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BERRY-ESSÉEN ESTIMATES IN HILBERT SPACE AND AN APPLICATION TO THE LAW OF THE ITERATED LOGARITHM

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We establish Berry-Esséen type estimates for random variables with values in a real separable Hilbert space H. These estimates are then used to prove the law of the iterated logarithm for sequences of H-valued random variables and also to prove a functional form of the law of the iterated logarithm for H-valued partial sums as given by Strassen. We also prove a log log result for H-valued symmetric stable random variables.

1. Introduction. Throughout the paper H will denote a real separable Hilbert space with norm $||\cdot||$ and all measures on H are assumed to be Borel measures.

The measure μ on H is said to be a mean zero Gaussian measure if every continuous linear functional f on H has a mean zero Gaussian distribution with variance $\int_H [f(x)]^2\mu(dx)$. The operator T defined on H by identifying H with its dual and given by

$$(Tf, g) = \int_H (f, x)(g, x)\mu(dx) \qquad (f, g \in H)$$

is called the covariance operator of μ. It is well known (see [12] for details) that a mean zero Gaussian measure on H is uniquely determined by its covariance operator and that the covariance operator of any Gaussian measure has a finite trace as well as the obvious properties of symmetry and nonnegativity. Conversely, any such operator is the covariance operator of a mean zero Gaussian measure on H and is commonly called an S-operator [12].

However, a mean zero Gaussian measure μ on H (or any real separable Banach space for that matter) can also be uniquely determined by a subspace H_μ of H which has a Hilbert space structure. The norm on H_μ will be denoted by $||\cdot||_\mu$ and it is well known that the H norm $||\cdot||$ is weaker than $||\cdot||_\mu$ on H_μ. In fact, $||\cdot||$ is a measurable norm on H_μ in the sense of [5] and the measure μ is the extension of the canonical normal distribution on H_μ to H. We describe this relationship by saying μ is generated by H_μ. Since $||\cdot||$ is weaker than $||\cdot||_\mu$ it follows that the topological dual of H can be embedded by the restriction map linearly into the dual of H_μ, and identifying H_μ and its dual we have the dual of H, call it H^*, linearly embedded in H_μ. As might be expected H_μ is intimately

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related to the covariance operator of μ, but for details on these matters as well as further references see [5], [6], [7].

Let $X_1, X_2, \cdots$ be independent identically distributed H-valued random variables with mean zero, covariance operator

$$(Tx, y) = E((X_1, x)(X_1, y)) \qquad (x, y \in H)$$

and such that $E\|X_1\|^2 < \infty$. Then T is an S-operator and if μ is the mean zero Gaussian measure on H with covariance operator T it is well known that

$$\mathscr{L}\left(\frac{X_1 + \cdots + X_n}{n^{\frac{1}{2}}}\right) \to_n \mu \tag{1.1}$$

in the sense of weak convergence. A natural question to ask is one regarding the Berry–Esséen estimates related to the convergence in (1.1) when H is infinite dimensional. The work of Sazonov in [13] and [14] is in this direction and was motivated by an attempt to obtain a rate of convergence result for the ω^2-test in statistics. In [13] he shows that for a particular sequence of mean zero bounded independent identically distributed H-valued random variables $\{Y_n\}$ one has

$$\sup_{t \geqq 0}\left|P\left(\left\|\frac{Y_1 + \cdots + Y_n}{n^{\frac{1}{2}}}\right\| \leqq t\right) - \mu(x\colon \|x\| \leqq t)\right| = O(n^{-1/10+\varepsilon}) \tag{1.2}$$

where $\varepsilon > 0$ is arbitrary. Using the same Y_n's and after considerable effort the estimate in (1.2) is improved to $O(n^{-1/6+\varepsilon})$ in [14] where again $\varepsilon > 0$ is arbitrary.

The results we obtain in section two are analogues of the Berry–Esséen estimates known when H is finite dimensional. Our results do not depend on previous results but are proved directly. Unfortunately, they fall short of what is known for finite dimensions, but when H is infinite dimensional we easily extend the results [13] and [14] as can be seen from Corollary 2.1 and Corollary 2.2.

In Section 3 we apply the estimates of Section 2 to obtain the law of the iterated logarithm for sequences of independent H-valued random variables which are not Gaussian as well as obtain a functional form of the law of the iterated logarithm for H-valued partial sums as in Strassen [15]. The idea of using Berry–Esséen type estimates in this connection appears in a number of places but perhaps most closely related to our results in [3]. Finally, in section four we discuss the analogue of [2] for sequences of independent H-valued symmetric stable laws.

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2. Rates of convergence for the central limit theorem in H. A nontrivial measure is any measure which is not concentrated at a single point.

LEMMA 2.1. *Let μ be a nontrivial mean zero Gaussian measure on a real separable Hilbert space H. Then there exists a constant C such that*

$$\mu(x \in H\colon a \leqq \|x\| \leqq a + \varepsilon) \leqq C \cdot \varepsilon \tag{2.1}$$

for all a and all $\varepsilon > 0$.

PROOF. Since μ is a mean zero Gaussian measure on H it is well known from, for example, [12] that there is a complete orthonormal sequence $\{e_n\}$ for H such that the functions $\{(x, e_n) : n \geqq 1\}$ are independent mean zero Gaussian random variables with variances $\lambda_n \geqq 0$ satisfying $\sum_n \lambda_n < \infty$. Further, μ nontrivial implies at least one $\lambda_n > 0$ and we have the Fourier transform of $||x||^2$ given by

$$\begin{aligned}\phi(t) = \int_H e^{it||x||^2}\mu(dx) &= \int_H \exp[it \textstyle\sum_n (x, e_n)^2\mu(dx) \\ &= \textstyle\prod_n (1 - 2it\lambda_n)^{-\frac{1}{2}} .\end{aligned} \tag{2.2}$$

If at least three λ_n's are positive then (by relabeling if necessary) we can assume $\lambda_1, \lambda_2, \lambda_3$ are positive and hence that

$$\begin{aligned}|\phi(t)| &\leqq 1 && \text{if} \quad |t| \leqq 1 ; \\ &\leqq (\lambda_1\lambda_2\lambda_3|t|^3)^{-\frac{1}{2}} && \text{if} \quad |t| > 1 .\end{aligned}$$

Hence $\int_{-\infty}^{\infty} |\phi(t)|\, dt < \infty$ and by the Fourier inversion theorem we have that $||x||^2$ has a bounded continuous density, call it $h(s)$, when at least three λ_n's are positive. We now assume $\lambda_1, \lambda_2, \lambda_3$ positive until further notice.

Since $\sum_n \lambda_n < \infty$ and the (x, e_n)'s are independent we have $\int_H ||x||^2\mu(dx) < \infty$ and

$$\begin{aligned}\phi'(t) = i \int_H ||x||^2 e^{it||x||^2}\mu(dx) &= i \textstyle\sum_n \int_H (x, e_n)^2 e^{it||x||^2}\mu(dx) \\ &= i \textstyle\sum_n (\prod_{k\neq n} (1 - 2it\lambda_k)^{-\frac{1}{2}}) \dfrac{\lambda_n}{(1 - 2it\lambda_n)^{\frac{3}{2}}} \\ &= i\phi(t) \textstyle\sum_n \dfrac{\lambda_n}{(1 - 2it\lambda_n)} .\end{aligned} \tag{2.3}$$

Now $\sum_n \lambda_n < \infty$ so $|\phi'(t)| \leqq C|\phi(t)|$ for all real t and hence $\int_{-\infty}^{\infty} |\phi'(t)|\, dt < \infty$. Thus the inverse Fourier transform

$$\rho(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-its}\phi'(t)\, dt \tag{2.4}$$

is a bounded continuous function of s. On the otherhand, the probability density of $||x||^2$, $h(s)$, is bounded and continuous and

$$\phi(t) = \int_{-\infty}^{\infty} e^{its}h(s)\, ds .$$

Now $\int_H ||x||^2\mu(dx) < \infty$ implies $\int_{-\infty}^{\infty} sh(s)\, ds < \infty$ and hence

$$\phi'(t) = i \int_{-\infty}^{\infty} e^{its}sh(s)\, ds . \tag{2.5}$$

By (2.4) and (2.5) and that $\int_{-\infty}^{\infty} |\phi'(t)|\, dt < \infty$ we have

$$\rho(s) = ish(s) ,$$

and since ρ is bounded and continuous $sh(s)$ is bounded and continuous on $(-\infty, \infty)$. Now the probability density of $||x||$, call it $g(s)$, satisfies

$$g(s) = \frac{d}{ds} P(||x|| \leqq s) = \frac{d}{ds} P(||x||^2 \leqq s^2) = 2sh(s^2) \qquad (s > 0) .$$

Since $g(s) = 0$ for $s < 0$ and $h(s)$ and $sh(s)$ are bounded and continuous on $(-\infty, \infty)$ we have

$$\sup_s |g(s)| = \sup_s |2sh(s^2)| \leqq 2 \sup_{s \leq 1} |h(s)| + 2 \sup_s |s^2h(s^2)| < \infty .$$

Thus (2.1) holds if at least three λ_n's are positive.

Since μ is nontrivial at least one $\lambda_n > 0$ and in case only one or only two λ_n's are positive then (2.1) is obvious since the density of $||x||$ is bounded in one or two dimensions.

LEMMA 2.2. *Let f be a r times continuously differentiable function on $(-\infty, \infty)$ such that $f^{(k)}(0) = 0$ for $k = 1, 2, \cdots, r$ and let $g(x + \lambda y) = f(||x + \lambda y||)$ for $x, y \in H$ and $(-\infty < \lambda < \infty)$. Then g is r times continuously differentiable as a function of λ on $(-\infty, \infty)$.*

PROOF. By the chain rule g is obviously r times continuously differentiable at all λ such that $x + \lambda y \neq 0$ since $\phi(\lambda) = ||x + \lambda y||$ is infinitely differentiable at all such λ. In fact, induction yields that

$$\frac{d^k}{d\lambda^k}\phi(\lambda) = \frac{d^k}{d\lambda^k}||x + \lambda y|| = \sum_{j=0}^{[k/2]} \alpha(j, k) \frac{(x + \lambda y, y)^{k-2j}(y, y)^j}{||x + \lambda y||^{2k-2j-1}} \tag{2.6}$$

where the $\alpha(j, k)$ are constants and $[\cdot]$ is the greatest integer function.

