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R-THEORY FOR MARKOV CHAINS ON A GENERAL STATE SPACE II: r-SUBINVARIANT MEASURES FOR r-TRANSIENT CHAINS

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This paper is a sequel to a previous paper of similar title. The structure of r-subinvariant measures for a Markov chain $\{X_n\}$ on a general state space $(\mathscr{X}, \mathscr{F})$ is investigated in the r-transient case, and a Martin boundary representation is found. Under certain continuity assumptions on the transition law of $\{X_n\}$ the elements of the Martin boundary are identified when $\mathscr{F}$ is countably generated, and a necessary and sufficient condition for an r-invariant measure for $\{X_n\}$ to exist is found. This generalizes the Harris-Veech conditions for countable $\mathscr{X}$.

7. Introduction. This paper is a sequel to Tweedie (1974), which we refer to as I, and whose notation and numbering is continued here. The object of the paper is to give a representation of the Martin boundary type for r-subinvariant measures (solutions of I (3.3)) when the Markov chain $\{X_n\}$ on $(\mathscr{X}, \mathscr{F})$ is r-transient. This is achieved in Theorems 9 and 10. Under a certain equicontinuity condition on the transition probabilities of $\{X_n\}$ it is shown that the elements of this representation are all σ-finite measures on $\mathscr{F}$, and further that if $\mathscr{F}$ is countably generated it is possible to identify the points in the Martin boundary as limits of ratios of transition probabilities (Theorem 11). This enables us to derive a necessary and sufficient condition for the existence of an r-invariant measure for $\{X_n\}$ in Theorem 12 analogous to that of Harris (1957) and Veech (1963) for $r = 1$ and $\mathscr{X} = \mathbb{Z}$.

8. Preliminaries and potentials. Throughout this paper we shall assume

ASSUMPTION 1. $\{X_n\}$ is r-transient: that is

$$G_r(x, A) = \sum_{n=1}^{\infty} P^n(x, A)r^n < \infty$$

for some $x \in \mathscr{X}$ and $A \in \mathscr{F}^+$;

ASSUMPTION 2. D is a fixed r-transient set in $\mathscr{F}^+$;

ASSUMPTION 3. An r-subinvariant measure Q for $\{X_n\}$ is a solution of $Q \geqq rQP$ (cf. I (3.3)) which also satisfies $Q(D) = 1$.

Because of Proposition 3.1 and Theorem 4 of I, the r-transient case is the only one that needs to be studied: hence our Assumption 1. Neither Assumption

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2 nor Assumption 3 in any sense restricts our results, and are mainly for notational convenience. We use also the notation

$$\mathscr{D} = \{\bar{D}(m,j),\, m, j = 1, 2, \cdots\} \tag{8.1}$$

where $\bar{D}(m,j)$ is defined by (1.5), and put

$$D_\infty = \{y \in \mathscr{X}\colon G_r(y, D) = \infty\}\,.$$

Condition I (I Section 1) ensures that $\mathscr{D}$ is a partition of $\mathscr{X}$, and Theorem 1 implies that $M(D_\infty) = 0$, since $\{X_n\}$ is r-transient, and also that every element of $\mathscr{D}$ is an r-transient set.

PROPOSITION 8.1. *Let Δ be any element of $\mathscr{D}$. There is a real number β_Δ, $0 < \beta_\Delta < \infty$, such that for every r-subinvariant measure Q for $\{X_n\}$,*

$$Q(\Delta) \leqq \beta_\Delta\,. \tag{8.2}$$

PROOF. Suppose $\Delta = \bar{D}(m,j)$ for some m, j. Since Q is r-subinvariant,

$$\begin{aligned} 1 = Q(D) &\geqq r^m \textstyle\int_{\bar{D}(m,j)} Q(dx)P^m(x, D) \\ &\geqq [r^m/(j+1)]Q(\bar{D}(m,j))\,; \end{aligned}$$

thus we can choose $\beta_\Delta = (j+1)/r^m$. □

For any r-subinvariant measure Q for $\{X_n\}$, the sequence

$$r^n \textstyle\int_{\mathscr{X}} Q(dx)P^n(x, A) \tag{8.3}$$

is non-increasing with n for any $A \in \mathscr{F}$; for $A \in \mathscr{F}_{\mathscr{D}}$, (8.3) is finite, from (8.2). We shall call a σ-finite measure Q a *potential* if Q is r-subinvariant and (8.3) tends to zero for every $A \in \mathscr{D}$ (and hence for every $A \in \mathscr{F}_{\mathscr{D}}$) as $n \to \infty$.

