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THE OPTIMAL REWARD OPERATOR IN DYNAMIC PROGRAMMING

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Consider a dynamic programming problem with analytic state space S, analytic constraint set A, and semi-analytic reward function $r(x, P, y)$ for $(x, P) \in A$ and $y \in S$: namely, $\{r > a\}$ is an analytic set for all a. Let Tf be the optimal reward in one move, with the modified reward function $r(x, P, y) + f(y)$. The optimal reward in n moves is shown to be $T^n 0$, a semi-analytic function on S. It is also shown that for any n and positive ε, there is an ε-optimal strategy for the n-move game, measurable on the σ-field generated by the analytic sets.

1. Introduction. There is a state space S, endowed with a σ-field $\sigma(S)$. Let $\pi(S)$ be the set of all probabilities on $\sigma(S)$. There is a given subset A of $S \times \pi(S)$, whose x-section A_x is nonempty for all $x \in S$. There is a nonnegative function $r(x, P, y)$ of $(x, P) \in A$ and $y \in S$, which is a $\sigma(S)$-measurable function of y for each pair (x, P). When you are at $x \in S$, you can select any $P \in A_x$, and move to a new state $y \in S$ selected at random according to P. You receive the reward $r(x, P, y)$. If you select P, your expected reward is $\int_S r(x, P, y)P(dy)$, and your optimal reward in one move is

$$u_1(x) = \sup_{P \in A_x} \int_S r(x, P, y)P(dy) .$$

Even under very stringent regularity conditions, u_1 need not be $\sigma(S)$-measurable: see example (45) below. To get around this for now, let $\int^*$ be the usual outer integral. Compare Dubins and Savage (1965) pages 8–9.

Suppose you are allowed to move twice. If your first move is to y, the most you can get on your second move is $u_1(y)$. So your optimal reward in two moves is

$$u_2(x) = \sup_{P \in A_x} \int_S^* [r(x, P, y) + u_1(y)]P(dy) ,$$

and so on. To study this formally, it is convenient to introduce the optimal reward operator T which transforms nonnegative functions f on S as follows:

$$(1) \qquad (Tf)(x) = \sup_{P \in A_x} \int_S^* [r(x, P, y) + f(y)]P(dy) .$$

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Let 0 be the function which vanishes identically. Then $T^n 0$ is your optimal reward in n moves, over the class of all strategies, measurable or not.

Fix $\mu \in \pi(S)$. Under mild regularity conditions, we will show that $T^n 0$ is μ-measurable; and there is a μ-measurable strategy whose expected reward in n moves is close to $(T^n 0)(x)$. The conditions are that S and A be analytic, and r be *semi-analytic*. A function f is semi-analytic if its domain is an analytic set, it takes nonnegative values, and $\{f > a\}$ is analytic for all $a > 0$. More precisely, we show that $T^n 0$ is semi-analytic, and that there are ε-optimal strategies measurable over the σ-field generated by the analytic sets. As usual, $\pi(S)$ is endowed with the weak * topology and the σ-field this topology generates.

If S, A, and r are Borel, Strauch (1966) showed that the optimal reward over measurable strategies is nearly measurable (an exact statement is too complicated to be helpful here) in the starting state, and can almost be realized by nearly measurable strategies of various special kinds. Strauch (1967) and Sudderth (1971) show that nearly measurable strategies do as well as nonmeasurable ones, when S, A, and r are Borel, and the total return is uniformly bounded. Our results hold for more general S, A, and r, and for unbounded returns. We construct analytically measurable strategies which are available everywhere, instead of Borel measurable strategies which are available a.e. And we provide a countably additive method for evaluating nonmeasurable strategies (which is at least as generous as the finitely additive methods). The main novelty, however, is showing how to compute the optimal reward from T, without leaving the class of functions which can be integrated by the ordinary countably additive method. Our work overlaps to some extent with unpublished notes by P. A. Meyer. The general issue of measurability of optimal rewards and existence of nearly optimal and measurable strategies was raised by Dubins and Savage (1965) pages 35–38.

Section 2 reviews some known facts about analytic sets. Section 3 reviews a selection theorem of von Neumann (1949) and Mackey (1957). Section 4 introduces the semi-analytic functions. Section 5 discusses compact metric state spaces. Section 6 establishes the dynamic programming results. Section 7 presents some examples.

2. Analytic sets. A detailed discussion of analytic sets, with proofs for most of the facts listed below, can be found in Kuratowski (1966). Let N be the set of sequences of positive integers, endowed with the product topology. So N is homeomorphic to the irrationals. Let A be a separable metric set. Then A is *analytic* provided there is a continuous function f on N whose range $f(N)$ is A.

(2) Any complete, separable metric set is analytic.

(3) Any Borel subset (that is, a set in the σ-field generated by the open sets) of an analytic set is analytic.

(4) Countable unions, intersections, and products of analytic sets are analytic.

(5) Let f be a Borel measurable mapping from the analytic set S into the

analytic set T. If A is an analytic subset of S, then $f(A)$ is an analytic subset of T. If B is an analytic subset of T, then $f^{-1}(B)$ is an analytic subset of S.

(6) Let A be an analytic subset of the analytic set S. Then A is universally measurable; that is, if μ is any probability on the Borel subsets of S, then A is μ-measurable. For $x \in N$, let $L(x) = \{y : y \in N \text{ and } y_i \leqq x_i \text{ for all } i\}$. Let f be a continuous function on N with $f(N) = A$. Then

$$\mu(A) = \sup_{x \in N} \mu\{f[L(x)]\}.$$

Facts (7)—(9) will be used only for constructing examples.

(7) There is an analytic subset of the unit interval whose complement is not analytic. In fact, there is a Borel subset of the unit square (even a G_δ), whose projection on the horizontal axis is analytic but not Borel.

The complement of an analytic set relative to a Borel subset of a complete separable metric set is called *complementary analytic.*

(8) There is a Borel subset B of the unit square, whose projection on the horizontal axis is the whole unit interval: but there is no Borel function f of the unit interval into itself, with $(x, f(x)) \in B$ for all x.

For a discussion of (8), see Blackwell (1968).

(9) According to Gödel (1938), it is consistent with the usual axioms of set theory to assume there is a complementary analytic subset of the unit square, whose projection on the horizontal axis is not Lebesgue measurable.