Now if $x + \lambda_0 y = 0$, then unless $y = 0$ we have $x + \lambda y \neq 0$ for all other λ. In case $y = 0$ the lemma is trivially true so assume $y \neq 0$. Then for $\lambda \neq \lambda_0$ we have by [1] that for $n = 1, 2, \cdots, r$

$$\frac{d^n}{d\lambda^n} g(x + \lambda y) = \sum_{k=1}^{n} \sum_{I(k,n)} \beta(\alpha_1, \cdots, \alpha_n)(\phi'(\lambda))^{\alpha_1} \cdots (\phi^{(n)}(\lambda))^{\alpha_n} f^{(k)}(\phi(\lambda)) \tag{2.7}$$

where $\beta(\alpha_1, \cdots, \alpha_n)$ are positive constants and $I(k, n)$ consists of all collections of nonnegative integers $\alpha_1, \cdots, \alpha_n$ such that $\alpha_1 + \cdots + \alpha_n = k$ and $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$.

Thus g will have r derivatives at $\lambda = \lambda_0$ all of which are zero if we show

$$\lim_{\lambda \to \lambda_0} \frac{d^n}{d\lambda^n} g(x + \lambda y) = 0 \qquad (n = 1, 2, \cdots, r) . \tag{2.8}$$

Now note that

$$\lim_{\lambda \to \lambda_0} \frac{f^{(k)}(\phi(\lambda))}{(\phi(\lambda))^{r-k}} = 0 \qquad (k = 1, \cdots, r) \tag{2.9}$$

since

$$f^{(k)}(\phi(\lambda)) = \int_0^{\phi(\lambda)} \int_0^{t_{r-k-1}} \cdots \int_0^{t_2} \int_0^{t_1} f^{(r)}(s)\, ds\, dt_1 \cdots dt_{r-k-1}$$

when $f^{(j)}(0) = 0$ for $j = 1, \cdots, r$. Furthermore, the general term in (2.7) can be estimated using (2.6) to obtain

$$|(\phi'(\lambda))^{\alpha_1} \cdots (\phi^{(n)}(\lambda))^{\alpha_n} f^{(k)}(\phi(\lambda))| = O\left(\frac{||y||^{\alpha_1+2\alpha_2+\cdots+n\alpha_n} f^{(k)}(\phi(\lambda))}{(\phi(\lambda))^{\alpha_2+2\alpha_3+\cdots+(n-1)\alpha_n}}\right) \tag{2.10}$$

where $\alpha_1 + \cdots + \alpha_n = k$, $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$. Thus $\alpha_2 + 2\alpha_3 + \cdots + (n-1)\alpha_n = n - k$ so (2.10) approaches zero as $\lambda \to \lambda_0$ by (2.9) as $n \leqq r$. Hence (2.8) holds proving the lemma.

LEMMA 2.3. *Suppose X and Y are independent H-valued random variables such that $E||X||^k < \infty$ and $E||Y||^k < \infty$. Then*

$$E(X, Y)^k = \lim_n \sum_{i_1,\cdots,i_k=1}^n E((X, e_{i_1}) \cdots (X, e_{i_k}))E((Y, e_{i_1}) \cdots (Y, e_{i_k}))$$

with respect to any complete orthonormal basis $\{e_i\}$ of H.

PROOF. Let P_n denote the projection onto the subspace generated by $\{e_1, \cdots, e_n\}$. Then

$$|(P_n X, Y)^k| \leqq ||P_n X||^k ||Y||^k \leqq ||X||^k ||Y||^k$$

and $E||X||^k||Y||^k = E||X||^k \cdot E||Y||^k$ by independence. Hence by the dominated convergence theorem

$$\begin{aligned} E(X, Y)^k &= \lim_n E(P_n X, Y)^k \\ &= \lim_n \sum_{i_1,\cdots,i_k=1}^n E((X, e_{i_1}) \cdots (X, e_{i_k}))E((Y, e_{i_1}) \cdots (Y, e_{i_k})) \end{aligned}$$

so the lemma is proved.

If X is an H-valued random variable and $\{e_n\}$ is a complete orthonormal basis for H, then the jth coordinate moments of X with respect to $\{e_n\}$ are the moments $E\{(X, e_{i_1}) \cdots (X, e_{i_j})\}$ for all possible choices of integers $i_1, \cdots, i_j$.

THEOREM 2.1. *Let $X_1, X_2, \cdots$ be independent H-valued random variables such that $E(X_n) = 0$ $(n = 1, 2, \cdots)$ and each X_n has common covariance operator T where*

$$(Tx, y) = E\{(X_n, x)(X_n, y)\} \qquad (x, y \in H;\ n = 1, 2, \cdots).$$

Let μ denote the mean zero Gaussian measure on H having covariance operator T and assume r is an integer such that $r \geqq 3$. Then:

(A) *If $\sup_n E||X_n||^r < \infty$ and the jth coordinates moments of each X_n with respect to some complete orthonormal basis $\{e_n\}$ in H agree with the corresponding coordinate moments for μ for all $j \leqq r - 1$, then*

$$\sup_{t\geq 0} \left| P\left(\left\| \frac{X_1 + \cdots + X_n}{n^{\frac{1}{2}}} \right\| \leqq t \right) - \mu(x : ||x|| \leqq t) \right| = O(n^{-\frac{1}{2}+3/2(r+1)}) . \tag{2.11}$$

(B) *If $\sup_n E||X_n||^{3r/2-1} < \infty$ and the jth coordinate moments of each X_n with respect to some complete orthonormal basis $\{e_n\}$ in H agree with the corresponding coordinate moments for μ for all $j \leqq r - 1$, then for each $\varepsilon > 0$*

$$\sup_{t\geq 0} \left| P\left(\left\| \frac{X_1 + \cdots + X_n}{n^{\frac{1}{2}}} \right\| \leqq t \right) - \mu(x : ||x|| \leqq t) \right| = O(n^{-\frac{1}{2}+1/r+\varepsilon}) . \tag{2.12}$$

REMARK. If $r = 3$, then each X_n having mean zero and common covariance operator T is equivalent to saying the jth coordinate moments of each X_n agree with the corresponding moments of μ for all $j \leqq 2$. Hence if $r = 3$ the extra hypothesis on the coordinate moments is vacuous, and our conditions reduce to the classical assumptions. The following corollaries are now immediate from Theorem 2.1 with $r = 3$.

COROLLARY 2.1. *Let $X_1, X_2, \cdots$ be independent H-valued random variables such* *that* $E(X_n) = 0$ $(n = 1, 2, \cdots)$, $\sup_n E\|X_n\|^3 < \infty$, *and each* X_n *has common covariance operator* T *where* $(Tx, y) = E\{(X_n, x)(X_n, y)\}$ *for* $x, y \in H$ *and* $n = 1, 2, \cdots$. *Then*

$$\sup_{t\geq 0}\left|P\left(\left\|\frac{X_1 + \cdots + X_n}{n^{\frac{1}{2}}}\right\| \leqq t\right) - \mu(x\colon \|x\| \leqq t)\right| = O(n^{-\frac{1}{8}}) \tag{2.13}$$

where μ *is the mean zero Gaussian measure on* H *with covariance operator* T.

COROLLARY 2.2. *Let* $\mu, X_1, X_2, \cdots$ *satisfy the conditions of the previous corollary and assume* $\sup_n E\|X_n\|^{3+\frac{1}{2}} < \infty$. *Then for each* $\varepsilon > 0$

$$\sup_{t\geq 0}\left|P\left(\left\|\frac{X_1 + \cdots + X_n}{n^{\frac{1}{2}}}\right\| \leqq t\right) - \mu(x\colon \|x\| \leqq t)\right| = O(n^{-\frac{1}{6}+\varepsilon})\,. \tag{2.14}$$

PROOF OF THEOREM 2.1. Fix $r \geqq 3$. First we will prove (A). For each integer $n \geqq 1$ and for each real number s define $f_{n,s}\colon (-\infty, \infty) \to [0, 1]$ such that $f_{n,s}$ is monotonic decreasing, $f_{n,s}(u) = 1$ for $u \leqq s$, $f_{n,s}(u) = 0$ for $u \geqq s + \delta_n$ where $\delta_n > 0$ will be specified later, $f_{n,s}^{(r)}$ is continuous on $(-\infty, \infty)$, and

$$|f_{n,s}^{(r)}(\cdot)| = O(\delta_n^{-r}\chi_{[s,s+\delta_n]}(\cdot)) \tag{2.15}$$

where χ_E denotes the indicator function of E. Furthermore we take $f_{n,s}(u) = f_{n,0}(u - s)$ so the bounding constant in (2.15) is uniform in s.