PROPOSITION 8.2. *Let Q be r-subinvariant for $\{X_n\}$. Then Q can be decomposed uniquely as*

$$Q(\cdot) = Q_P(\cdot) + Q_I(\cdot) \tag{8.4}$$

where

(i) *either $Q_P(\cdot) \equiv 0$, or $Q_P(\cdot)/Q_P(D)$ is a potential*
(ii) *either $Q_I(\cdot) \equiv 0$, or $Q_I(\cdot)/Q_I(D)$ is r-invariant for $\{X_n\}$.*

PROOF. Since (8.3) is monotone decreasing,

$$Q_I(A) = \lim_{n\to\infty} r^n \textstyle\int_{\mathscr{X}} Q(dx)P^n(x, A) \tag{8.5}$$

exists for each $A \in \mathscr{F}_{\mathscr{D}}$. A basic theorem (cf. Gänssler (1971) 1.10) then ensures that Q_I is a measure on each of the σ-fields $\mathscr{F}_\Delta$, $\Delta \in \mathscr{D}$. There is then a unique extension of Q_I to a measure on $\mathscr{F}$, given by $Q_I(A) = \sum_{\Delta \in \mathscr{D}} Q_I(A \cap \Delta)$, $A \in \mathscr{F}$.

For any $A \in \mathscr{F}_{\mathscr{D}}$, the monotonicity and finiteness of the limit (8.5) shows

$$\begin{aligned} Q_I(A) &= \lim_{n\to\infty} r \textstyle\int_{\mathscr{X}} [r^{n-1} \int_{\mathscr{X}} Q(dx)P^{n-1}(x, dy)]P(y, A) \\ &= r \textstyle\int_{\mathscr{X}} Q_I(dy)P(y, A) \end{aligned} \tag{8.6}$$

and since (8.6) extends by definition to arbitrary $A \in \mathscr{F}$, either Q_I is identically zero or $Q_I(\cdot)/Q_I(D)$ is r-invariant. Define $Q_P(\cdot) = Q(\cdot) - Q_I(\cdot)$. Either $Q_P(\cdot) \equiv 0$; or, from (8.6) and the r-subinvariance of Q, $Q_P(\cdot)/Q_P(D)$ is r-subinvariant; in the latter case, (8.5) ensures that $Q_P(\cdot)/Q_P(D)$ is a potential.

The decomposition is unique; for suppose $Q = Q_P' + Q_I'$, with Q_I' and Q_P' satisfying (i) and (ii). Then

$$\begin{aligned} Q_I'(A) &= \lim_{n\to\infty} r^n \int_{\mathscr{X}} Q_I'(dy)P^n(y, A) \\ &= \lim_{n\to\infty} r^n \int_{\mathscr{X}} Q(dy)P^n(y, A) \\ &= Q_I(A) \end{aligned}$$

for all $A \in \mathscr{F}$. □

In the next section we shall derive an integral representation for potentials, and in the following sections an integral representation for r-invariant measures. From Proposition 8.2, this suffices to give a representation for arbitrary r-subinvariant measures for $\{X_n\}$.

9. The representation of potentials. Write, for $x \in \mathscr{X}$,

$$K_r(x, A) = \sum_0^\infty P^n(x, A)r^n = \delta(x, A) + G_r(x, A)\,;$$

from r-transience, $K_r(x, A) < \infty$ for $x \notin D_\infty$, $A \in \mathscr{F}_{\mathscr{D}}$. For $x \notin D_\infty$, put

$$M_x(A) = K_r(x, A)/K_r(x, D)\,; \tag{9.1}$$

note that the dependence of M_x on r and D has been suppressed. As in Proposition 3.2, M_x is r-subinvariant, $x \notin D_\infty$; moreover, since

$$r^n \int_{\mathscr{X}} M_x(dy)P^n(y, A) = \sum_n^\infty r^m P^m(x, A)/K_r(x, D)\,,$$

M_x is a potential for $x \notin D_\infty$.