We will use the notion of the weak* topology on $\pi(S)$, and review the main points here. For a detailed discussion, see Parthasarathy (1967) Sec. II. 6. Let S be a separable metric set. Let P_n and P be probabilities on the Borel subsets of S. By definition, $P_n \to P$ in the weak* topology, provided $\int_S f\,dP_n \to \int_S f\,dP$ for all bounded, continuous f.

(10) If S is compact metric, so is $\pi(S)$ in the weak* topology.

(11) If f is bounded and continuous on S, there are bounded, uniformly continuous f_n on S with $f_n \uparrow f$.

(12) S can be homeomorphically embedded in a compact metric set S^*. After the embedding, $\pi(S)$ becomes the set of $\mu \in \pi(S^*)$ which assign outer measure 1 to S, with the relative topology.

(13) $\pi(S)$ is separable metric. The Borel σ-field on $\pi(S)$ is the σ-field generated by the weak* topology.

(14) The Borel σ-field is also generated by the functions $\mu \to \int_S f\,d\mu$, for

(a) all bounded, continuous f.

(b) all indicator functions f of any class of sets which generates the Borel σ-field in S.

If S is Borel (that is, a Borel subset of a complete separable metric set), then $\pi(S)$ is Borel. If S is analytic, we will show $\pi(S)$ is analytic. If S is a universally measurable subset of S^*, we will show $\pi(S)$ is a universally measurable subset of $\pi(S^*)$.

3. A selection theorem.

(15) PROPOSITION. *Let S and T be analytic sets. Let $\mathscr{A}$ be the σ-field in S generated by the analytic sets. Let p project $S \times T$ onto S. Let A be an analytic subset of $S \times T$. Then there is an $\mathscr{A}$-measurable function t from pA to T, with $(x, t(x)) \in A$ for all $x \in pA$.*

NOTE. *$\mathscr{A}$-measurable* means $t^{-1}B \in \mathscr{A}$, for all Borel subsets B of T.

PROOF. Let f be a continuous function from N to $S \times T$, whose range is A. So $p \circ f$ is a continuous function on N whose range is pA; and $f^{-1}p^{-1}(x)$ is a nonempty closed subset of N for $x \in pA$. Let $\phi(x)$ be the least element of $f^{-1}p^{-1}(x)$, in the lexicographic order, for $x \in pA$. So ϕ maps pA into N. Now $pA \in \mathscr{A}$ by (5). You can check that ϕ is $pA \cap \mathscr{A}$-measurable. Finally, t is the projection of $f \circ \phi$ on T. □

This result is due to Mackey (1957) and von Neumann (1949). Even if S, T, and A are Borel, and $\text{proj}_S A = S$, there may not be a Borel selector (8).

4. Semi-analytic functions. Let f be a nonnegative, real-valued function defined on the analytic set S. Then f is *semi-analytic* provided the set $\{x : f(x) > a\}$ is analytic for all nonnegative a. See Kuratowski (1966) Sec. 35, XI. You can verify the following:

(16) If f is semi-analytic on S, then $\{x : f(x) \geqq a\}$ is analytic.

(17) Any nonnegative Borel measurable function on S is semi-analytic.

(18) Suppose f is semi-analytic on S, and T is an analytic subset of S. The restriction of f to T is semi-analytic on T.

(19) Suppose f and g are semi-analytic functions on S. Then $f+g$, $\max\{f, g\}$, $\min\{f, g\}$ and fg are also semi-analytic. If $c \geqq 0$, then cf is also semi-analytic.

(20) Let f_n be semi-analytic on S for each n. Suppose $f_n \uparrow f$ or $f_n \downarrow f$. Then f is semi-analytic.

(21) Let f be a semi-analytic function on S. Let g be a Borel measurable function from the analytic set T into S. Then $f \circ g$ is semi-analytic on T.

(22) If f is semi-analytic on S, then f is universally measurable; so f can be integrated with respect to any probability on the Borel subsets of S.

(23) EXAMPLE. Let A be an analytic subset of the unit interval, whose complement is not analytic. Then 1_A is semi-analytic, but $1 - 1_A$ is not semi-analytic.

(24) EXAMPLE. A Borel measurable function composed with a semi-analytic function need not be semi-analytic. Again, let A be an analytic subset of the unit interval, whose complement A^c is not analytic. Then $1_{\{0\}} \circ 1_A = 1_{A^c}$, which is not semi-analytic.

5. The compact metric case. Let S be a compact metric set. Let $\pi(S)$ be the set of probability measures on the σ-field of Borel subsets of S. Endow $\pi(S)$ with the weak* topology, so $\pi(S)$ is again compact metric. For a detailed discussion of $\pi(S)$, see Dubins and Freedman (1965) or Parthasarathy (1967).

(25) LEMMA. *If A is an analytic subset of S, and a is a nonnegative real number, then $\{\mu : \mu \in \pi(S)$ and $\mu(A) > a\}$ is an analytic subset of $\pi(S)$.*

NOTE. $\mu(A)$ is well defined in (6).

PROOF. Let 2^S be the set of nonempty, compact subsets of S, endowed with the usual compact metric topology (Hausdorff (1957) Section 28). If K_n and K are elements of 2^S, then $K_n \to K$ in this topology provided

(26) for each $s \in K$, there are $s_n \in K_n$ with $s_n \to s$, and

(27) if $s_n \in K_n$ and $s_n \to s$, then $s \in K$.

Since A is analytic, there is a continuous function f on N with $f(N) = A$. Define $L(x)$ as in (6). We claim that the function $x \to f(L(x))$ is continuous from N to 2^S. Indeed, $L(x)$ is compact, so $f(L(x)) \in 2^S$. Suppose $x(n) \to x$ in N. We have to show $f[L[(x(n)]] \to f[L(x)]$. To verify (26), suppose $y \in L(x)$ and $s = f(y)$. Let $y(n)_i = \min [x(n)_i, y_i]$. Then $y(n) \in L(x(n))$ and $y(n) \to y$, so $f(y(n)) \to f(y)$. To verify (27), let $z_i = \sup_n x(n)_i$, so $z \in N$. Let $y(n) \in L(x(n))$, and suppose $s_n = f(y(n)) \to s$ in S. Now $y(n) \in L(z)$, which is compact. By passing to a subsequence, suppose $y(n) \to y$ in N. Clearly, $y \in L(x)$. But $s = \lim f(y(n)) = f(y)$.