Let $g_{n,s}(x) = f_{n,s}(\|x\|)$ for each $x \in H$ and let $W_n = (X_1 + \cdots + X_n)/n^{\frac{1}{2}}$ for $n = 1, 2, \cdots$. Then, if $P(\|W_n\| \leqq t) \geqq \mu(x\colon \|x\| \leqq t)$ we have

$$\begin{aligned} &|P(\|W_n\| \leqq t) - \mu(x\colon \|x\| \leqq t)| \\ &\qquad \leqq E(g_{n,t}(W_n)) - \textstyle\int_H g_{n,t}(x)\mu(dx) + \mu(x\colon t \leqq \|x\| \leqq t + \delta_n)\,. \end{aligned} \tag{2.16}$$

If $P(\|W_n\| \leqq t) < \mu(x\colon \|x\| \leqq t)$, then

$$\begin{aligned} |P(\|W_n\| \leqq t) &- \mu(x\colon \|x\| \leqq t)| \\ &= P(\|W_n\| > t) - \mu(x\colon \|x\| > t) \\ &\leqq E(1 - g_{n,t-\delta_n}(W_n)) - \textstyle\int_H (1 - g_{n,t-\delta_n}(x))\mu(dx) \\ &\qquad + \mu(x\colon t - \delta_n \leqq \|x\| \leqq t) \\ &= \textstyle\int_H g_{n,t-\delta_n}(x)\mu(dx) - E(g_{n,t-\delta_n}(W_n)) \\ &\qquad + \mu(x\colon t - \delta_n \leqq \|x\| \leqq t)\,. \end{aligned} \tag{2.17}$$

Now by Lemma 2.1 $\mu(x\colon s \leqq \|x\| \leqq t) = O(|s - t|)$ so (2.11) follows from (2.16) and (2.17) if we show

$$\sup_{t\geq 0}|E(g_{n,t}(W_n)) - \textstyle\int_H g_{n,t}(x)\mu(dx)| = O(n^{-\frac{1}{2}+3/2(r+1)}) \tag{2.18}$$

with $\delta_n = n^{-\frac{1}{2}+3/2(r+1)}$. The supremum need only be taken over $t \geqq 0$ since for $s \leqq 0$ we have

$$\begin{aligned} |E(g_{n,s}(W_n)) &- \textstyle\int_H g_{n,s}((x)\mu(dx)| \\ &\leqq E(g_{n,s}(W_n)) + \textstyle\int_H g_{n,s}(x)\mu(dx) \\ &\leqq |E(g_{n,0}(W_n)) - \textstyle\int_H g_{n,0}(x)\mu(dx) + 2\mu(x\colon \|x\| \leqq \delta_n) \end{aligned}$$

since $g_{n,s} \leqq g_{n,t}$ for $s \leqq t$.

Let $X_1^*, X_2^*, \cdots$ be independent random variables each with Gaussian law μ as given in the theorem, and assume they are independent of $X_1, X_2, \cdots$. Then $\mathscr{L}((X_1^* + \cdots + X_n^*)/n^{\frac{1}{2}}) = \mu$ and if $Z_n = (X_1^* + \cdots + X_n^*)/n^{\frac{1}{2}}$ then (2.18) holds if

$$\sup_{t\geq 0} |E(g_{n,t}(W_n)) - E(g_{n,t}(Z_n))| = O(n^{-\frac{1}{2}+3/2(r+1)}) . \tag{2.19}$$

Now

$$g_{n,t}(W_n) - g_{n,t}(Z_n) = \sum_{k=1}^n V_k \tag{2.20}$$

where

$$\begin{aligned} V_k &= g_{n,t}\left(\frac{X_1 + \cdots + X_k + X_{k+1}^* + \cdots + X_n^*}{n^{\frac{1}{2}}}\right) \\ &\quad - g_{n,t}\left(\frac{X_1 + \cdots + X_{k-1} + X_k^* + \cdots + X_n^*}{n^{\frac{1}{2}}}\right) \\ &= g_{n,t}(U_k + X_k/n^{\frac{1}{2}}) - g_{n,t}(U_k + X_k^*/n^{\frac{1}{2}}) \end{aligned}$$

and

$$U_k = (X_1 + \cdots + X_{k-1} + X_{k+1}^* + \cdots + X_n^*)/n^{\frac{1}{2}} .$$

Let $h(\lambda) = g_{n,t}(U_k + \lambda X_k/n^{\frac{1}{2}})$ for $-\infty < \lambda < \infty$. Then by Lemma 2.2, (2.6), (2.7) and Taylor's formula we have

$$\begin{aligned} &g_{n,t}(U_k + X_k/n^{\frac{1}{2}}) \\ &= \sum_{j=0}^{r-1} \frac{h^{(j)}(0)}{j!} + \frac{h^{(r)}(\lambda)}{r!} \qquad (0 < \lambda < 1) \\ &= g_{n,t}(U_k) + \sum_{j=1}^{r-1} \frac{1}{j!} [\sum_{i=1}^{j} \{\sum_{I(i,j)} J(\alpha_1, \cdots, \alpha_j, \phi, 0)\} f_{n,t}^{(i)}(\phi(0))] \\ &\quad + \frac{1}{r!} \sum_{i=1}^{r} \sum_{I(i,r)} J(\alpha_1, \cdots, \alpha_r, \phi, \lambda) f_{n,t}^{(i)}(\phi(\lambda)) \end{aligned} \tag{2.21}$$

where λ is a random variable such that $0 < \lambda < 1$, $\phi(\lambda) = ||U_k + \lambda X_k/n^{\frac{1}{2}}||$, $J(\alpha_1, \cdots, \alpha_j, \phi, \lambda) = \beta(\alpha_1, \cdots, \alpha_j)(\phi'(\lambda))^{\alpha_1} \cdots (\phi^{(j)}(\lambda))^{\alpha_j}$ for $j = 1, \cdots, r$, and $I(i, j)$ is defined as in (2.7).

A similar expansion holds for $g_{n,t}(U_k + X_k^*/n^{\frac{1}{2}})$ with X_k replaced by X_k^* and λ by λ^* with $0 < \lambda^* < 1$. Hence

$$\begin{aligned} V_k &= g_{n,t}(U_k + X_k/n^{\frac{1}{2}}) - g_{n,t}(U_k + X_k^*/n^{\frac{1}{2}}) \\ &= \sum_{j=1}^{r-1} \frac{1}{j!} \{\sum_{i=1}^{j} [\sum_{I(i,j)} [J(\alpha_1, \cdots, \alpha_j, \phi, 0) \\ &\quad - J(\alpha_1, \cdots, \alpha_j, \phi_*, 0)]] f_{n,t}^{(i)}(\phi(0))\} \\ &\quad + \eta_{n,t}(U_k, \lambda X_k/n^{\frac{1}{2}}) - \eta_{n,t}(U_k, \lambda^* X_k^*/n^{\frac{1}{2}}) \end{aligned} \tag{2.22}$$

where $\phi_*(\lambda) = ||U_k + \lambda X_k^*/n^{\frac{1}{2}}||$,

$$\eta_{n,t}(U_k, \lambda X_k/n^{\frac{1}{2}}) = \frac{1}{r!} \sum_{i=1}^{r} [\sum_{I(i,r)} J(\alpha_1, \cdots, \alpha_r, \phi, \lambda)] f_{n,t}^{(i)}(\phi(\lambda)) , \tag{2.23}$$

and a similar expression holds for $\eta_{n,t}(U_k, \lambda^* X_k^*/n^{\frac{1}{2}})$ with λ, ϕ replaced by λ^* and ϕ^*.

Now $\phi(0) = \phi_*(0) = ||U_k||$, U_k is independent of X_k and X_k^*, and the jth coordinate moments of each X_k agree with the corresponding coordinate moment of X_k^* for all $j \leqq r - 1$ so we have

$$E\{[J(\alpha_1, \cdots, \alpha_j, \phi, 0) - J(\alpha_1, \cdots, \alpha_j, \phi_*, 0)]f_{n,t}^{(i)}(||U_k||)\} = 0 \tag{2.24}$$

for all nonnegative integers $\alpha_1, \cdots, \alpha_j$ such that $\alpha_1 + \cdots + \alpha_j = i$, $\alpha_1 + 2\alpha_2 + \cdots + j\alpha_j = j$, and $j = 1, \cdots, r - 1$. To establish (2.24) estimates as used in (2.9) and (2.10) along with the boundedness of each $f^{(i)}$ assure integrability, and the independence of U_k, X_k, and X_k^* along with (2.6) yield (2.24) since the jth coordinate moments of X_k and X_k^* agree for $j \leqq r - 1$.

That is, using (2.6) and expanding the derivatives in (2.24) to their various powers and collecting similar terms a typical difference in (2.24) is of the form

$$c \cdot E\left\{\frac{f_{n,t}^{(i)}(||U_k||)}{||U_k||^w}[(U_k, X_k)^u||X_k||^v - (U_k, X_k^*)^u||X_k^*||^v]\right\} \tag{2.25}$$

where c is a constant and u, v, w are integers such that

$$\begin{aligned} u &= \alpha_1 + \textstyle\sum_{s=1}^{\alpha_2}(2 - 2u_{2,s}) + \sum_{s=1}^{\alpha_3}(3 - 2u_{3,s}) + \cdots + \sum_{s=1}^{\alpha_j}(j - 2u_{j,s}) \\ v &= 2\textstyle\sum_{s=1}^{\alpha_2} u_{2,s} + 2\sum_{s=1}^{\alpha_3} u_{3,s} + \cdots + 2\sum_{s=1}^{\alpha_j} u_{j,s} \\ w &= \alpha_1 + \textstyle\sum_{s=1}^{\alpha_2}(4 - 2u_{2,s} - 1) + \cdots + \sum_{s=1}^{\alpha_j}(2j - 2u_{j,s} - 1) \end{aligned}$$

with $0 \leqq u_{i,s} \leqq [i/2]$ for $i = 2, \cdots, j$ and $s = 1, \cdots, \alpha_i$.

Now expand U_k, X_k, and X_k^* about the orthonormal basis $\{e_n\}$ with respect to which X_k and X_k^* have equal coordinate moments to obtain from Lemma 2.3 that

$$\begin{aligned} &E(f_{n,t}^{(i)}(||U_k||)(U_k, X_k)^u||X_k||^v/||U_k||^w) \\ &= \lim_m \textstyle\sum_{i_1,\cdots,i_u=1}^m E\left(\frac{f_{n,t}^{(i)}(||U_k||)}{||U_k||^w}(U_k, e_{i_1}) \cdots (U_k, e_{i_u})\right) \\ &\qquad \times E(||X_k||^v(X_k, e_{i_1}) \cdots (X_k, e_{i_u})) \\ &= \lim \textstyle\sum_{i_1,\cdots,i_u=1}^m E\left(\frac{f_{n,t}^{(i)}(||U_k||)}{||U_k||^w}(U_k, e_{i_1}) \cdots (U_k, e_{i_u})\right) \\ &\qquad \times E(||X_k^*||^v(X_k^*, e_{i_1}) \cdots (X_k^*, e_{i_u})) \\ &= E\left(\frac{f_{n,t}^{(i)}(||U_k||)}{||U_k||^w}(U_k, X_k^*)^u||X_k^*||^v\right). \end{aligned} \tag{2.26}$$

Here we used the fact that $u + v = \alpha_1 + 2\alpha_2 + \cdots + j\alpha_j = j \leqq r - 1$ and $w - u = \alpha_2 + 2\alpha_3 + \cdots + (j-1)\alpha_j = j - i$ along with (2.9) to assure the integrability of the terms involved, and that X_k and X_k^* have equal jth coordinate moments $j \leqq r - 1$ yielding

$$E(||X_k||^v(X_k, e_{i_1}) \cdots (X_k, e_{i_u})) = E(||X_k^*||^v(X_k^*, e_{i_1}) \cdots (X_k^*, e_{i_u}))$$

for all $i_1, \cdots, i_u$.

