alpha}}|(X_n + \cdots + X_{n+j}) \\ - (X_{n_k} + \cdots + X_{n_k+j})|) \leqq \delta \quad \text{a.e.}\end{aligned} \tag{21}$$

From (20) and (21), the assertion (18) follows. □

LEMMA 1. *Let $Y_1, Y_2, \cdots, Z_1, Z_2, \cdots$ be random variables such that for each n, Z_n is independent of $(Y_1, \cdots, Y_n)$. Suppose $Z_n \to_P 0$. Then for any real number c,*

$$\limsup_{n\to\infty}(Y_n + Z_n) \leqq c \quad \text{a.e.} \Rightarrow \limsup_{n\to\infty} Y_n \leqq c \quad \text{a.e.} \tag{22}$$

$$\liminf_{n\to\infty}(Y_n + Z_n) \geqq c \quad \text{a.e.} \Rightarrow \liminf_{n\to\infty} Y_n \geqq c \quad \text{a.e.} \tag{23}$$

$$\limsup_{n\to\infty}|Y_n + Z_n| \leqq c \quad \text{a.e.} \Rightarrow \limsup_{n\to\infty}|Y_n| \leqq c \quad \text{a.e.} \quad \textit{and} \quad \limsup_{n\to\infty}|Z_n| \leqq 2c \quad \text{a.e.} \tag{24}$$

PROOF. Given any $\varepsilon > 0$, let $\tau_m = \inf\{n \geqq m : Y_n \geqq c + 2\varepsilon\}$. Let $P[|Z_n| < \varepsilon] \geqq \frac{1}{2}$ for $n \geqq m_0$. Then for $m \geqq m_0$,

$$\begin{aligned} P(\textstyle\bigcup_{n=m}^{\infty} [Y_n + Z_n \geqq c + \varepsilon]) &\geqq P(\textstyle\bigcup_{n=m}^{\infty} [\tau_m = n, |Z_n| < \varepsilon]) \\ &= \textstyle\sum_{n=m}^{\infty} P[\tau_m = n] P[|Z_n| < \varepsilon], \\ &\qquad \text{since } Z_n \text{ is independent of } (Y_1, \cdots, Y_n) \\ &\geqq \tfrac{1}{2} \textstyle\sum_{n=m}^{\infty} P[\tau_m = n] = \tfrac{1}{2} P(\textstyle\bigcup_{n=m}^{\infty} [Y_n \geqq c + 2\varepsilon]) . \end{aligned} \tag{25}$$

If $\limsup_{n\to\infty} (Y_n + Z_n) \leqq c$ a.e., then $\lim_{m\to\infty} P(\bigcup_{n=m}^{\infty} [Y_n + Z_n \geqq c + \varepsilon]) = 0$, and so it follows from (25) that $\lim_{m\to\infty} P(\bigcup_{n=m}^{\infty} [Y_n \geqq c + 2\varepsilon]) = 0$. Hence $\limsup_{n\to\infty} Y_n \leqq c + 2\varepsilon$ a.e. Since ε is arbitrary, we obtain (22). In a similar way, we can prove (23), and (24) follows easily from (22) and (23). □

THEOREM 2. *Let $X_1, X_2, \cdots$ be* i.i.d. *random variables and let $\sigma > 0$, $0 < \alpha < 1$. Then the following statements are equivalent*:

(26) $EX_1 = 0$, $EX_1^2 = \sigma^2$ *and* $E|X_1|^{2/\alpha}(\log^+ |X_1| + 1)^{-1/\alpha} < \infty$.

(27) *The limit relations* (8) *and* (9) *both hold.*

(28) *The limit relation* (10) *holds.*

(29) $\limsup_{n\to\infty} |X_1 + \cdots + X_n| / \{2n \log\log n\}^{\frac{1}{2}} = \sigma$ a.e. *and* $\lim_{n\to\infty} n^{-\alpha/2}(\log n)^{-\frac{1}{2}} X_n = 0$ a.e.

PROOF. It is well known that (26) and (29) are equivalent. Now assume (26). To prove both (27) and (28), given $\delta > 0$, we choose an integer $k > 1$ such that $k - \alpha k > 1$ and pick $\varepsilon > 0$ such that $\varepsilon k < \delta$. Define

$$\begin{aligned} X_n' &= X_n I_{[|X_n| \geqq \varepsilon n^{\alpha/2} (\log n)^{\frac{1}{2}}]} \\ X_n'' &= X_n I_{[|X_n| \leqq n^{\alpha/2}/(\log n)]} \\ X_n''' &= X_n - X_n' - X_n'' . \end{aligned}$$