The function $(\mu, K) \to \mu(K)$ is upper semi-continuous from $\pi(S) \times 2^S$ to the unit interval, by Dubins and Freedman (1965) Theorem 3.8. So the composition $(\mu, x) \to \mu\{f[L(x)]\}$ is upper semi-continuons from $\pi(S) \times N$ to the unit interval. In particular,

$$B = \{(\mu, x) : \mu\{f[L(x)]\} > a\}$$

is Borel in $\pi(S) \times N$; in fact, B is an F_σ. Now $\{\mu : \mu(A) > a\}$ is the projection of B on the μ-axis by (6), and is analytic by (5). □

Let λ be Lebesgue measure on the unit interval. The next result is a known consequence of Fubini's theorem.

(28) LEMMA. *Let $(S, \mathscr{A}, P)$ be a probability triple. Let f be an $\mathscr{A}$-measurable function from S to the unit interval. Then $\{(y, z) : f(y) > z\}$ is a product-measurable subset of $S \times [0, 1]$. And*

$$\int_S f(y)P(dy) = (P \times \lambda)\{(y, z) : f(y) > z\} .$$

(29) LEMMA. *Let $r(x, Q, y)$ be a semi-analytic function from $S \times \pi(S) \times S$ to*

the unit interval. Let

$$h(x, Q, P) = \int_S r(x, Q, y)P(dy) .$$

Then h is semi-analytic on $S \times \pi(S) \times \pi(S)$.

PROOF. Let $\Gamma = S \times \pi(S) \times S \times [0, 1]$, a compact metric set in the product topology. Define the function g from $S \times \pi(S) \times \pi(S)$ to $\pi(\Gamma)$ as follows: $g(x, Q, P)$ is $P \times \lambda$ installed on the (x, Q)-slice of Γ. More formally, if ϕ is a continuous function on Γ, then the $g(x, Q, P)$-integral of ϕ is

$$\int_{S\times[0,1]} \phi(x, Q, y, z)(P \times \lambda)(dy, dz) .$$

We say that g is continuous. To check this, approximate ϕ by finite linear combinations of functions ψ of the form $\phi_1(x)\phi_2(Q)\phi_3(y)\phi_4(z)$, where ϕ_1 and ϕ_3 are continuous functions on S, while ϕ_2 is a continuous function on $\pi(S)$, and ϕ_4 is a continuous function on the unit interval. This is possible by Stone–Weierstrass. The $g(x, Q, P)$-integral of ψ depends continuously on (x, Q, P). Therefore, so does the $g(x, Q, P)$-integral of ϕ.

Let $A = \{(x, Q, y, z): r(x, Q, y) > z\}$. We say A is an analytic subset of Γ. Indeed, $A = \bigcup_t A_t$, where t is a positive rational, and $A_t = \{r(x, Q, y) > t\} \cap \{t > z\}$. Each A_t is analytic, so A is analytic, using (4). Let $A_{(x,Q)}$ be the (x, Q)-section of A. Use (28), with the σ-field generated by the analytic sets for $\mathscr{A}$:

$$h(x, Q, P) = (P \times \lambda)(A_{(x,Q)}) = g(x, Q, P)(A) .$$

So

$$\{h > a\} = g^{-1}\{\mu: \mu \in \pi(\Gamma) \text{ and } \mu(A) > a\} ,$$

which is analytic by (25) and (5). □

(30) COROLLARY. *Let $r(x, Q, y)$ be a semi-analytic function from $S \times \pi(S) \times S$ to the unit interval. Let*

$$h(x, P) = \int_S r(x, P, y)P(dy) .$$

Then h is semi-analytic on $S \times \pi(S)$.

PROOF. The map $(x, P) \to (x, P, P)$ is Borel measurable, from $S \times \pi(S)$ to $S \times \pi(S) \times \pi(S)$. Compose the h of (29) with this map, and use (21). □

(31) COROLLARY. *Let $r(x, Q, y)$ be a semi-analytic function on $S \times \pi(S) \times S$. Let*

$$h(x, P) = \int_S r(x, P, y)P(dy) .$$

Then h is semi-analytic on $S \times \pi(S)$.

PROOF. Let $r_n = \min\{r, n\}$. Then r_n and $n^{-1}r_n$ are semi-analytic by (19). And $0 \leqq n^{-1}r_n \leqq 1$. So

$$h_n(x, P) = n \int n^{-1}r_n(x, P, y)P(dy)$$

is semi-analytic by (30). But $h_n \uparrow h$ by monotone convergence. So h is semi-analytic by (20). □

(32) COROLLARY. *Let A be an analytic subset of $S \times \pi(S)$. Let $r(x, Q, y)$ be a semi-analytic function on $A \times S$. Let*

$$h(x, P) = \int_S r(x, P, y)P(dy)$$
$$h^*(x) = \sup_{P \in A_x} h(x, P) .$$

Then h is semi-analytic on A, and h^* is semi-analytic on $\mathrm{proj}_S A$.

PROOF. Let $r^*(x, P, y) = r(x, P, y)$ for $(x, P) \in A$ and $y \in S$ and let $r^*(x, P, y) = 0$ elsewhere on $S \times \pi(S) \times S$. Then r^* is semi-analytic. So (31) makes $\int_S r^*(x, P, y)P(dy)$ semi-analytic. But h is the restriction of this function to A: use (18) to make h semi-analytic. Then $\{h^* > a\} = \mathrm{proj}_S \{h > a\}$ is analytic by (5). □

6. Dynamic programming. Return to the dynamic programming problem of Section 1. A *strategy* s of length n is a sequence of functions

$$x_1 \to s_{x_1}, (x_1, x_2) \to s_{x_1 x_2}, \cdots, (x_1, \cdots, x_n) \to s_{x_1, \cdots, x_n}$$

from $S, S^2, \cdots, S^n$ to $\pi(S)$, subject to the constraint

$$s_{x_1, \cdots, x_i} \in A_{x_i} \qquad \text{for } i = 1, \cdots, n .$$

For the moment, there are no measurability conditions. The *reward* of s at $\mathbf{x} = (x_1, \cdots, x_{n+1})$ is

$$r_n(s, \mathbf{x}) = r(x_1, s_{x_1}, x_2) + \cdots + r(x_n, s_{x_1, \cdots, x_n}, x_{n+1}) .$$