From (2.26) we get (2.25) equal to zero and since (2.24) is the sum of terms of the type in (2.25) we have (2.24) holding. Thus by (2.22) we get

$$(2.27) \qquad |E(V_k)| \leqq E(|\eta_{n,t}(U_k, \lambda X_k/n^{\frac{1}{2}})|) + E(|\eta_{n,t}(U_k, \lambda^* X_k^*/n^{\frac{1}{2}})|)$$

where $\eta_{n,t}$ is defined in (2.23).

Since $f_{n,s}^{(j)}(0) = 0$ for all $s \geqq 0$, $j = 1, \cdots, r$ and (2.15) holds we have (since $f_{n,s}$ is constant for $u \notin [s, s+\delta_n]$) that

$$(2.28) \qquad |f_{n,s}^{(i)}(|u|)| = O(\delta_n^{-r}|u|^{r-i}\chi_{[s,s+\delta_n]}^{(|u|)}) \qquad (i = 1, \cdots, r; -\infty < s < \infty)\,.$$

Thus by (2.23)

$$(2.29) \qquad E(|\eta_{n,t}(U_k, \lambda X_k/n^{\frac{1}{2}})|) = O(\sup_{1 \leqq i \leqq r} |E\{[\phi'(\lambda)]^{\alpha_1} \cdots [\phi^{(r)}(\lambda)]^{\alpha_r} f_{n,t}^{(i)}(\phi(\lambda))\}|)$$

where $\phi(\lambda) = \|U_k + \lambda X_k/n^{\frac{1}{2}}\|$, $\alpha_1 + \cdots + \alpha_r = i$, and $\alpha_1 + 2\alpha_2 + \cdots + r\alpha_r = r$. From (2.6), (2.9) and (2.28) we have uniformly in $t \geqq 0$ that

$$(2.30) \qquad \begin{aligned} &|E\{[\phi'(\lambda)]^{\alpha_1} \cdots [\phi^{(r)}(\lambda)]^{\alpha_r} f_{n,t}^{(i)}(\phi(\lambda))\}| \\ &= O\left(E\left(\frac{\|X_k/n^{\frac{1}{2}}\|^{\alpha_1+2\alpha_2+\cdots+r\alpha_r}\delta_n^{-r}\|U_k + \lambda X_k/n^{\frac{1}{2}}\|^{r-i}}{\|U_k + \lambda X_k/n^{\frac{1}{2}}\|^{\alpha_2+\cdots+(r-1)\alpha_r}}\right)\right) \\ &= O(n^{-r/2}\delta_n^{-r}) \end{aligned}$$

since $\alpha_1 + 2\alpha_2 + \cdots + r\alpha_r = r$ and $\alpha_1 + \cdots + \alpha_r = i$. Thus (2.30) and (2.29) imply

$$E(|\eta_{n,t}(U_k, \lambda X_k/n^{\frac{1}{2}})|) = O(n^{-r/2}\delta_n^{-r})$$

uniformly in $t \geqq 0$. Similar estimates hold when λ and X_k are replaced by λ^* and X_k^* so we have from (2.27) and (2.20) that

$$(2.31) \qquad \begin{aligned} \sup_{t \geqq 0} |E(g_{n,t}(W_n) - g_{n,t}(Z_n))| &= O(n^{-r/2+1}\delta_n^{-r}) \\ &= O(n^{-\frac{1}{2}+3/2(r+1)}) \end{aligned}$$

since $\delta_n = n^{-\frac{1}{2}+3/2(r+1)}$. Thus (2.19) holds and (A) follows.

Now we establish (B). Fix $\varepsilon > 0$ and define functions $f_{n,s}$ as above with δ_n to be specified later. Then (2.12) will hold (if we argue as above) provided

$$(2.32) \qquad \sup_{t \geqq 0} |E(g_{n,t}(W_n)) - E(g_{n,t}(Z_n))| = O(n^{-\frac{1}{2}+1/r+\varepsilon})$$

and δ_n is sufficiently small. To verify (2.32) we proceed as before finding that it suffices to show

$$(2.33) \qquad E(|\eta_{n,t}(U_k, \lambda X_k/n^{\frac{1}{2}})|) = O(n^{-\frac{3}{2}+1/r+\varepsilon})$$

uniformly in t $\geqq 0$ since similar estimates will hold when λ^* and X_k^* replace λ and X_k. Now (2.23) implies

$$(2.34) \qquad E(|\eta_{n,t}(U_k, \lambda X_k/n^{\frac{1}{2}})|) = O(\sup_{1 \leqq i \leqq r} \sup_{I(i,r)} E|J(\alpha_1, \cdots, \alpha_r, \phi, \lambda) f_{n,t}^{(i)}(\phi(\lambda))|)\,.$$

From (2.6) (if $\alpha_1 + \cdots + \alpha_r = i, \alpha_1 + 2\alpha_2 + \cdots + r\alpha_r = r$, $i = 1, \cdots, r$) we have by (2.28) that

$$E|J(\alpha_1, \cdots, \alpha_r, \phi, \lambda)f_{n,t}^{(i)}(\phi(\lambda))| \tag{2.35}$$
$$= O\left(E\left|\frac{||X_k/n^{\frac{1}{2}}||^{\alpha_1+2\alpha_2+\cdots+r\alpha_r}f_{n,t}^{(i)}(||U_k + \lambda X_k/n^{\frac{1}{2}}||)}{||U_k + \lambda X_k/n^{\frac{1}{2}}||^{\alpha_2+2\alpha_3+\cdots+(r-1)\alpha_r}}\right|\right)$$
$$= O(E|||X_k/n^{\frac{1}{2}}||^{\alpha_1+2\alpha_2+\cdots+r\alpha_r}\delta_n^{-r}\chi_{[t,t+\delta_n]}(||U_k + \lambda X_k/n^{\frac{1}{2}}||)|)$$
$$= O(\delta_n^{-r}n^{-r/2}E\{||X_k||^r[\chi_{[t-\delta_n,t+2\delta_n)}(||U_k||) + \chi_{[\delta_n,\infty)}(||\lambda X_k/n^{\frac{1}{2}}||)]\}) .$$

Thus by (A), Lemma 2.1, and U_k being independent of X_k we have (if $\delta_n = n^{-\frac{1}{2}+1/(r+1)+3/2(r+1)^2}$)

$$\sup_{1\leq i\leq r}\sup_{I(i,r)}\sup_{t\geq 0} E|J(\alpha_1, \cdots, \alpha_r, \phi, \lambda)f_{n,t}^{(i)}(\phi(\lambda))| \tag{2.36}$$
$$= O(\delta_n^{-r}n^{-r/2}\{E(||X_k||^r)[C_1\delta_n + C_2 n^{-\frac{1}{2}+3/2(r+1)}]$$
$$+ E(||X_k||^r\chi_{[n^{\frac{1}{2}}\delta_n,\infty)}(||X_k||))\})$$
$$= O(n^{-\frac{3}{2}+1/r}) + O(n^{-\frac{3}{2}+1/(r+1)+3/2(r+1)^2})$$

since

$$E(||X_k||^r\chi_{[n^{1/r},\infty)}(||X_k||)) \leqq E(||X_k||^r||X_k||^{(r-2)/2}/n^{(r-2)/2r}) = O(n^{-\frac{1}{2}+1/r})$$

when $\sup_k E||X_k||^{3r/2-1} < \infty$. Thus

$$\sup_{1\leq i\leq r}\sup_{I(i,r)}\sup_{t\geq 0} E|J(\alpha_1, \cdots, \alpha_r, \phi, \lambda)f_{n,t}^{(i)}(\phi(\lambda))| = O(n^{-\frac{3}{2}+1/(r+1)+3/2(r+1)^2}) . \tag{2.37}$$

Now (2.37) implies

$$\sup_{t\geq 0} E|\eta_{n,t}(U_k, \lambda X_k/n^{\frac{1}{2}})| = O(n^{-\frac{3}{2}+1/(r+1)+3/2(r+1)^2}) \tag{2.38}$$

and hence we have by the reasoning used in (A)

$$\sup_{t\geq 0}\left|P\left(\left\|\frac{X_1 + \cdots + X_n}{n^{\frac{1}{2}}}\right\| \leqq t\right) - \mu(x: ||x|| \leqq t)\right| = O(n^{-\frac{1}{2}+1/(r+1)+3/2(r+1)^2}) . \tag{2.39}$$

Using (2.39) in (2.36) we see that at the next step with $\delta_n = n^{-\frac{1}{2}+1/(r+1)+1/(r+1)^2+3/2(r+1)^3}$ that we have

$$\sup_{1\leq i\leq r}\sup_{I(i,r)}\sup_{t\geq 0} E|J(\alpha_1, \cdots, \alpha_r, \phi, \lambda)f_{n,t}^{(i)}(\phi(\lambda))| = O(n^{-\frac{3}{2}+1/(r+1)+1/(r+1)^2+3/2(r+1)^3}) .$$

Hence we obtain

$$\sup_{t\geq 0}\left|P\left(\left\|\frac{X_1 + \cdots + X_n}{n^{\frac{1}{2}}}\right\| \leqq t\right) - \mu(x: ||x|| \leqq t)\right| = O(n^{-\frac{1}{2}+1/(r+1)+1/(r+1)^2+3/2(r+1)^3}) . \tag{2.40}$$

Continuing in this way we see at the k iteration that

$$\sup_{1\leq i\leq r}\sup_{I(i,r)}\sup_{t\geq 0} E|J(\alpha_1, \cdots, \alpha_r, \phi, \lambda)f_{n,t}^{(i)}(\phi(\lambda))| = O(n^{-\frac{3}{2}+1/(r+1)+1/(r+1)^2+\cdots+1/(r+1)^k+3/2(r+1)^{k+1}}) . \tag{2.41}$$

Now $\varepsilon > 0$ and $\sum_{j=1}^{\infty} 1/(r+1)^j = 1/r$ imply that for k sufficiently large we have (2.33) with $\delta_n = n^{-\frac{1}{2}+(1/r)+\varepsilon}$. Thus (2.32) holds with $\delta_n = n^{-\frac{1}{2}+(1/r)+\varepsilon}$ and (2.12) holds so (B) follows.