THEOREM 9. *A σ-finite measure Q is a potential for $\{X_n\}$ if and only if there is a probability measure λ on $\mathscr{F}$ with $\lambda(D_\infty) = 0$, such that*

$$Q(A) = \int_{\mathscr{X}} M_x(A)\lambda(dx)\,, \qquad A \in \mathscr{F}. \tag{9.2}$$

PROOF. Our proof follows Moy (1967). Suppose Q is a potential, and put, for $A \in \mathscr{F}$,

$$H(A) = Q(A) - r \int_{\mathscr{X}} Q(dx)P(x, A)\,; \tag{9.3}$$

for $A \in \mathscr{F}_{\mathscr{D}}$, $0 \leqq H(A) \leqq Q(A) < \infty$, and

$$\begin{aligned} \int_{\mathscr{X}} H(dx)K_r(x, A) &= \lim_{N\to\infty} \int_{\mathscr{X}} H(dx) \sum_0^N P^n(x, A)r^n \\ &= Q(A) - \lim_{N\to\infty} \int_{\mathscr{X}} Q(dx)P^{N+1}(x, A)r^{N+1} \\ &= Q(A)\,. \end{aligned} \tag{9.4}$$

Since $Q(D) = 1$, (9.4) implies $H(D_\infty) = 0$; and as $\bar{D}_\infty \subseteqq D_\infty$ (cf. the proof of Proposition 2.1) (9.4) further implies that

$$Q(D_\infty) = 0\,. \tag{9.5}$$

Define, for each $A \in \mathscr{F}$

$$\lambda(A) = \int_A H(dw)K_r(w, D)\,; \tag{9.6}$$

from the above remark λ is a measure on $\mathscr{F}$ with $\lambda(D_\infty) = 0$, and λ satisfies (9.2) by virtue of (9.4); finally, since $Q(D) = 1$, (9.4) with $A = D$ shows that λ is a probability measure on $\mathscr{F}$.

Conversely, if λ is a probability measure on $\mathscr{F}$ with $\lambda(D_\infty) = 0$ and Q is defined by (9.2), Q is r-subinvariant since each M_x is r-subinvariant; and for any $A \in \mathscr{F}_{\mathscr{D}}$

$$r^n \int_{\mathscr{X}} Q(dy)P^n(y, A) = \int_{\mathscr{X}} [r^n \int_{\mathscr{X}} M_x(dy)P^n(y, A)]\lambda(dx)\,,$$

which tends to zero by monotone convergence, since each M_x, $x \notin D_\infty$, is a potential. Hence Q is a potential. □

Write, for $x \notin D_\infty$,

$$N_x(\cdot) = G_r(x, \cdot)/G_r(x, D)\,. \tag{9.7}$$

From Proposition 3.2, $N_x(\cdot)$ is r-subinvariant for $x \notin D_\infty$, and as with $M_x(\cdot)$, $N_x(\cdot)$ is a potential. The next result follows easily from Theorem 9, and we omit the proof.

PROPOSITION 9.1. *If Q is a potential, then there is a probability measure μ on $\mathscr{F}$ with $\mu(D_\infty) = 0$ such that*

$$r \int_{\mathscr{X}} Q(dy)P(y, A)/r \int_{\mathscr{X}} Q(dy)P(y, D) = \int_{\mathscr{X}} N_x(A)\mu(dx)\,, \qquad A \in \mathscr{F}. \tag{9.8}$$

Conversely, if μ is a probability measure with $\mu(D_\infty) = 0$, there is a potential Q such that (9.8) *holds.* □

If Q is a potential, so is $r \int_{\mathscr{X}} Q(dy)P(y, \cdot)/r \int_{\mathscr{X}} Q(dy)P(y, D)$; we shall show that any r-invariant measure which is null on D_∞ can be approximated by such "second-order" potentials, and for this reason we give the representation (9.8).