The *upper expected reward* $\rho_n(s, x_1)$ of s starting from $x_1 \in S$ is the n-fold iterated upper integral of $r_n(s, \mathbf{x})$ with respect to the n measures

$$s_{x_1, \cdots, x_n}(dx_{n+1}), \cdots, s_{x_1}(dx_2) .$$

In the absence of measurability conditions, these n measures on S cannot be combined into a single measure on S^n. If $x \in S$ and s is a strategy of length $n + 1$, the x-section s^x of s is this strategy of length n:

$$x_1 \to s_{x x_1}, (x_1, x_2) \to s_{x x_1 x_2}, \cdots, (x_1, \cdots, x_n) \to s_{x x_1, \cdots, x_n} .$$

If s is a strategy of length $k + 1$,

$$\rho_{k+1}(s, x) = \int_S^* [r(x, s_x, y) + \rho_k(s^x, y)] s_x(dy) . \tag{33}$$

The optimal reward operator T was defined in (1); and 0 vanishes identically; and $u_k = T^k 0$. The next result shows that u_n is the optimal reward in n moves, with nonmeasurable strategies allowed.

(34) PROPOSITION. (a) *$u_n(x) \geqq \rho_n(s, x)$ for all $x \in S$ and strategies s of length n.*
(b) *Fix n and positive ε. There is a strategy s of length n, such that*

$$\rho_n(s, x) > u_n(x) - \varepsilon \qquad \textit{when} \quad u_n(x) < \infty$$
$$> 1/\varepsilon \qquad \textit{when} \quad u_n(x) = \infty .$$

PROOF. *Claim* (a). This is clear for $n = 1$. Suppose it for $n = k$. Using (33),

$$\rho_{k+1}(s, x) \leqq \int_S^* [r(x, s_x, y) + u_k(y)]s_x(dy)$$
$$\leqq (Tu_k)(x) = u_{k+1}(x) .$$

Claim (b). This is clear for $n = 1$. Suppose it for $n = k$. Abbreviate $\theta_\varepsilon(z) = z - \varepsilon$ for $z < \infty$, and $\theta_\varepsilon(z) = 1/\varepsilon$ for $z = \infty$. Use the case $n = 1$ on the reward function $r(x, P, y) + u_k(y)$ to get a strategy t of length 1, with

$$\int_S^* [r(x, t_x, y) + u_k(y)]t_x(dy) > \theta_\varepsilon[u_{k+1}(x)] .$$

If $f_n \uparrow f$ and f_1 is bounded below, then $\int^* f_n \uparrow \int^* f$. Consequently, there is a $j(x)$ so large that

$$\int_S^* [r(x, t_x, y) + \theta_{1/j(x)}(u_k(y))]t_x(dy) > \theta_\varepsilon[u_{k+1}(x)] .$$

Use the induction hypothesis to generate a strategy t^x of length k, with

$$\rho_k(t^x, y) > \theta_{1/j(x)}[u_k(y)] \qquad \text{for all } y .$$

Outer integrals are order-preserving, so

$$\int_S^* [r(x, t_x, y) + \rho_k(t^x, y)]t_x(dy) > \theta_\varepsilon[u_{k+1}(x)] .$$

Let s be the strategy of length $k + 1$, with $s_x = t_x$ and $s^x = t^x$. Use (33). □

For the rest of this section, make the following measurability assumptions.

(35) CONDITIONS. The state space S is analytic, and is endowed with the Borel σ-field. Embed S homeomorphically into a compact metric set S^*. Let $\pi(S)$ be the set of probabilities on S, with the weak* topology. Then $\pi(S)$ is homeomorphic to $\{\mu : \mu \in \pi(S^*) \text{ and } \mu(S) = 1\}$, with the relative topology, by (12). Here, $\mu(S)$ is well defined by (6). So $\pi(S)$ is analytic by (25) and (4). And $S \times \pi(S)$ is analytic in the product topology. The contraint set A is assumed analytic. So, $A \times S$ is analytic in the product topology. Remember that the x-section A_x of A is nonempty for all $x \in S$. The reward function r is assumed semi-analytic on $A \times S$.

The optimal reward operator T is defined for semi-analytic functions f by

$$(36) \qquad (Tf)(x) = \sup_{P \in A_x} \int_S [r(x, P, y) + f(y)]P(dy) .$$

(37) THEOREM. *If conditions* (35) *hold, Tf is well defined by* (36) *and is semi-analytic on S, for each semi-analytic function f on S.*

PROOF. For a moment, fix (x, P). Then $r(x, P, \cdot)$ is semi-analytic on S, so $r(x, P, \cdot) + f$ is semi-analytic on S by (19). Consequently, the integral is defined by (22). This proves Tf well-defined. Why is it semi-analytic? The map $(x, P, y) \to y$ is Borel. So $(x, P, y) \to f(y)$ is semi-analytic by (21). And

$$r_1(x, P, y) = r(x, P, y) + f(y)$$

is semi-analytic by (19). Now use (32) on r_1. □

(38) COROLLARY. *If conditions* (35) *hold*, $T^n f$ *is well defined by* (36) *and is semi-analytic on* S, *for each semi-analytic function* f *on* S.

This is one of the main results of the paper.

The next main result (43) shows there are nearly optimal analytically measurable strategies. Here are some preliminaries.

(39) LEMMA. *Let* S, T, *and* U *be analytic sets. Let* f *map* S *into* T, *and let* g *map* T *into* U; *so* $g \circ f$ *maps* S *into* U. *Let* $X = S$, T, *or* U. *Let* $\mathscr{B}_X$ *be the Borel* σ-*field in* X; *let* $\mathscr{A}_X$ *be the* σ-*field generated by the analytic subsets of* X; *let* $\mathscr{M}_X$ *be the* σ-*field of universally measurable subsets of* X. *So* $\mathscr{B}_X \subset \mathscr{A}_X \subset \mathscr{M}_X$. *Suppose* g *is analytically measurable*: $g^{-1}(\mathscr{B}_U) \subset \mathscr{A}_T$.