REMARK. The key idea of our proof of Theorem 2.1 is not new and appears, for example, in a proof of the central limit theorem on page 167 of L. Breiman's book entitled *Probability*.

3. The law of the iterated logarithm. We use the notation LL n for log log n throughout the remainder of the paper. The results of this section establish the law of the iterated logarithm for sequences of H-valued random variables obtaining an extension of Corollary 4 of [9] or the main theorem of [11] which handled only Gaussian random variables. In [9] and [11], however, the Gaussian random variables take values in an arbitrary real separable Banach space and in that sense are more general. We also obtain the analogue of Strassen's functional version of the log log law ([15], Theorem 3) for H-valued random variables.

If F is any subset of H we define for each x in H

$$\|x - F\| = \inf_{y \in F} \|x - y\| . \tag{3.1}$$

If $\varepsilon > 0$ we also define

$$F^\varepsilon = \{y : \|y - F\| < \varepsilon\} . \tag{3.2}$$

THEOREM 3.1. *Let $X_1, X_2, \cdots$ be independent H-valued random variables such that $E(X_n) = 0$ $(n = 1, 2, \cdots)$, $\sup_n E\|X_n\|^3 < \infty$, and each X_n has common covariance operator T where*

$$(Tx, y) = E\{(X_n, x)(X_n, y)\} \qquad (x, y \in H, n = 1, 2, \cdots) .$$

Let μ be the mean zero Gaussian measure on H with covariance operator T and let K denote the unit ball of the Hilbert space H_μ which generates μ on H. Then

$$P\left(\lim \left\|\frac{X_1 + \cdots + X_n}{(2n \,\mathrm{LL}\, n)^{\frac{1}{2}}} - K\right\| = 0\right) = 1 , \tag{3.3}$$

and, in fact, with probability one the sequence $\{(X_1 + \cdots + X_n)/(2n \,\mathrm{LL}\, n)^{\frac{1}{2}}\}$ accumulates at every point of K.

REMARK 3.1. It is known from, for example, ([9] Lemma 3) that K is a compact subset of H. Thus the conclusions of Theorem 3.1 are equivalent to saying that with probability one the sequence of points $\{(X_1 + \cdots + X_n)/(2n \,\mathrm{LL}\, n)^{\frac{1}{2}} : n \geqq 3\}$ is conditionally compact in H and that with probability one the set of limit points of each sequence is precisely K.

PROOF. Let $S_n = X_1 + \cdots + X_n$ for $n \geqq 1$. To verify (3.3) it suffices to prove for each rational $\varepsilon > 0$ that with probability one $S_n/(2n \,\mathrm{LL}\, n)^{\frac{1}{2}} \notin K^\varepsilon$ only finitely often. Fix $\varepsilon > 0$ and let $A_n = \{S_n/(2n \,\mathrm{LL}\, n)^{\frac{1}{2}} \notin K^\varepsilon\}$ for $n \geqq 3$. Let $n_r = [\beta^r]$ where $[\cdot]$ denotes the greatest integer function and $\beta > 1$. Then for all integers

r such that $n_r \geqq 3$ we define

$$B_r = \left\{\frac{S_n}{(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}} \notin K^\varepsilon \text{ for some } n : n_r \leqq n < n_{r+1}\right\}.$$

Then $\limsup_n A_n \subseteqq \limsup_r B_r$.

Now if $Y_1, \cdots, Y_N$ are successive sums of independent H-valued random variables such that

$$\sup_{1 \leq j \leq N} P(\|Y_N - Y_j\| > \varepsilon/2) = c < 1,$$

then by the same proof as used when $H = (-\infty, \infty)$ we have

$$P(\textstyle\bigcup_{j=1}^N \{Y_j \notin K^\varepsilon\}) \leqq \frac{1}{1-c} P(Y_N \notin K^{\varepsilon/2}).$$

Hence

$$P(B_r) \leqq \frac{1}{1-d} P\left(\frac{S_{n_{r+1}-1}}{(n_{r+1}-1)^{\frac{1}{2}}} \notin (n_r/(n_{r+1}-1))^{\frac{1}{2}}(2 \operatorname{LL} n_r)^{\frac{1}{2}} K^{\varepsilon/2}\right) \tag{3.4}$$

where

$$d = \sup_{n_r \leq n < n_{r+1}} P(\|S_{n_{r+1}-1} - S_n\| > \varepsilon(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}/2) < \tfrac{1}{2}$$

for all r sufficiently large by applying Chebyshev's inequality and that $\beta > 1$.

Now for any fixed $\varepsilon > 0$ there exists $\beta > 1$ sufficiently close to one and a $\delta > 0$ such that for all r sufficiently large

$$(n_r/(n_{r+1}-1))^{\frac{1}{2}} K^{\varepsilon/2} \supseteqq K^\delta \cdot \tag{3.5}$$

To see (3.5) holds choose $0 < \delta < \varepsilon/4$ and assume $x \in ((n_{r+1}-1)/n_r)^{\frac{1}{2}} K^\delta$. Then there is an $h \in K$ such that $\|(n_r/(n_{r+1}-1))^{\frac{1}{2}} x - h\| \leqq \delta$ which implies

$$\begin{aligned} \|x - h\| &\leqq \|x - ((n_{r+1}-1)/n_r)^{\frac{1}{2}} h\| + |((n_{r+1}-1)/n_r)^{\frac{1}{2}} - 1| \|h\| \\ &\leqq \delta((n_{r+1}-1)/n_r)^{\frac{1}{2}} + |((n_{r+1}-1)/n_r)^{\frac{1}{2}} - 1| \sup_{h \in K} \|h\| \\ &\leqq \varepsilon/2 \end{aligned}$$

if $\beta > 1$ is sufficiently close to one since $\sup_{h \in K} \|h\|$ is finite (recall K is compact in H) and $0 < \delta < \varepsilon/4$. Thus $x \in K^{\varepsilon/2}$ and (3.5) holds.

Choosing δ and $\beta > 1$ so that (3.5) holds we obtain from (3.4) that for all r sufficiently large

$$P(B_r) \leqq \frac{1}{1-d} P\left(\frac{S_{n_{r+1}-1}}{(n_{r+1}-1)^{\frac{1}{2}}} \notin (2 \operatorname{LL} n_r)^{\frac{1}{2}} K^\delta\right). \tag{3.6}$$

Let $\alpha_k = \lambda_k^{\frac{1}{2}} e_k$ where $\{e_k\}$ is the sequence of orthonormal eigenvectors of the S-operator which is the covariance operator of μ, and $\{\lambda_k\}$ the corresponding non-zero eigenvalues. Let

$$\Pi_N(x) = \textstyle\sum_{k=1}^N (x, e_k) e_k \qquad (N = 1, 2, \cdots).$$

Then $\{\alpha_k\}$ is a complete orthonormal sequence in H_μ which lies in H^* under the embedding of H^* in H_μ discussed in the introduction, and we have

$$\Pi_N(x) = \sum_{k=1}^N \frac{(x, e_k)}{\lambda_k^{\frac{1}{2}}} \alpha_k = \sum_{k=1}^N \alpha_k(x)\alpha_k \qquad (N = 1, 2, \cdots)$$

where $\alpha_k(x)$ is the evaluation of the linear functional corresponding to α_k at x.

Then for $N = 1, 2, \cdots$

$$\begin{aligned} (3.7)\qquad & P\left(\frac{S_{n_{r+1}-1}}{(n_{r+1}-1)^{\frac{1}{2}}} \notin (2\,\mathrm{LL}\,n_r)^{\frac{1}{2}} K^\delta\right) \\ & \leqq P\left(\Pi_N\left(\frac{S_{n_{r+1}-1}}{(n_{r+1}-1)^{\frac{1}{2}}}\right) \notin (2\,\mathrm{LL}\,n_r)^{\frac{1}{2}}\Pi_N(K^{\delta/2})\right) \\ & \quad + P\left(\left\|(I-\Pi_N)\left(\frac{S_{n_{r}-1-1}}{(n_{r+1}-1)^{\frac{1}{2}}}\right)\right\| \geqq \frac{\delta}{2}(2\,\mathrm{LL}\,n_r)^{\frac{1}{2}}\right) \\ & \leqq P(\Pi_N Z \notin (2\,\mathrm{LL}\,n_r)^{\frac{1}{2}}\Pi_N(K_\mu{}^\gamma)) \\ & \quad + P\left(\|(I-\Pi_N)Z\| \geqq (2\,\mathrm{LL}\,n_r)^{\frac{1}{2}}\frac{\delta}{2}\right) + O((n_r)^{-\frac{1}{8}}) \end{aligned}$$

where $Ix = x$, $\mathscr{L}(Z) = \mu$, $\gamma > 0$ is such that if $K_\mu{}^\gamma = \{x \in H_\mu : \|x - K\|_\mu < \gamma\}$ then $\Pi_N K_\mu{}^\gamma \subseteqq \Pi_N K^{\delta/2}$. The existence of such a $\gamma > 0$ is obvious since $\Pi_N H$ is finite dimensional so the norms $\|\cdot\|$ and $\|\cdot\|_\mu$ are equivalent on $\Pi_N H$, and finally we note that

$$\Pi_N K_\mu{}^\gamma = \{x \in \Pi_N H_\mu : \|x\|_\mu \leqq 1 + \gamma\}$$

hence the estimates of Corollary 2.1 apply to obtain (3.7).