10. Approximation by potentials and related results.

PROPOSITION 10.1. *If Q is any r-invariant measure for $\{X_n\}$ with $Q(D_\infty) = 0$, then there is a sequence of measures Q_n on $\mathscr{F}$ such that*

(i) *Q_n is a potential for each n;*
(ii) *for each $A \in \mathscr{F}$, as $n \to \infty$, $Q_n(A) \to Q(A)$;*
(iii) *as $n \to \infty$, for each $A \in \mathscr{F}$,*

$$\frac{r \int_{\mathscr{X}} Q_n(dy)P(y, A)}{r \int_{\mathscr{X}} Q_n(dy)P(y, D)} \to Q(A)\,. \tag{10.1}$$

PROOF. Let Q be r-invariant with $Q(D_\infty) = 0$, and write D_0 for the union of those elements of $\mathscr{D}$ which are Q-null. Let $\{D_1, D_2, \cdots\}$ be an ordering of the remaining elements of $\mathscr{D}$. From Proposition 8.1, Q is finite on each D_j. Define a set function λ_Q on $\mathscr{F}_{\mathscr{D}}$ by setting

$$\lambda_Q(A) = 2^{-j}Q(A)/Q(D_j)\,, \qquad A \subseteq D_j,\ A \in \mathscr{F}, j = 1, 2, \cdots$$
$$\lambda_Q(A) = 0\,, \qquad A \subseteq D_0,\ A \in \mathscr{F},$$

and extend λ_Q to $\mathscr{F}$ by setting, for $A \in \mathscr{F}$, $\lambda_Q(A) = \sum_j \lambda_Q(A \cap D_j)$. Thus defined, λ_Q is a probability measure on $\mathscr{F}$, and $\lambda_Q(D_\infty) = Q(D_\infty) = 0$. Utilising the converse statement of Theorem 9, construct a potential by setting

$$\mu(A) = \int_{\mathscr{X}} M_x(A)\lambda_Q(dx)\,, \qquad A \in \mathscr{F}\,. \tag{10.2}$$

Suppose $\mu(A) = 0$; since $M_x(A) \geqq \delta(x, A)/K_r(x, D) > 0$, $x \in A$, (10.2) shows that $\lambda_Q(A) = 0$, and so $Q(A) = 0$. Hence $\mu \gg Q$; write $q(\cdot)$ for a density of Q with respect to μ, and define

$$q_n(x) = \min\,(q(x), n)\,, \qquad x \in \mathscr{X}\,.$$

Put, for any $A \in \mathscr{F}$ and $n \geqq 1$, $\mu_n(A) = \int_A q_n(y)\mu(dy)$. If $B(n) = \{y : q_n(y) = n\}$, $B(q) = \mathscr{X}\backslash B(n)$, then we have, for $A \in \mathscr{F}_{B(n)}$,

$$\begin{aligned}\mu_n(A) = n\mu(A) &\geqq r \textstyle\int_{\mathscr{X}} [n\mu(dy)]P(y, A) \\ &\geqq r \textstyle\int_{\mathscr{X}} \mu_n(dy)P(y, A)\,,\end{aligned}$$

whilst for $A \in \mathscr{F}_{B(q)}$,

$$\begin{aligned}\mu_n(A) = Q(A) &= r \textstyle\int_{\mathscr{X}} Q(dy)P(y, A) \\ &\geqq r \textstyle\int_{\mathscr{X}} \mu_n(dy)P(y, A)\,.\end{aligned}$$

Thus $Q_n(\cdot) = \mu_n(\cdot)/\mu_n(D)$ is r-subinvariant; and Q_n is a potential since

$$r^m \int_{\mathscr{X}} Q_n(dy)P^m(y, A) \leqq nr^m \int_{\mathscr{X}} \mu(dy)P^m(y, A)/\mu_n(D)\,,$$

a potential. Moreover, $q_n(x) \uparrow q(x)$ as $n \to \infty$; it follows by monotone convergence that for all A,

$$\mu_n(A) \uparrow Q(A)\,, \qquad n \to \infty \tag{10.3}$$

and so, since $P(\cdot, A)$ is bounded,

$$r \int_{\mathscr{X}} \mu_n(dy)P(y, A) \uparrow r \int_{\mathscr{X}} Q(dy)P(y, A)\,, \qquad n \to \infty\,. \tag{10.4}$$