(a) *If* f *is Borel, that is,* $f^{-1}(\mathscr{B}_T) \subset \mathscr{B}_S$, *then* $f^{-1}(\mathscr{A}_T) \subset \mathscr{A}_S$.

(b) *If* f *is Borel, then* $g \circ f$ *is analytically measurable.*

(c) *If* f *is universally measurable, that is,* $f^{-1}(\mathscr{B}_T) \subset \mathscr{M}_S$, *then* $g \circ f$ *is universally measurable.*

PROOF. *Claim* (a). With the help of (5),

$$f^{-1}(B \backslash C) = (f^{-1}(B)) \backslash (f^{-1}(C)) \in \mathscr{A}_S,$$

for analytic subsets B and C in T. So $f^{-1}(D) \in \mathscr{A}_S$ for all sets D in the algebra generated by the analytic sets. Here, $B \backslash C$ is the set of points in B but not in C.

Claim (b). This follows from (a).

Claim (c). Let μ be a probability on $\mathscr{B}_S$. There is a μ-null $N \in \mathscr{B}_S$ and a Borel function f' with $f = f'$ off N. Now $g \circ f'$ is analytically, and so, universally measurable by (b), and $g \circ f' = g \circ f$ off N. □

(40) PROPOSITION. *Let* U *and* V *be analytic sets. Let* $f \geqq 0$ *be universally measurable on* $U \times V$ *and let* $\phi \geqq 0$ *be Borel measurable on* $U \times V$. *Let* $u \to t_u$ *be universally measurable from* U *to* $\pi(V)$, *where* $\pi(V)$ *has the Borel* σ-*field generated by the weak* topology. Let* μ *be a probabiliy on* U.

(a) $u \to \int_V \phi(u, v) t_u(dv)$ *is universally measurable.*

(b) *There is a unique probability* $\mu \times t$ *on* $U \times V$ *such that*

$$\int_{U \times V} \phi \, d(\mu \times t) = \int_U \int_V \phi(u, v) t_u(dv) \mu(du)$$

for all nonnegative Borel ϕ.

(c) $v \to f(u, v)$ *and* $u \to \int_V f(u, v) t_u(dv)$ *are universally measurable.*

(d) $\int_{U \times V} f \, d(\mu \times t) = \int_U \int_V f(u, v) t_u(dv) \mu(du)$.

PROOF. *Claim* (a). It is enough to do this when ϕ is the indicator of a Borel rectangle. Then $(u, P) \to \int \phi(u, v) P(dv)$ is Borel on $U \times \pi(V)$. Compose this function with the universally measurable map $u \to (u, t_u)$ and use (39c).

Claim (b). Use (a) and monotone convergence.

Claim (c). For the first claim, consider measures concentrated on $\{u\} \times V$.

For the second, there is a Borel function f' and a Borel set G, such that $(\mu \times t)(G) = 1$ and $f = f'$ on G. Let G_u be the u-section of G, and let $G^* = \{u: t_u(G_u) = 1\}$. Then G^* is universally measurable by (a), $\mu(G^*) = 1$ by (b). Fix $u \in G^*$. Now $f(u, v) = f'(u, v)$ for all $v \in G_u$, which has t_u-measure 1. So

$$\int_V f(u, v)t_u(dv) = \int_V f'(u, v)t_u(dv) \qquad \text{for } u \in G^* .$$

The right side is universally measurable in u by (a), and $\mu(G^*) = 1$, so the left side is μ-measurable; but μ is arbitrary.

Claim (d). Continuing the same argument, $\mu(G^*) = 1$ shows

$$\begin{aligned}\int_U \int_V f(u, v)t_u(dv)\mu(du) &= \int_U \int_V f'(u, v)t_u(dv)\mu(du)\\ &= \int_{U\times V} f'\, d(\mu \times t) \qquad \text{by (b)}\\ &= \int_{U\times V} f\, d(\mu \times t)\,,\end{aligned}$$

where the last line follows because $f = f'$ with $(\mu \times t)$-probability 1. □

It will be helpful to establish this result in several dimensions.

(41) COROLLARY. *Let $U_1, \cdots, U_{n+1}$ be analytic sets. Endow $\pi(U_i)$ with the Borel σ-field generated by the weak* topology. Let*

$$u_1 \to t_{u_1}, (u_1, u_2) \to t_{u_1u_2}, \cdots, (u_1, \cdots, u_n) \to t_{u_1,\cdots,u_n}$$

be universally measurable functions from $U_1, U_1 \times U_2, \cdots, U_1 \times \cdots \times U_n$ to $\pi(U_2), \pi(U_3), \cdots, \pi(U_{n+1})$. If f is a function on $U_1 \times \cdots \times U_{n+1}$, its u-section f_u by $u \in U_1$ is this function on $U_2 \times \cdots \times U_{n+1}$:

$$(u_2, \cdots, u_{n+1}) \to f(u, u_2, \cdots, u_{n+1}) .$$

(a) *For each $u \in U_1$, there is a unique probability $t_{\mathbf{u}}$ on $U_2 \times \cdots \times U_{n+1}$, such that*

$$\int_{U_2\times\cdots\times U_{n+1}} \phi_u\, dt_{\mathbf{u}} = \int_{U_2} \cdots \int_{U_{n+1}} \phi(u, u_2, \cdots, u_{n+1})t_{uu_2,\cdots,u_n}(du_{n+1}) \cdots t_u(du_2)\,,$$

for all nonnegative Borel ϕ on $U_1 \cdots U_{n+1}$. The integral is a universally measurable function of u.

(b)

$$\int_{U_2\times\cdots\times U_{n+1}} f_u\, dt_{\mathbf{u}} = \int_{U_2} \cdots \int_{U_{n+1}} f(u, u_2, \cdots, u_{n+1})t_{uu_2,\cdots,u_n}(du_{n+1}) \cdots t_u(du_2)$$

for all nonnegative universally measurable f on $U_1 \times \cdots \times U_{n+1}$. The integral is a universally measurable function of u.