Fix $c > 1$ and choose λ so that $2\lambda(\delta/2)^2 \geqq c$. Now choose N_0 sufficiently large so that $N \geqq N_0$ implies $E(\exp\{\lambda\|(I - \Pi_N)Z\|^2\}) < \infty$ (this follows by straightforward calculations since with probability one $Z = \sum_k \lambda_k^{\frac{1}{2}}\alpha_k(Z)e_k$ provided $\{\lambda_k\}$, $\{\alpha_k\}$, and $\{e_k\}$ are related as indicated above). For fixed $N \geqq N_0$ and λ we then obtain

$$(3.8)\qquad P\left(\|(I-\Pi_N)Z\| \geqq \frac{\delta}{2}(2\,\mathrm{LL}\,n_r)^{\frac{1}{2}}\right) \leqq \exp\{-c\,\mathrm{LL}\,n_r\}.$$

By applying Lemma 5 of [10] at $t = 1$ or Proposition 1 of [11] we have a $d > 1$ such that for all r sufficiently large

$$(3.9)\qquad P(\Pi_N Z \notin (2\,\mathrm{LL}\,n_r)^{\frac{1}{2}}\Pi_N(K_\mu{}^\gamma)) \leqq \exp\{-d\,\mathrm{LL}\,n_r\}.$$

Combining (3.6), (3.7), (3.8), and (3.9) we see

$$(3.10)\qquad \textstyle\sum_r P(B_r) < \infty .$$

Then $P(\limsup_n A_n) = 0$ and (3.3) holds.

To show that $\{S_n/(2n\,\mathrm{LL}\,n)^{\frac{1}{2}} : n \geqq 3\}$ accumulates at every point of K with probability one it suffices by the separability of K to prove that if $h \in K$, $\|h\|_\mu < 1$, and $\varepsilon > 0$, then there is a $\beta > 1$ so that with probability one

$$(3.11)\qquad \left\|\frac{S_{n_r}}{(2n_r\,\mathrm{LL}\,n_r)^{\frac{1}{2}}} - h\right\| < \varepsilon$$

for infinitely many r where, as above, $n_r = [\beta^r]$. Now

$$(3.12)\quad \left\|\frac{S_{n_r}}{(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}} - h\right\| \leqq \left\|(I - \Pi_N)\frac{S_{n_r}}{(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}}\right\| + \left\|\Pi_N\left(\frac{S_{n_r}}{(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}}\right) - \Pi_N h\right\| + \|\Pi_N h - h\|.$$

Fixing $\beta > 1$ and $\varepsilon > 0$ there exists an N_0 such that $N \geqq N_0$ implies $\|\Pi_N(h) - h\| < \varepsilon$ and with probability one

$$(3.13)\quad \left\|(I - \Pi_N)\frac{S_{n_r}}{(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}}\right\| < \frac{\varepsilon}{3}$$

for all r sufficiently large where (3.13) holds by the argument used in (3.7) and (3.8). Hence it suffices to show there exists a $\beta > 1$ such that with probability one

$$(3.14)\quad \left\|\Pi_N \frac{S_{n_r}}{(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}} - \Pi_N h\right\| < \frac{\varepsilon}{3}$$

for infinitely many r.

Now set $\gamma = \|h\|_\mu$ and note $\gamma < 1$. Fix $\beta \geqq 3$ such that $\gamma/(1 - 1/\beta) < 1$ and β is an integer. Now fix N such that $\|\Pi_N h - h\| < \varepsilon$ and (3.13) holds. Let $n_r = [\beta^r]$ and define for $r = 2, 3, \cdots$

$$A_r = \left\{\left\|\Pi_N\left(\frac{S_{n_r} - S_{n_{r-1}}}{(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}}\right) - \Pi_N h\right\| < \frac{\varepsilon}{6}\right\}.$$

Then $A_2, A_3, \cdots$ are independent and we will show

$$(3.15)\quad \textstyle\sum_{r \geqq 2} P(A_r) = +\infty$$

obtaining $P(\limsup_r A_r) = 1$.

Using the definition of Π_N and that N is fixed we see that

$$(3.16)\quad P(A_r) = \mu\left(x\colon \left\|\frac{\Pi_N x}{t_r} - \Pi_N h\right\| < \frac{\varepsilon}{6}\right) + O(n_r^{-\frac{1}{8}})$$

where $t_r = (2n_r \operatorname{LL} n_r/(n_r - n_{r-1}))^{\frac{1}{2}}$. The validity of (3.16) follows immediately from Corollary 2.1 if $\Pi_N h = 0$ or since N is fixed, by applying similar estimates to spheres centered at $\Pi_N h$ if $\Pi_N h \neq 0$ where we can assume, for these purposes, that $\varepsilon < \|\Pi_N h\|$. Since N is fixed, however, we could also apply the finite dimensional estimates used in [13] or [14] to obtain (3.16). Hence by (3.16) we have

$$P(A_r) \geqq \textstyle\prod_{j=1}^N \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{t_r|h_j|}^{t_r(|h_j| + \varepsilon/6NM)} e^{-s^2/2}\, ds + O(n_r^{-\frac{1}{8}})$$

where $M = \sup_{1 \leqq j \leqq N} \|\alpha_j\|$ and $h_j = (h, \alpha_j)$. Thus

$$P(A_r) \geqq \text{constant} \cdot \textstyle\prod_{j=1}^N \frac{\exp\{-t_r^2 h_j^2/2\}}{t_r}$$

for r sufficiently large since

$$\textstyle\int_a^b e^{-s^2/2}\, ds \geqq \frac{1}{b} e^{-a^2/2}[1 - e^{-(b^2 - a^2)/2}] \qquad (0 \leqq a < b)$$

$$\geqq \text{constant} \cdot \exp\left\{-\gamma \operatorname{LL} n_r \Big/ \left(1 - \frac{1}{\beta}\right)\right\} \Big/ (\operatorname{LL} n_r)^N$$

for r sufficiently large since $\lim_r \dfrac{n_r}{n_r - n_{r-1}} = \dfrac{1}{1 - 1/\beta}$

$$\geqq \text{constant} \cdot \frac{1}{[r \log \beta]^{\gamma/(1-1/\beta)+\delta}} \quad \text{for any} \quad \delta > 0$$

$$\geqq \text{constant} \cdot \frac{1}{r^\alpha} \quad \text{where} \quad \alpha = \frac{\gamma}{1 - 1/\beta} + \delta < 1 \quad \text{provided}$$

$\delta > 0$ is sufficiently small since $0 \leqq \dfrac{\gamma}{1 - 1/\beta} < 1$.

Thus (3.15) holds.

Let

$$B_r = \left\{\left\|\frac{\Pi_N S_{n_r}}{(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}} - \Pi_N h\right\| < \frac{\varepsilon}{6} + \frac{M_1}{\beta^{\frac{1}{2}}}\right\}$$

$$D_r = \left\{\left\|\frac{\Pi_N S_{n_{r-1}}}{(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}}\right\| < \frac{M_1}{\beta^{\frac{1}{2}}}\right\}$$

where M_1 is any constant satisfying

$$\frac{M_1}{2} \geqq \sup_{x \in K} \|x\| .$$

Clearly M_1 can be taken to be finite since as was mentioned previously K is compact in H.

By the previous arguments we know that for all w in a set E of probability one we have an integer $J_{(w)}$ such that $w \in D_r$ for all $r \geqq J(w)$. Thus for all $w \in E \cap \limsup_r A_r$ we have $w \in \limsup_r B_r$ and hence

$$P(\limsup_r B_r) = 1 \tag{3.17}$$

for all β sufficiently large so that (3.15) holds. Choosing β sufficiently large so that (3.15) holds and $M_1/\beta^{\frac{1}{2}} < \varepsilon/6$ we have by (3.17) that (3.14) holds for infinitely many r. This completes the proof.

We now turn to the functional form of the law of the iterated logarithm for the partial sum processes generated by the sequence $\{X_k\}$. More precisely, let $X_1, X_2, \cdots$ be independent H-valued random bariables satisfying the conditions of Theorem 3.1 and assume μ is the mean zero Gaussian measure on H having covariance operator T. Let C_H denote the real separable Banach space of continuous functions on [0, 1] into H which vanish at zero under the norm

$$\|f\|_\infty = \sup_{0 \leqq t \leqq 1} \|f(t)\| .$$

Let $\mathscr{H}$ denote the Hilbert space in C_H which generates the Brownian motion in H induced by μ up to time 1 (see [7] or [9] for details). Then by [7] or [9] we know

$$\mathscr{H} = \left\{f \in C_H : f(t) = \textstyle\sum_{j \leq 1} \int_0^t \frac{d}{ds}\, \alpha_j(f(s))\, ds\, \alpha_j \quad (0 \leqq t \leqq 1)\right\}$$

where the convergence of the series is in the H_μ norm for each t and $\{\alpha_j\}$ is any complete orthonormal basis for H_μ which is also a subset of H^* under the embedding of H^* in H_μ as discussed in the introduction. The norm on $\mathscr{H}$ is given by $\|f\|_{\mathscr{H}}^2 = \sum_{j\geq 1} \int_0^1 [(d/ds)\alpha_j(f(s))]^2\, ds$.

Let $\mathscr{K}$ denote the unit ball of $\mathscr{H}$ and define for each w

$$f_n(k/n, w) = \frac{S_k(w)}{(2n\,\mathrm{LL}\,n)^{\frac{1}{2}}} \qquad (k = 0, 1, \cdots, n;\, n \geqq 3)\,.$$

Let $f_n(t, w)$ be linear over the subintervals $k/n \leqq t \leqq (k+1)/n$ for $k = 0, \cdots, n-1$. Here, of course, $S_k = X_1 + \cdots + X_k$ for $k \geqq 1$ and $S_0 = 0$.