Since $Q(D) = 1$, (ii) follows from (10.3), and because Q is r-invariant, (iii) follows from (10.4). □

The assumption that $Q(D_\infty) = 0$ in this proposition is necessary as well as sufficient, since from (9.5), all potentials allot zero probability to D_∞. However, in the following two cases any r-invariant measure Q satisfies $Q(D_\infty) = 0$:

(i) $r = 1$ and $\{X_n\}$ is 1-transient; Theorem 1 in I proves that D_∞ is empty in this case;

(ii) $R \geqq r \geqq 1$ and $\{X_n\}$ satisfies Condition I′ of I, Section 1; as remarked after Lemma 3.3 in I, Condition I′ ensures that all r-invariant measures are equivalent to M, and $M(D_\infty) = 0$ from Theorem 1.

If Q is r-invariant and $Q(D_\infty) > 0$, write $Q'(A) = Q(A)$, $A \subseteq D_\infty{}^c$, $Q'(D_\infty) = 0$: it is easy to check that Q' is r-subinvariant for $\{X_n\}$. From Proposition 8.2 we can decompose Q' into a linear combination of a potential and an r-invariant measure, both of which are zero on D_∞, and so as a corollary to Proposition 10.1 we have

PROPOSITION 10.2. *If Q is r-invariant, then there exist nonnegative constants α, β with $\alpha + \beta = 1$ and potentials $Q_0, Q_1, \cdots$ such that*

$$Q(A) = \alpha Q_0(A) + \beta \lim_{n\to\infty} Q_n(A)$$
$$= \alpha Q_0(A) + \beta \lim_{n\to\infty} \frac{r \int_{\mathscr{X}} Q_n(dy)P(y, A)}{r \int_{\mathscr{X}} Q_n(dy)P(y, D)}$$

for every $A \in \mathscr{F}$ with $A \subseteq D_\infty{}^c$.

To show that the condition $Q(D_\infty)$ is not trivial under Condition I (and thus to show that r-invariant measures need not be equivalent to M), use Example 1 with $\{Y_n\}$ an r-transient chain, $r > 1$, and $\alpha = r^{-1}$, and $P\{0, A\} = [1 - r^{-1}]P\{1, A\}$ rather than $\beta\delta(1, A)$. Since $P^n(0, D) \geqq r^{-n+m}P^m(1, D)[1 - r^{-1}]$, $D_\infty = \{0\} \neq \varnothing$. Define Q by

$$Q(A) = G_r(1, A)/G_r(1, D) \qquad A \subseteq Z\backslash\{0\} .$$
$$= [(1 - r^{-1})G_r(1, D)]^{-1} \qquad A = \{0\} :$$

thus $Q(D_\infty) > 0$, and it is easily checked that Q is r-invariant for $\{X_n\}$. In this case Q restricted to $D_\infty{}^c = \{1, 2, \cdots\}$ is a pure potential.

We conclude this section by proving two results stated without proof in I.

PROPOSITION 10.3. *Suppose $\{X_n\}$ is r-transient and Q is an r-subinvariant measure for $\{X_n\}$. Then any $A \in \mathscr{F}$ such that $Q(A) < \infty$ is r-transient.*