(c) *$u \to t_{\mathbf{u}}$ is universally measurable.*

Suppose $n \geqq 2$. For $u \in U_1$, let t^u be the sequence of functions

$$u_2 \to t_{uu_2}, (u_2, u_3) \to t_{uu_2u_3}, \cdots, (u, u_2, \cdots, u_n) \to t_{uu_2,\cdots,u_n}$$

from $U_2, U_2 \times U_3, \cdots, U_2 \times \cdots \times U_n$ to $\pi(U_3), \pi(U_4), \cdots, \pi(U_{n+1})$.

(d) *$(u, v) \to t_v{}^u$ is universally measurable on $U_1 \times U_2$.*

PROOF. This is a straightforward but messy induction from (40). You can reduce the number of factors by one if you group the first two together. □

Now return to dynamic programming under conditions (35). A strategy s of length n and its x-section s^x and its reward $r_n(s, x)$ at $x = (x_1, \cdots, x_{n+1})$ are defined as in the general case. The strategy s is called analytically or universally measurable if all the component functions are; as usual, $\pi(S)$ is endowed with the Borel σ-field generated by the weak* topology.

(42) LEMMA. *Suppose* (35). *Let s be a universally measurable strategy of length n. Define the probability s_x on S^n by* (41).

(a) $r_n(s, x_1, x_2, \cdots, x_{n+1})$ *is a universally measurable function on* S^{n+1}.

(b) $\rho_n(s, x) = \int_{S^n} r_n(s, x, x_2, \cdots, x_{n+1}) s_x(dx_2, \cdots, dx_{n+1})$ *is a universally measurable function of* $x \in S$.

(c) *Suppose* $n \geqq 2$. *Then* $\rho_{n-1}(s^x, y)$ *is a universally measurable function of* $(x, y) \in S^2$, *and*

$$\rho_n(s,x) = \int_S [r(x, s_x, y) + \rho_{n-1}(s^x, y)] s_x(dy) .$$

PROOF. For (a), use (39c). Then (b) follows from (41b), and (c) from (41d). □

The next theorem is the second main result of the paper. Remember $u_n = T^n 0$ is the optimal reward in n moves, even allowing nonmeasurable strategies.

(43) THEOREM. *Suppose conditions* (35). *Fix a positive integer n, and a positive ε. There is an analytically measurable strategy s of length n, such that*

$$\begin{aligned} \rho_n(s, x) &> u_n(x) - \varepsilon \qquad \textit{when} \quad u_n(x) < \infty \\ &> 1/\varepsilon \qquad\qquad \textit{when} \quad u_n(x) = \infty . \end{aligned}$$

PROOF. *The case $n = 1$.* Let

$$h(x, P) = \int_S r(x, P, y) P(dy) \qquad \text{for } (x, P) \in A .$$

So h is semi-analytic by (32). And $u_1(x) = \sup_{P \in A_x} h(x, P)$. Fix a positive integer k with $k > 1/\varepsilon$. For $j = 0, 1, \cdots$, let

$$A_j = \{(x, P) : (x, P) \in A \text{ and } h(x, P) \geqq j/k\} .$$

So A_j is analytic by (16), and $A = A_0 \supset A_1 \supset \cdots$. Let B_j be the projection of A_j on S, so B_j is analytic by (5), and $S = B_0 \supset B_1 \supset \cdots$. Let $B^* = \bigcap_j B_j$, which is also analytic by (4).

Remember that $\mathscr{A}$ is the σ-field generated by the analytic subsets of S. Use (15) to find $\mathscr{A}$-measurable functions t_j from B_j to $\pi(S)$, with $(x, t_j(x)) \in A_j$ for all $x \in B_j$. Use (15) again to find an $\mathscr{A}$-measurable function t^* from B^* to $\pi(S)$ with $(x, t^*(x)) \in A_k$ for all $x \in B^*$. Let

$$\begin{aligned} s_x &= t_j(x) \qquad \text{for} \quad x \in B_j \backslash B_{j+1} \quad \text{and} \quad j = 0, 1, \cdots \\ &= t^*(x) \qquad \text{for} \quad x \in B^* . \end{aligned}$$

This s is an analytically measurable strategy of length one. We say it is ε-optimal. Indeed, $\rho_1(s, x) = h(x, s_x)$. Fix $j = 0, 1, \cdots$ and $x \in B_j \backslash B_{j+1}$. Then

$h(x, P) < (j+1)/k$ for all $P \in A_x$, because $x \notin B_{j+1} = \text{proj}_S\{h \geqq (j+1)/k\}$. So $u_1(x) \leqq (j+1)/k$. But $(x, s_x) \in A_j = \{h \geqq j/k\}$, so

$$h(x, s_x) \geqq j/k > u_1(x) - \varepsilon .$$

Finally, $B^* = \{u_1 = \infty\}$. If $x \in B^*$, then $(x, s_x) \in A_k$, so $h(x, s_x) \geqq k > 1/\varepsilon$.

The induction. Suppose $k \geqq 1$ and the theorem holds for $n = k$. We must get it for $n = k + 1$. Fix $e > 0$; we will construct an ε-optimal analytically measurable strategy s. The construction is made separately on the sets $\{u_{k+1} < \infty\}$ and $\{u_{k+1} = \infty\}$. The second set is analytic by (38), so the first is CA.

The set $\{u_{k+1} < \infty\}$. The modified reward function $r(x, P, y) + u_k(y)$ is semi-analytic by (19) and (38). If you integrate out y with respect to P, and sup out $P \in A_x$, you get $u_{k+1}(x)$. So, you can use the case $n = 1$ with the modified reward function to generate a $\frac{1}{2}\varepsilon$-optimal analytically measurable strategy t of length one. Let t^* be a $\frac{1}{2}\varepsilon$-optimal analytically measurable strategy of length k, for the original reward function, generated by the induction hypothesis.