THEOREM 3.2. *Let X_1, X_2 be independent H-valued random variables satisfying the conditions of Theorem* 3.1. *Let $\mathscr{K}$ and $\{f_n\colon n \geqq 3\}$ be defined as indicated above. Then*

$$P(\lim_n \|f_n(t, w) - \mathscr{K}\|_\infty = 0) = 1 \tag{3.18}$$

and, in fact, with probability one the sequence $\{f_n\colon n \geqq 3\}$ accumulates at every point of $\mathscr{K}$ with respect to the norm $\|\cdot\|_\infty$.

REMARK 3.2. Since $\mathscr{K}$ is a compact subset of C_H ([9] Lemma 3) Theorem 3.2 is equivalent to saying that with probability one the sequence of functions $\{f_n\colon n \geqq 3\}$ is conditionally compact in C_H and that with probability one the set of limit points of each sequence is precisely $\mathscr{K}$. Further, it is easy to see that Theorem 3.1 follows immediately from Theorem 3.2 since $f_n(1, w) = S_n/(2n\,\mathrm{LL}\,n)^{\frac{1}{2}}$ and $K = \{h \in H\colon h = f(1) \text{ where } f \in \mathscr{K}\}$. We proved it directly as it provides insight into the more detailed proof required for Theorem 3.2, and because some of the arguments used in Theorem 3.1 can be alluded to, thus shortening the proof of Theorem 3.2. As mentioned in the introduction, the ideas of [3] are used extensively.

LEMMA 3.1. *With probability one the sequence $\{f_n\colon n \geqq 3\}$ is equicontinuous uniformly on* $[0, 1]$.

The proof of Lemma 3.1 can be carried out exactly as Theorem 2 of [3] except that the estimates of Corollary 2.1 must be used and Lemma 1 of [3] must be replaced by estimates of the type used in (3.4). We omit further details.

For any integer m and $f \in C_H$ let $\Gamma_m f$ denote the piecewise linear approximation to f such that

$$\Gamma_m f(k/m) = f(k/m) \qquad (k = 0, 1, \cdots, m)$$

and $\Gamma_m f$ is linear on each of the subintervals $[k/m, (k+1)/m]$ for $k = 0, \cdots, m-1$.

In view of the equicontinuity in Lemma 3.1 we have:

LEMMA 3.2. *With probability one and for any $\varepsilon > 0$ there is an integer $m_0 = m_0(w, \varepsilon)$ such that*

$$\|\Gamma_m f_n - f_n\| < \varepsilon$$

for all $m, n \geqq m_0$.

LEMMA 3.3. *With probability one and for any* $\varepsilon > 0$ *there is an integer* $n_0 = n_0(w, \varepsilon)$ *such that*

$$\|f_n - f_{n'}\|_\infty < \varepsilon$$

for all $n, n' \geqq n_0$ *provided* $|1 - n'/n| < 1/n_0$.

The proof of Lemma 3.3 is exactly as that of Corollary 2 in [3] so we omit it.

LEMMA 3.4. *Under the hypothesis of Theorem* 3.2 *we have* (3.18) *holding.*

PROOF. In view of Lemma 3.2 and Lemma 3.3 we have (3.18) if for $\varepsilon > 0$, $\beta > 1$ and all integers $m \geqq 1$ we have $\Gamma_m f_{[\beta^r]} \in \mathscr{K}^\varepsilon$ for all r sufficiently large on a set of probability one. We let $n_r = [\beta^r]$.

Now for $N = 1, 2, \cdots$

$$P(\Gamma_m f_{n_r} \notin \mathscr{K}^\varepsilon) \leqq P(\Pi_N \Gamma_m f_{n_r} \notin \mathscr{K}^{\varepsilon/2}) + P(\|(I - \Pi_N)\Gamma_m f_{n_r}\|_\infty \geqq \varepsilon/2) \tag{3.19}$$

where Π_N is defined as in the proof of Theorem 3.1. Now

$$\begin{aligned} P(\|(I - \Pi_N)\Gamma_m f_{n_r}\|_\infty > \varepsilon/2) \\ &= P(\sup_{1\leqq k\leqq m} \|(I - \Pi_N) f_{n_r}(k/m, w)\| > \varepsilon/2) \\ &\leqq P(\sup_{1\leqq k\leqq m} \|(I - \Pi_N) S_{\rho_k}(w)/(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}\| > \varepsilon/4) \\ &\quad + P(\sup_{1\leqq k\leqq m} \|(I - \Pi_N) S_{\lambda_k}(w)/(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}\| > \varepsilon/4) \end{aligned} \tag{3.20}$$

where $\rho_k = [n_r k/m]$ $(k = 0, 1, \cdots, m)$ and $\lambda_k = \rho_{k+1}$ for $k = 0, 1, \cdots, m - 1$ and $\lambda_m = \rho_m = n_r$. Now each of the terms on the right-hand side of (3.20) can be estimated using an inequality as developed for (3.4). Using the approximation of Corollary 2.1 on these estimates we obtain

$$\begin{aligned} &P(\|(I - \Pi_N)\Gamma_m f_{n_r}\|_\infty > \varepsilon/2) \\ &\quad \leqq 2P\left(\|(I - \Pi_N)Z\| > \frac{\varepsilon}{4} (2 \operatorname{LL} n_r)^{\frac{1}{2}}\right) + O(n_r^{-\frac{1}{8}}) \end{aligned} \tag{3.21}$$

where $\mathscr{L}(Z) = \mu$.

Thus for fixed $\beta > 1$ and $c > 1$ we can argue as in (3.8) to obtain N_0 such that $N \geqq N_0$ implies

$$P(\|(I - \Pi_N)\Gamma_m f_{n_r}\|_\infty > \varepsilon/2) < 2 \exp\{-c \operatorname{LL} n_r\} + O(n_r^{-\frac{1}{8}}) . \tag{3.22}$$

Combining (3.19) and (3.22) we get $\Gamma_m f_{n_r} \in \mathscr{K}^\varepsilon$ for all r sufficiently large on a set of probability one if for any $N \geqq N_0$

$$\textstyle\sum_r P(\Pi_N \Gamma_m f_{n_r} \notin \mathscr{K}^{\varepsilon/2}) < \infty . \tag{3.23}$$

Since the H-norm is weaker than the norm $\|\cdot\|_\mu$ on H_μ there exists a $\delta > 0$ such that if $\mathscr{D} = \{g \in \mathscr{H}: \|g - \mathscr{K}\|_{\infty,\mu} < \delta\}$ (where $\|g - f\|_{\infty,\mu} = \sup_{0\leqq t\leqq 1} \|g(t) - f(t)\|_\mu$ for $f, g \in \mathscr{H}$) then $\mathscr{D} \subseteq \mathscr{K}^{\varepsilon/2}$. Thus (3.23) holds if we show

$$\textstyle\sum_r P(\Pi_N \Gamma_n f_{n_r} \notin \mathscr{D}) < \infty .$$

Let $E = \Pi_N(H) = \Pi_N(H_\mu)$ and define

$$K_m = \{(k_1, \cdots, k_m): k_i \in E \ \ (i = 1, \cdots, m) \ \text{ and } \ \textstyle\sum_{i=1}^m \|k_i\|_\mu^2 \leqq 1\} .$$

Let ρ_k and λ_k be as in (3.20). Then

$$\begin{aligned}(3.24)\qquad & P(\Pi_N \Gamma_m f_{n_r} \notin \mathscr{D}) \\ & \leqq P((\Pi_N S_{\rho_1}, \Pi_N(S_{\rho_2} - S_{\lambda_1}), \cdots, \Pi_N(S_{\rho_m} - S_{\lambda_{m-1}})) \\ & \quad \notin (2n_r \operatorname{LL} n_r/m)^{\frac{1}{2}} K_m^{\delta/2}) \\ & \quad + \sum_{k=1}^m P\left(\|\Pi_N X_{\lambda_k}\|_\mu + \|\Pi_N X_{\lambda_{k-1}}\|_\mu > \frac{\delta(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}}{2m}\right)\end{aligned}$$

where $K_m^{\delta/2} = \{(k_1, \cdots, k_m) : k_i \in E\ (i = 1, \cdots, m), |||(k_1, \cdots, k_m) - K_m||| < \delta/2\}$ and $|||(k_1, \cdots, k_m)|||^2 = \sum_{i=1}^m \|k_i\|_\mu^2$. That is, if $(\Pi_N(S_{\rho_1}(w)), \Pi_N(S_{\rho_2}(w) - S_{\lambda_1}(w)), \cdots, \Pi_N(S_{\rho_m}(w) - S_{\lambda_{m-1}}(w)))$ is in $(2n_r \operatorname{LL} n_r/m)^{\frac{1}{2}} K_m^{\delta/2}$ and $\|\Pi_N X_{\lambda_k}(w)\|_\mu + \|\Pi_N X_{\lambda_{k-1}}(w)\|_\mu \leqq \delta(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}/2m$ for $k = 1, \cdots, m$, then

$$\sum_{k=1}^m \left\|\frac{\Pi_N(S_{\rho_k}(w) - S_{\lambda_{k-1}}(w))}{(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}}\right\|_\mu^2 \leqq \frac{1}{m}(1 + \delta/2)^2$$

and hence

$$\begin{aligned}& \int_0^1 \left\|\frac{d}{dt} \Pi_N \Gamma_m f_{n_r}(t, w)\right\|_\mu^2 dt \\ & \leqq m \sum_{k=1}^m \left[\frac{\|\Pi_N(S_{\rho_k}(w) - S_{\lambda_{k-1}}(w)\|_\mu + \|\Pi_N X_{\lambda_k}(w)\|_\mu + \|\Pi_N X_{\lambda_{k-1}}(w)\|_\mu}{(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}}\right]^2 \\ & \leqq (1 + \delta)^2\end{aligned}$$

which implies $\Pi_N \Gamma_m f_{n_r}(t, w) \in \mathscr{D}$.