PROOF. If $A \subseteq D_\infty$, then $M(A) = 0$, and from Condition I, $G_r(x, A) = 0$ for all $x \notin D_\infty$, and A is trivially r-transient. Suppose $A \subseteq D_\infty{}^c$, $A \in \mathscr{F}$, and $Q(A) < \infty$. Let α, β and $Q_0, Q_1, \cdots$ be as in Proposition 10.2. If $\alpha > 0$, then for some probability measure λ_0 on $\mathscr{X}$

$$\infty > Q_0(A) = \int_{\mathscr{X}} M_x(A)\lambda_0(dy)$$
$$= \int_{\mathscr{X}} [K_r(x, A)/K_r(x, D)]\lambda_0(dx)$$

and so for some $x \in \mathscr{X}$, $G_r(x, A) < \infty$, and A is r-transient. Similarly, if $\beta > 0$, for some sequence $\{\lambda_n\}$ of probability measures

$$\infty > \lim_{n\to\infty} \int_{\mathscr{X}} M_x(A)\lambda_n(dx) ,$$

and again $G_r(x, A) < \infty$ for λ_n-almost all $x \in \mathscr{X}$. □

COROLLARY. *If $\{X_n\}$ is R-recurrent, and Q is the unique R-invariant measure (cf. Theorem 4) then any set A such that $Q(A) < \infty$ is an R-recurrent set.*

PROOF. For any $r < R$, Q is r-subinvariant; hence $Q(A) < \infty$ implies A is r-transient for all $r < R$, which is the definition of an R-recurrent set. □

This proposition and its corollary extend the result of Šidák (1967), mentioned at the end of I, Section 2. The corollary gives us a constructive proof that an R-recurrent chain partitions the space into R-recurrent sets; previously (I, Proposition 2.2) only a non-constructive proof was given.

PROPOSITION 10.4. *If U is an r-subinvariant measure for $\{X_n\}$ not necessarily finite on D, and h is an r-superinvariant function for $\{X_n\}$ then $\int_{\mathscr{X}} h(x)U(dx)$ is divergent when $\{X_n\}$ is r-transient.*

PROOF. The definition of r-superinvariance (I (4.10)) is that, for almost all $x \in \mathscr{X}$

$$0 < h(x) \leqq r \int_{\mathscr{X}} P(x, dy)h(y) . \tag{10.5}$$

Iterating (10.5) implies that for all $n \geqq 1$ and almost all x

$$r^n \int_{\mathscr{X}} P^n(x, dy)h(y) \leqq r^{n+1} \int_{\mathscr{X}} P^{n+1}(x, dy)h(y) ; \tag{10.6}$$

summing (10.6) for $n \geqq 1$ and adding (10.5) gives

$$h(x) + \int_{\mathscr{X}} G_r(x, dy)h(y) \leqq \int_{\mathscr{X}} G_r(x, dy)h(y) ,$$

and so for almost all x,

$$\int_{\mathscr{X}} G_r(x, dy)h(y) = \infty . \tag{10.7}$$

Let N be the M-null set on which (10.7) fails, and put $N' = N \cup \bar{N}$; if U is r-subinvariant, so is $U'(\cdot) \equiv U(\cdot)$ on $\mathscr{X}\backslash N'$, $U'(N') = 0$. As in the preceding proof, Proposition 10.2 and the potential representation, together with (10.7), imply

$$\int_{\mathscr{X}} h(x)U'(dx) = \infty ,$$

and the proposition follows. □

This result was stated in I, Proposition 4.3. The proof of (10.7) is due, when $\mathscr{X} = \mathbb{Z}$, to Vere-Jones (1967), who uses a different method of showing that it implies the result of the proposition in the countable case.

11. The representation of r-invariant measures. In the light of the previous section, *we shall assume from now on that D_∞ is empty*, or equivalently that we alter our state space to $\mathscr{X}\backslash D_\infty$. Proposition 10.2 shows that this does not alter our results significantly, and it follows from (10.1), (9.8) and (9.7) that if Q is r-invariant for $\{X_n\}$, there is a sequence of probability measures $\{\mu_n\}$ on $\mathscr{F}$ such that

$$Q(A) = \lim_{n\to\infty} \int_{\mathscr{X}} N_x(A)\mu_n(dx) , \qquad A \in \mathscr{F} \tag{11.1}$$

where N_x is defined by (9.7) for every $x \in \mathscr{X}$. The purpose of this section is to derive from (11.1) an integral representation for r-invariant measures analogous to the classical Martin boundary representation: for $\mathscr{X} = \mathbb{Z}$ this is derived by Moy (1967) for general r, following Doob (1959) for $r = 1$.