Let $U_n = X_n'' - EX_n''$. Then $EU_n = 0$, $EU_n^2 = \sigma_n^2 \to \sigma^2$ and $|U_n| \leqq 2n^{\alpha/2}/(\log n) = o(n^{\alpha/2}(\log n)^{-\frac{1}{2}})$. Hence the sequence U_j satisfies the hypothesis of Theorem 1 (cf. [8] page 255), and so with probability one,

$$\begin{aligned} \limsup_{n\to\infty} (U_n + \cdots + U_{n+[n^\alpha]}) / \{2(1-\alpha) n^\alpha \log n\}^{\frac{1}{2}} \\ = \sigma \\ = \limsup_{n\to\infty} \max_{0 \leqq j \leqq n^\alpha} |U_n + \cdots + U_{n+j}| / \{2(1-\alpha) n^\alpha \log n\}^{\frac{1}{2}} , \end{aligned} \tag{30}$$

$$\liminf_{n\to\infty} (U_n + \cdots + U_{n+[n^\alpha]}) / \{2(1-\alpha) n^\alpha \log n\}^{\frac{1}{2}} = -\sigma . \tag{31}$$

We note that since $EX_1 = 0$, we have for all large n,

$$\begin{aligned} E|X_n''| &= |\textstyle\int_{[|X_1| > n^{\alpha/2}/(\log n)]} X_1 \, dP| \\ &= O(n^{\alpha/2 - 1} (\log n)^{3/\alpha - 1}) . \end{aligned}$$

Therefore

$$\lim_{n\to\infty} \textstyle\sum_{j=0}^{[n^\alpha]} E|X_{n+j}''| / \{n^\alpha \log n\}^{\frac{1}{2}} = 0 . \tag{32}$$

Since $|X_n'''| < \varepsilon n^{\alpha/2}(\log n)^{\frac{1}{2}}$, we have for all large n

$$\begin{aligned} P[|X_n'''| + \cdots + |X'''_{n+[n^\alpha]}| &\geqq \delta(n^\alpha \log n)^{\frac{1}{2}}] \\ &\leqq P[\textstyle\sum_{j=0}^{[n^\alpha]} |X'''_{n+j}|/\{\varepsilon n^{\alpha/2}(\log n)^{\frac{1}{2}}\} \geqq k] \\ &\leqq P[X'''_{n+j} \neq 0 \text{ for at least } k \text{ of the indices } j = 0, \cdots, [n^\alpha]] \\ &\leqq \tbinom{[n^\alpha]+1}{k} P^k[|X_1|^{2/\alpha}(\log^+|X_1| + 1)^{-1/\alpha} \geqq n(\log n)^{-3/\alpha}] \\ &= O(n^{\alpha k}\{(\log n)^{3/\alpha}/n\}^k) , \qquad \text{by the Markov inequality.} \end{aligned}$$

Since $k - \alpha k > 1$, an application of the Borel–Cantelli lemma gives

$$\limsup_{n\to\infty} \textstyle\sum_{j=0}^{[n^\alpha]} |X'''_{n+j}|/\{n^\alpha \log n\}^{\frac{1}{2}} \leqq \delta \quad \text{a.e.} \tag{33}$$

As δ as arbitrary, we obtain (27) and (28) from (30), (31), (32) and (33), noting that $P[X_n' \neq 0 \text{ i.o.}] = 0$.

We now show that (27) implies (26). Define

$$\begin{aligned} Y_n &= (X_n + \cdots + X_{n+[n^\alpha]-1})/\{2(1-\alpha)n^\alpha \log n\}^{\frac{1}{2}} , \\ Z_n &= X_{n+[n^\alpha]}/\{2(1-\alpha)n^\alpha \log n\}^{\frac{1}{2}} . \end{aligned}$$

We note that (27) implies that $\limsup_{n\to\infty} |Y_n + Z_n| \leqq \sigma$ a.e. Since Z_n is independent of $(Y_1, \cdots, Y_n)$ and $Z_n \to_P 0$, we obtain from Lemma 1 that $\limsup_{n\to\infty} |Z_n| \leqq 2\sigma$ a.e. But $\limsup_{n\to\infty} |Z_n|$ can only be ∞ a.e. or 0 a.e., and so $\lim_{n\to\infty} Z_n = 0$ a.e. This implies that $E|X_1|^{2/\alpha}(\log^+|X_1| + 1)^{-1/\alpha} < \infty$. Letting $\mu = EX_1$, $\tau^2 = \operatorname{Var} X_1$, it then follows from our preceding proof that

$$\limsup_{n\to\infty} (X_n + \cdots + X_{n+[n^\alpha]} - n^\alpha\mu)/\{2(1-\alpha)n^\alpha \log n)\}^{\frac{1}{2}} = \tau \quad \text{a.e.} \tag{34}$$