Now the first term on the right-hand side of (3.24) can be dominated by a Gaussian probability plus an error term of size $O(n_r^{-\frac{1}{2}})$ by applying Corollary 2.1 to the Hilbert space $H_m = \{(k_1, \cdots, k_m) : k_i \in E\ (i = 1, \cdots, m)\}$ with norm $|||\cdot|||$, and the second term can be estimated using Chebyshev's inequality yielding

$$\begin{aligned}(3.25)\qquad & \sum_{k=1}^m P\left(\|\Pi_N X_{\lambda_k}\|_\mu + \|\Pi_N X_{\lambda_{k-1}}\|_\mu > \frac{\delta(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}}{2m}\right) \\ & = O(\delta^{-2}(n_r \operatorname{LL} n_r)^{-1}) .\end{aligned}$$

Arguing as in (3.9) we find that the Gaussian probabilities sum to something finite as r goes to infinity and since the error terms plus the bounds in (3.25) also sum we have $\sum_r P(\Pi_N \Gamma_m f_{n_r} \in \mathscr{D}) < \infty$. This proves that (3.18) holds.

LEMMA 3.5. *With probability one the sequence* $\{f_n(t, w) : n \geqq 3\}$ *accumulates at every point of* $\mathscr{K}$ *with respect to the norm* $\|\cdot\|_\infty$.

PROOF. Since $\mathscr{K}$ is separable in the norm $\|\cdot\|_\infty$ (it is compact by [9]) it suffices to prove for each $g \in \mathscr{K}$, $\|g\|_{\mathscr{H}} < 1$, and $\varepsilon > 0$ there is a $\beta > 1$ so that with probability one

$$\|f_{n_r} - g\|_\infty < \varepsilon$$

for infinitely many r. By Lemma 3.1 and Lemma 3.2 there is an integer $m = m(\varepsilon, w)$ such that with probability one $\|f_{n_r}(t, w) - \Gamma_m f_{n_r}(t, w)\|_\infty < \varepsilon/2$ for all sufficiently large r and also $||g - \Gamma_m g||_\infty < \varepsilon/2$. Thus it suffices to show that with probability one $||\Gamma_m f_{n_r} - \Gamma_m g||_\infty < \varepsilon$ for infinitely many r.

Set $\beta = m$. Then $n_r = m^r$ and

$$||\Gamma_m f_{n_r} - \Gamma_m g||_\infty = \sup_{1\leq k\leq m}\left|\left|\frac{S_{kn_{r-1}}}{(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}} - g\left(\frac{k}{m}\right)\right|\right|$$

$$\leq \sum_{k=1}^m \left|\left|\frac{(S_{kn_{r-1}} - S_{(k-1)n_{r-1}})}{(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}} - \left(g\left(\frac{k}{m}\right) - g\left(\frac{k-1}{m}\right)\right)\right|\right| \tag{3.26}$$

$$\leq m^{\frac{1}{2}}|||\Gamma_m(f_{n_r} - g)|||_1$$

where $|||\Gamma_m f|||_1^2 = \sum_{k=1}^m ||f(k/m) - f((k-1)/m)||^2$. Furthermore, $g \in \mathscr{K}$ and $||g||_{\mathscr{H}} < 1$ implies

$$\sum_{k=1}^m \left|\left|g\left(\frac{k}{m}\right) - g\left(\frac{k-1}{m}\right)\right|\right|_\mu^2 = \sum_{j\geq 1}\sum_{k=1}^m \left(\int_{(k-1)/m}^{k/m} \frac{d}{ds}\alpha_j(g(s))\,ds\right)^2$$

$$\leq \sum_{j\geq 1}\sum_{k=1}^m \int_{(k-1)/m}^{k/m}\left[\frac{d}{ds}\alpha_j(g(s))\right]^2 ds \cdot \frac{1}{m} \tag{3.27}$$

$$\leq \frac{1}{m}\sum_{j\geq 1}\int_0^1\left[\frac{d}{ds}\alpha_j(g(s))\right]^2 ds$$

$$\leq \frac{1}{m}||g||_{\mathscr{H}} < \frac{1}{m}.$$

Set $\tilde{f}_{n_r} = (m^{\frac{1}{2}}/(2n_r \operatorname{LL} n_r)^{\frac{1}{2}})(S_{n_{r-1}}, S_{2n_{r-1}} - S_{n_{r-1}}, \cdots, S_{mn_{r-1}} - S_{(m-1)n_{r-1}})$ and $\tilde{g} = m^{\frac{1}{2}}(g(1/m), g(2/m) - g(1/m), \cdots, g(1) - g((m-1)/m))$.

Then (3.27) implies $\tilde{g}$ is in the unit ball of the Hilbert space obtained by taking a product of H_μ m times with the norm

$$|||(f_1, \cdots, f_m)|||_\mu = (\textstyle\sum_{k=1}^m ||f_k||_\mu^2)^{\frac{1}{2}}. \tag{3.28}$$

Taking the product of H m times with norm $|||(f_1, \cdots, f_m)||| = (\sum_{k=1}^m ||f_k||^2)^{\frac{1}{2}}$ we see that $|||\tilde{f}_{n_r} - \tilde{g}||| = m^{\frac{1}{2}}|||\Gamma_m(f_{n_r} - g)|||_1$, and hence (3.26) implies that

$$||\Gamma_m f_{n_r} - \Gamma_m g||_\infty < \varepsilon$$

infinitely often with probability one if

$$|||\tilde{f}_{n_r} - \tilde{g}||| < \varepsilon \tag{3.29}$$

infinitely often with probability one. Now (3.29) occurs infinitely often with probability one by applying the argument of the second part of Theorem 3.1 in the product Hilbert spaces. Thus Lemma 3.5 holds.

The proof of Theorem 3.2 now follows by combining Lemma 3.4 and Lemma 3.5.

4. The log log law for symmetric stable random variables. A random variable X induces a stable measure or law on H if $X_1, \cdots, X_n$ being independent copies of X implies

$$\mathscr{L}(X_1 + \cdots + X_n) = \mathscr{L}(a_n X + b_n)$$

where a_n is a positive constant and $b_n \in H$. Stable random variables with values in H can be characterized by their type ($0 < \alpha \leqq 2$) as in the case $H = (-\infty, \infty)$ and, for example, being of type 2 means that the random variable has a Gaussian law. For details see [8]. For our immediate purposes a useful fact about stable laws in H was obtained in [10] and is given in the next lemma without proof.

LEMMA 4.1. *If $\mu = \mathscr{L}(X)$ is a stable law of type α ($0 < \alpha < 2$) on H, then*

$$P(\|x\| > t) = \mu(x \in H: \|x\| > t) \sim C/t^\alpha \qquad \text{as } t \to \infty \tag{4.1}$$

where $C > 0$.

Lemma 4.1 now enables us to generalize the result of [2] to H-valued independent identically distributed symmetric stable laws of type α, $0 < \alpha \leqq 2$.

THEOREM 4.1. *Let $\{X_k: k \geqq 1\}$ be a sequence of independent identically distributed symmetric stable laws of type α, $0 < \alpha < 2$, with values in H and let $S_n = \sum_{k=1}^n X_k$. Then*

$$P(\limsup_n \|S_n/n^{1/\alpha}\|^{1/\mathrm{LL}\, n} = e^{1/\alpha}) = 1\,. \tag{4.2}$$

If $\mu = \mathscr{L}(X_1)$ is symmetric and of type 2, i.e. *μ is mean zero Gaussian, then*

$$P\left(\limsup_n \left\|\frac{S_n}{(2n\,\mathrm{LL}\, n)^{\frac{1}{2}}}\right\| = \rho^{\frac{1}{2}}\right) = 1 \tag{4.3}$$

where ρ is the maximal eigenvalue of the covariance operator T of μ.

PROOF. For (4.2) to hold if suffices to show that for any $\varepsilon > 0$ we have with probability one that

$$\|S_n/n^{1/\alpha}\| > (\log n)^{(1+\varepsilon)/\alpha} \quad \text{finitely often} \tag{4.4}$$

and

$$\|S_n/n^{1/\alpha}\| > (\log n)^{(1-\varepsilon)/\alpha} \quad \text{infinitely often.} \tag{4.5}$$

To verify (4.4) and (4.5) the proof proceeds exactly as in [2] except that one needs Lemma 4.1 and the inequality $P(\sup_{1\leqq k\leqq n} \|S_k\| > t) \leqq 2P(\|S_n\| > t)$ which follows as in the case $H = (-\infty, \infty)$ using the symmetry of the random variables. Further details are omitted but can be found in [2].

To establish (4.3) which is the analogue of (4.2) for symmetric Gaussian laws we apply Theorem 3.1 to obtain

$$P\left(\limsup_n \left\|\frac{S_n}{(2n\,\mathrm{LL}\, n)^{\frac{1}{2}}}\right\| = \sup_{x\in K} \|x\|\right) = 1 \tag{4.6}$$

where K is the unit ball of the Hilbert space H_μ which generates μ on H. Furthermore, if T is the covariance operator of μ on H, then $Tx = \sum_k \lambda_k(x, e_k)e_k$ where $\lambda_k > 0$, $\sum \lambda_k < \infty$, and $\{e_n\}$ is an orthonormal set in H. If M equals the closed subspace generated by $\{e_n\}$ then from [6] we know that

$$H_\mu = \{x \in M: \textstyle\sum_k (x, e_k)^2/\lambda_k < \infty\}\,.$$

Further, the inner product for H_μ is given by

$$(x, y)_\mu = \sum_k \frac{(x, e_k)(y, e_k)}{\lambda_k}$$

and hence

$$K = \left\{x \in M\colon \sum_k \frac{(x, e_k)^2}{\lambda_k} \leqq 1\right\}.$$

Now if $x \in K$ then $x \in M$ so

$$||x||^2 = \sum_k (x, e_k)^2 = \sum_k \lambda_k \frac{(x, e_k)^2}{\lambda_k} \leqq \sup_k \lambda_k \sum_k \frac{(x, e_k)^2}{\lambda_k} \leqq \sup_k \lambda_k$$

and hence

$$\sup_{x \in K} ||x|| \leqq (\sup_k \lambda_k)^{\frac{1}{2}} = \rho^{\frac{1}{2}}.$$

By choosing $x = \rho^{\frac{1}{2}} e_j$ where $\rho = \lambda_j$ we find $||x|| = \rho^{\frac{1}{2}}$ and $x \in K$ so (4.3) holds and the theorem is proved.

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