For this paragraph, restrict x to $\{u_{k+1} < \infty\}$. The strategy s is defined starting from such x as follows: $s_x = t_x$ and $s^x = t^*$. Clearly, it is analytically measurable. To see why it is ε-optimal, notice that

$$u_{k+1}(x) \geqq \textstyle\int_S [r(x, t_x, y) + u_k(y)]t_x(dy) > u_{k+1}(x) - \frac{1}{2}\varepsilon .$$

Since $u_{k+1}(x) < \infty$, $t_x\{u_k = \infty\} = 0$. Using (42c),

$$\begin{aligned}\rho_{k+1}(s, x) &= \textstyle\int_S [r(x, t_x, y) + \rho_k(t^*, y)]t_x(dy) \\ &> \textstyle\int_{\{u_k<\infty\}} [r(x, t_x, y) + u_k(y) - \frac{1}{2}\varepsilon]t_x(dy) \\ &= \textstyle\int_S [r(x, t_x, y) + u_k(y)]t_x(dy) - \frac{1}{2}\varepsilon \\ &> u_{k+1}(x) - \tfrac{1}{2}\varepsilon - \tfrac{1}{2}\varepsilon = u_{k+1}(x) - \varepsilon .\end{aligned}$$

The set $\{u_{k+1} = \infty\}$. Let

$$h(x, P) = \textstyle\int_S [r(x, P, y) + u_k(y)]P(dy) ,$$

a semi-analytic function of x and P by (38) and (31). Let $n \geqq 2$. Let

$$A_n = \left\{(x, P) : (x, P) \in A \text{ and } h(x, P) > \frac{1}{\varepsilon} \text{ and } P\{u_k = \infty\} > \frac{1}{n}\right\} .$$

So A_n is analytic by (16) and (25). Let $B_n = \text{Proj}_S A_n$, analytic by (5). Use (15) to construct an analytically measurable function ϕ_n on B_n, with $(x, \phi_n(x)) \in A_n$ for $x \in B_n$. Let θ be an ε/n-optimal analytically measurable strategy of length k, generated by the induction hypothesis. Clearly, $B_n \subset \{u_{k+1} = \infty\}$. Define s on $\bigcup_{n=2}^{\infty} B_n$ as follows: if $x \in B_n \backslash B_{n-1}$, let $s_x = \phi_n(x)$ and $s^x = \theta$. Clearly, s is analytically measurable. Why is it ε-optimal? If $x \in B_n \backslash B_{n-1}$, then $(x, s_x) \in A_n$, so $s_x\{u_k = \infty\} > 1/n$.

Continuing with the help of (42c),

$$\begin{aligned}\rho_{k+1}(s, x) &= \textstyle\int_S [r(x, s_x, y) + \rho_k(\theta, y)]s_x(dy) \\ &\geqq \textstyle\int_{\{u_k=\infty\}} \rho_k(\theta, y)s_x(dy) \\ &> \frac{n}{\varepsilon}s_x\{u_k = \infty\} > \frac{n}{\varepsilon} \cdot \frac{1}{n} = \frac{1}{\varepsilon} .\end{aligned}$$

To define s on $\{u_{k+1} = \infty\}\backslash\bigcup_{n=2}^{\infty} B_n$, let

$$A^* = \left\{(x, P): (x, P) \in A \text{ and } h(x, P) > \varepsilon + \frac{1}{\varepsilon}\right\}.$$

As before, A^* is analytic. Clearly, $\{u_{k+1} = \infty\} \subset \mathrm{Proj}_S A^*$. Use (15) to construct an analytically measurable function ϕ on $\{u_{k+1} = \infty\}$, such that $(x, \phi(x)) \in A^*$ when $u_{k+1}(x) = \infty$. Let t^* be an ε-optimal analytically measurable strategy of length k, generated by the induction hypothesis. For $x \in \{u_{k+1} = \infty\}\backslash\bigcup_{n=2}^{\infty} B_n$, let $s_x = \phi(x)$ and $s^x = t^*$. Clearly, s is analytically measurable. Why is it ε-optimal? Since $(x, s_x) \in A^*$ but $x \notin B_n$, it follows that $(x, s_x) \notin A_n$, so $s_x\{u_k = \infty\} = 0$. Using (42c).

$$\begin{aligned}\rho_{k+1}(s, x) &= \textstyle\int_S [r(x, s_x, y) + \rho_k(t^*, y)]s_x(dy)\\ &> \textstyle\int_S [r(x, s_x, y) + u_k(y) - \varepsilon]s_x(dy)\\ &> \frac{1}{\varepsilon} + \varepsilon - \varepsilon = \frac{1}{\varepsilon}.\end{aligned}$$

This completes the construction of s. It is ε-optimal and analytically measurable since all three pieces are. □

NOTES. (a) In fact, our $s_{x_1,\cdots,x_i}$ is $\mathscr{A}^i$-measurable. If S is uncountable, $\mathscr{A}^i$ is smaller than the σ-field generated by the analytic subsets of S^i.

(b) If r is uniformly bounded, the argument generates Markovian strategies: $s_{x_1,\cdots,x_i}$ depends only on i and x_i.

(c) Clearly, u_n is non-decreasing in n. Let $u_\infty = \lim_n u_n$. Our argument will produce an analytically measurable strategy for the infinite game, which stops everywhere, and whose expected reward starting from x exceeds $\theta_\varepsilon[u_\infty(x)]$. Namely, let $n(x)$ be the least n with $u_n(x) > \theta_\varepsilon[u_\infty(x)]$. So $n(x)$ is analytically measurable in x. Use the theorem separately on each piece $\{n(x) = n\}$, to construct an ε-optimal strategy of length n. We define the reward of a nonmeasurable strategy in the infinite game as the limit in n of what it can accomplish in n moves. Then we can do as well with analytically measurable strategies.

(d) From (36)—(37),

$$u_{n+1}(x) = \sup_{P \in A_x} \textstyle\int_S [r(x, P, y) + u_n(y)]P(dy).$$

By monotone convergence, $u_\infty \geqq Tu_\infty$. The argument in (c) shows that equality holds. Indeed, suppose, s is the ε-optimal strategy for the infinite game. Let $\rho_0 = 0$. Then

$$\textstyle\int_S [r(x, s_x, y) + \rho_{n(x)-1}(s^x, y)]s_x(dy) > \theta_\varepsilon[u_\infty(x)].$$

Using (34a),

$$(Tu_\infty)(x) \geqq \textstyle\int_S [r(x, s_x, y) + u_\infty(y)]s_x(dy) > \theta_\varepsilon[u_\infty(x)].$$

Let $\varepsilon \to 0$. That is, under condition (35), the optimal reward in the infinite game is u_∞, a semi-analytic function; u_∞ satisfies the optimality equation $Tu_\infty = u_\infty$; and there is an ε-optimal strategy which stops everywhere and is analytically measurable.