From (27) and (34), it is clear that $\mu = 0$ and $\tau = \sigma$. Therefore we have proved that (27) implies (26). Since (28) also implies that $\limsup_{n\to\infty} |Y_n + Z_n| \leqq \sigma$ a.e., it is easy to see that (28) implies (26). □

3. The convergence rate of $P[|S_n| > b_n]$ for certain upper-class sequences (b_n). In [2], Baum and Katz have proved that if $X_1, X_2, \cdots$ are i.i.d. then for $\alpha > \frac{1}{2}$ and $p > 1/\alpha$,

$$\begin{aligned} &E|X_1|^p < \infty , \quad \text{and for the case} \quad \alpha \leqq 1 , \quad EX_1 = 0 \\ &\Leftrightarrow \textstyle\sum n^{p\alpha-2}P[|S_n| > \varepsilon n^\alpha] < \infty \qquad \text{for all } \varepsilon > 0 \\ &\Leftrightarrow \textstyle\sum n^{p\alpha-2}P[\sup_{k\geqq n} |S_k/k^\alpha| > \varepsilon] < \infty \qquad \text{for all } \varepsilon > 0 . \end{aligned} \tag{35}$$

By making use of Theorem 2, we can easily prove the following theorem which can be regarded as some sort of limiting case of (35) when $\alpha = \frac{1}{2}$ and $p > 2$.

THEOREM 3. *Suppose $X_1, X_2, \cdots$ are* i.i.d. *random variables and $p > 2$. If $EX_1 = 0$, $EX_1^2 = \sigma^2$ and $E|X_1|^p(\log^+|X_1| + 1)^{-p/2} < \infty$, then for any $\varepsilon > \sigma(p-2)^{\frac{1}{2}}$,*

$$\textstyle\sum n^{p/2-2}P[|S_n| > \varepsilon(n \log n)^{\frac{1}{2}}] < \infty \tag{36}$$

$$\textstyle\sum n^{p/2-2}P[\sup_{k\geqq n} |S_k/(k \log k)^{\frac{1}{2}}| > \varepsilon] < \infty . \tag{37}$$

Conversely, if for some $\varepsilon > 0$, either (36) *or* (37) *holds, then $EX_1 = 0$ and $E|X_1|^p(\log^+|X_1| + 1)^{-p/2} < \infty$.*

PROOF. Let $\alpha = 2/p$. Then $0 < \alpha < 1$. Set $n_k = [k^{1/(1-\alpha)}]$, and note that $(S_{n_k, n_k^\alpha}: k \geqq k_0)$ is an independent sequence. Therefore by the Borel–Cantelli lemma, the following two statements are equivalent:

$$P[|S_{n_k, n_k^\alpha}| > \delta(n_k^\alpha \log n_k)^{\frac{1}{2}} \text{ i.o.}] = 0 \tag{38}$$

$$\sum_{k=1}^\infty P[|S_{n_k^\alpha}| > \delta(n_k^\alpha \log n_k)^{\frac{1}{2}}] < \infty . \tag{39}$$

For any real number $t \geqq 1$, define $S_t = X_1 + \cdots + X_{[t]}$. Then (39) is equivalent to

$$\int_1^\infty P[|S_{t^{\alpha/(1-\alpha)}}| > \delta\{t^{\alpha/(1-\alpha)} \log (t^{1/(1-\alpha)})\}^{\frac{1}{2}}]\, dt < \infty . \tag{40}$$

Applying a change of variable $u = t^{\alpha/(1-\alpha)}$ to the integral in (40), we then find that (39) is equivalent to

$$\sum_{n=1}^\infty n^{1/\alpha - 2} P[|S_n| > (\delta/\alpha^{\frac{1}{2}})(n \log n)^{\frac{1}{2}}] < \infty . \tag{41}$$

Suppose $EX_1 = 0$, $EX_1^2 = \sigma^2$ and $E|X_1|^p(\log^+ |X_1| + 1)^{-p/2} < \infty$. Then by Theorem 2, (38) holds for all $\delta > \sigma\{2(1-\alpha)\}^{\frac{1}{2}}$. Therefore (41) holds for $\delta/\alpha^{\frac{1}{2}} > \sigma\{2(1-\alpha)/\alpha\}^{\frac{1}{2}} = \sigma(p-2)^{\frac{1}{2}}$, and so we have proved that (36) holds for $\varepsilon > \sigma(p-2)^{\frac{1}{2}}$.