(e) We used the condition that r is semi-analytic in two critical places. First, we need it to prove that u_n is semi-analytic in (38). Second, we need r and u_k to be semi-analytic in order to use the selection procedure (15) on $\{(x, P)$: $(x, P) \in A$ and $\int [r(x, P, y) + u_k(y)]P(dy) \geqq a\}$ in the proof of (43).

(f) Dynamic programming problems are often formulated this way. There is a state space T, with σ-field $\sigma(T)$, and an action set Θ with σ-field $\sigma(\Theta)$. There is a transition probability $q(\cdot \mid t, \theta')$ on $\sigma(T)$, for each $t \in T$ and $\theta' \in \Theta$. There is a reward function $f(t, \theta', t')$ on $T \times \Theta \times T$. The transition probability and reward function are product measurable. Informally, if you are $t \in T$, you can chose any $\theta' \in \Theta$, and move to $t' \in T$ chosen at random from $q(\cdot \mid t, \theta')$; you receive the reward $f(t, \theta' t')$.

Such a problem can be reformulated in the constraint-set terms of the present paper as follows. The new state space S is $T \times \Theta$, with the product σ-field. The constraint set A is the range of the map

$$(t, \theta, \theta') \rightarrow [(t, \theta), q(\cdot \mid t, \theta') \times \delta_{\theta'}]$$

from $T \times \Theta \times \Theta$ to $S \times \pi(S)$; here, $\delta_{\theta'}$ is point mass at θ'. The new reward function is

$$r[(t, \theta), P, (t', \theta')] = f(t, \theta', t') .$$

An original starting state t is translated into the starting state (t, θ_0), where θ_0 is a fixed (arbitrary) point of Θ. Suppose T and Θ are analytic, $\sigma(T)$ and $\sigma(\Theta)$ are the Borel σ-fields, and f is semi-analytic. Then (35) is satisfied, for A is analytic by (5).

Similarly, any constraint-set problem (S, A, r) can be formulated as an action-set problem, by introducing a new absorbing state $\infty \notin S$. The appropriate action set is all of $\pi(S)$. The transition probability q is

$$\begin{aligned} q(\cdot \mid x, a) &= a \qquad \text{when} \quad (x, a) \in A \\ &= \delta_\infty \qquad \text{when} \quad (x, a) \notin A . \end{aligned}$$

And $r(x, a, y)$ is set equal to zero if $(x, a) \notin A$ or $y = \infty$. Again, δ_∞ is point mass at ∞.

7. Examples.

(45) EXAMPLE. There is a dynamic programming problem with S, A, and r Borel, but u_1 not Borel.

PROOF. Let S be the unit interval. Let δ_z be point mass at z, so $z \rightarrow \delta_z$ is continuous, and $\pi_0(S) = \{\delta_z : z \in S\}$ is compact. Let $A = S \times \pi_0(S)$. Let B be a Borel subset of the unit square, whose projection on the x-axis B^* is not Borel. Let $r(x, \delta_z, y) = 1_B(x, z)$. Then $\int r(x, \delta_z, y)\delta_z(dy) = 1_B(x, z)$, so

$$u_1(x) = \sup_z 1_B(x, z) = 1_{B^*}(x) . \qquad \square$$

(46) EXAMPLE. There is a dynamic programming problem with S, A, and r Borel, and A a product set, but no $\frac{1}{2}$-optimal Borel strategies of length one.

PROOF. Use the construction for (45). Suppose f is a function from S into S, and $x \to \delta_{f(x)}$ is $\frac{1}{2}$-optimal. If $x \in B^*$, then $1_B(x, f(x))$ exceeds $\frac{1}{2}$, so is 1. If $x \notin B^*$, clearly, $1_B(x, f(x)) = 0$. That is, $1_B(x, f(x)) = 1_{B^*}(x)$, which is not a Borel function. So f is not Borel. □

(47) EXAMPLE. There is a dynamic programming problem with S, A, and r Borel, and $u_1(x) = 1$. But there are no Borel strategies of length one.

PROOF. Let S be the unit interval. Let B be a Borel subset of the unit square, whose projection on the x-axis is S, but which includes no Borel graph. Let $A = \{(x, \delta_y) : (x, y) \in B\}$, and $r \equiv 1$. □

(48) EXAMPLE. Consistent with the axioms of set theory, there is a dynamic programming problem with Borel S, A, and r, and a universally measurable f, such that

$$\int [r(x, P, y) + (Tf)(y)]P(dy) \tag{49}$$

is undefined for some $(x, P) \in A$.

PROOF. Let S be the closed unit interval, and let $r \equiv 0$. Let p project the closed unit square onto the x-axis. Let ϕ be a one-to-one bimeasurable map of the open unit square onto the open unit interval. Let $g = p \circ \phi^{-1}$, a measurable map of the open unit interval onto itself. Let B be the graph of g, visualized as a function from the y-axis to the x-axis:

$$B = \{(x, y) : 0 < x, y < 1, \text{ and } x = g(y)\}.$$

Let A consist of the pairs (x, δ_z) with $(x, z) \in B$, together with the pairs (0, Lebesgue) and $(1, \delta_1)$.

Let E be a complementary analytic subset of the open unit square, with pE not Lebesgue measurable (9). Let $C = \phi(E)$, a complementary analytic subset of the open unit interval. Let $f = 1_C$. Then f is universally measurable. Clearly, $(Tf)(0) =$ Lebesgue (C) and $(Tf)(1) = f(1)$. Suppose $0 < x < 1$. Then

$$(Tf)(x) = \sup_{P \in A_x} \int f(y)P(dy) = \sup_{z \in B_x} 1_C(z).$$

So $(Tf)(x) = 1$ if there is a $z \in C$ with $(x, z) \in B$, that is, with $x = g(z) = p(\phi^{-1}(z))$: otherwise, $(Tf)(x) = 0$. In other terms, $(Tf)(x) = 1$ iff $x \in p(\phi^{-1}(C)) = pE$. That is, $Tf = 1_{pE}$. So (49) is undefined at $x = 0$ and $P =$ Lebesgue. □

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