Given $\xi > \varepsilon > \sigma(p-2)^{\frac{1}{2}}$, choose $c > 1$ such that $\xi > \varepsilon c^{\frac{1}{2}}$. By the Lévy inequality,

$$\begin{aligned} &P[\max_{c^j \leqq k \leqq c^{j+1}} |S_k/(k \log k)^{\frac{1}{2}}| > \xi] \\ &\quad \leqq 2P[|S_{[c^{j+1}]}| > \xi\{c^j \log c^j\}^{\frac{1}{2}} - \{2 \textstyle\sum_{i=1}^{[c^{j+1}]} EX_i^2\}^{\frac{1}{2}}] \\ &\quad \leqq 2P[|S_{[c^{j+1}]}| > \varepsilon\{c^{j+1} \log c^{j+1}\}^{\frac{1}{2}}] \qquad \text{for all large } j . \end{aligned}$$

Therefore

$$\begin{aligned} &\sum_{n \geqq n_0} n^{p/2-2} P[\sup_{k \geqq n} |S_k/(k \log k)^{\frac{1}{2}}| > \xi] \\ &\quad \leqq M \sum_{i=i_0}^\infty c^{(p/2-1)i} \sum_{j=i}^\infty P[\max_{c^j \leqq k \leqq c^{j+1}} |S_k/(k \log k)^{\frac{1}{2}}| > \xi] \\ &\quad \leqq M' \sum_{j=1}^\infty c^{(p/2-1)j} P[|S_{[c^{j+1}]}| > \varepsilon\{c^{j+1} \log c^{j+1}\}^{\frac{1}{2}}] \end{aligned} \tag{42}$$

where M, M' are positive constants. Since we have proved that (36) holds, it follows that the last series in (42) is finite. Hence we have proved (37).

Now assume that (36) holds for some $\varepsilon > 0$. Then (41) and therefore (38) holds for some $\delta > 0$. We first prove that $EX_1^2 < \infty$. Without loss of generality, we can assume that X_1 is symmetric. If $EX_1^2 = \infty$, then we can choose c such that $EX_1^2 I_{[|X_1| \leqq c]} > \delta^2/(1-\alpha)$. Let $X_n' = X_n I_{[|X_n| \leqq c]}$. It then follows from the proof of Theorem 1 that

$$\limsup_{k\to\infty} (X'_{n_k} + \cdots + X'_{n_k + [n_k^\alpha] - 1})/\{2(1-\alpha)n_k^\alpha \log n_k\}^{\frac{1}{2}} \geqq \delta/(1-\alpha)^{\frac{1}{2}} \quad \text{a.e.}$$

(cf. (15)). Therefore using an argument of Feller (cf. [5] page 346),

$$\limsup_{k\to\infty} S_{n_k, n_k^\alpha}/(n_k^\alpha \log n_k)^{\frac{1}{2}} \geqq 2^{\frac{1}{2}}\delta \quad \text{a.e.} ,$$

contradicting (38). Hence $EX_1^2 < \infty$. It is also easy to see from (38) that $EX_1 = 0$. Therefore

$$\lim_{n\to\infty} P[|S_n| > \varepsilon(n \log n)^{\frac{1}{2}}] = 0 \tag{43}$$

$$\lim_{n\to\infty} nP[|X_1| > \varepsilon(n \log n)^{\frac{1}{2}}] = 0 = \lim_{n\to\infty} nP[|X_1|^2 > n] . \tag{44}$$

Let $B_n = [|S_n| > \varepsilon(n \log n)^{\frac{1}{2}}]$, and for $k = 1, \cdots, n$, define

$$R_k^{(n)} = [|X_k| > 2\varepsilon(n \log n)^{\frac{1}{2}}] , \qquad T_k^{(n)} = [|\sum_{1 \leq i \leq n, i \neq k} X_i| < \varepsilon(n \log n)^{\frac{1}{2}}] .$$

Then by an argument due to Erdös [4], we have

$$P(B_n) \geqq \sum_{k=1}^{n} \{P(T_k^{(n)}) - nP(R_1^{(n)})\} P(R_1^{(n)}) .$$

From (43) and (44), we can therefore find $\lambda > 0$ such that

$$P(B_n) \geqq \lambda n P(R_1^{(n)}) . \tag{45}$$

Using (36) and (45), we then obtain that $\sum n^{p/2-1} P[|X_1| > 2\varepsilon(n \log n)^{\frac{1}{2}}] < \infty$ and so $E|X_1|^p (\log^+ |X_1| + 1)^{-p/2} < \infty$. □

In [4], Erdös has obtained the following result: If $X_1, X_2, \cdots$ are i.i.d. with $EX_1 = 0$ and $EX_1^4 < \infty$, then there exists a constant r such that $\sum P[|S_n| > n^{\frac{1}{2}}(\log n)^r] < \infty$. We note that his result follows immediately from Theorem 3 with $p = 4$. Theorem 3 also gives a complete answer to the problem of finiteness of moments of the random variable $N(\varepsilon) = \sum_1^\infty I_{[|S_n| \geq \varepsilon(\log n)^{\frac{1}{2}}]}$ recently considered by Stratton [11]. We shall present this and other related results elsewhere.

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