Let Q be a fixed r-invariant measure for the remainder of this section, and let Δ be an element of $\mathscr{D}$. Let β_Δ be the bound corresponding to Δ in Proposition 8.1. We shall write Ω_Δ for the space of measures ω on $\mathscr{F}_\Delta$ satisfying $\omega(\Delta) \leqq \beta_\Delta$; from Proposition 8.1, any r-subinvariant measure belongs to Ω_Δ, and so in particular $Q \in \Omega_\Delta$ and for every $x \in \mathscr{X}$, $N_x \in \Omega_\Delta$.

Now define the product space Ω_Δ' by

$$\Omega_\Delta' = [0, \beta_\Delta]^{\mathscr{F}_\Delta}. \tag{11.2}$$

Write $\pi_A : \Omega_\Delta' \to [0, \beta_\Delta]$, $A \in \mathscr{F}_\Delta$, for the projection map which takes $\omega' \in \Omega_\Delta'$ to its 'Ath' coordinate. There is then a natural embedding of Ω_Δ in Ω_Δ' given by $\omega \to \omega'$ where $\pi_A \omega' = \omega(A)$.

Further, define the measure space $\Omega_{\mathscr{D}}$ and the product space $\Omega_{\mathscr{D}}'$ by

$$\Omega_{\mathscr{D}} = \{\omega : \omega \text{ is a measure on } \mathscr{F}, \text{ and } \omega \text{ restricted to } \mathscr{F}_\Delta \text{ is in } \Omega_\Delta, \text{ for all } \Delta \in \mathscr{D}\} \tag{11.3}$$

$$\Omega_{\mathscr{D}}' = \prod_{\Delta \in \mathscr{D}} \Omega_\Delta' = \prod_{\Delta \in \mathscr{D}} [0, \beta_\Delta]^{\mathscr{F}_\Delta}, \tag{11.4}$$

and $\Phi : \Omega_{\mathscr{D}} \to \Omega_{\mathscr{D}}'$ by $\Phi(\omega) = \omega'$ where $\pi_A(\omega') = \omega(A)$ for all $A \in \mathscr{F}_{\mathscr{D}}$. The map Φ is an embedding of $\Omega_{\mathscr{D}}$ in $\Omega_{\mathscr{D}}'$.

With this notation, we can define a map $\tilde{N} : \mathscr{X} \to \Omega_{\mathscr{D}}'$ by setting, for each $x \in \mathscr{X}$,

$$\tilde{N}(x) = \Phi(N_x(\cdot)). \tag{11.5}$$

Because N_x is r-subinvariant, N_x is in $\Omega_{\mathscr{D}}$, and so (11.5) is well defined. Let $\mathscr{B}^\pi$ be the product σ-field on $\Omega_{\mathscr{D}}'$. Since each of the functions $x \to N_x(A)$, $A \in \mathscr{F}$, is measurable, the function $\tilde{N}$ defined by (11.5) is measurable with respect to $\mathscr{B}^\pi$. Unless $N_x = N_y$ implies $x = y$, $\tilde{N}$ will not embed $\mathscr{X}$ in $\Omega_{\mathscr{D}}'$, but rather will embed the set of equivalence classes of points in $\mathscr{X}$, equivalent under the relationship $x \sim y$ when $N_x = N_y$. Since the only functions we shall consider on $\tilde{N}(\mathscr{X})$ are those which are constant on such equivalence classes, this does not affect our analysis.

Write $\mathscr{T}$ for the product topology on $\Omega_{\mathscr{D}}'$; by Tychonoff's Theorem, $\Omega_{\mathscr{D}}'$ is compact under $\mathscr{T}$, and hence the image $\tilde{N}(\mathscr{X})$ of $\mathscr{X}$ is relatively compact as a subset of $\Omega_{\mathscr{D}}'$. Write $\tilde{\mathscr{X}}$ for the $\mathscr{T}$-closure of $\tilde{N}(\mathscr{X})$ in $\Omega_{\mathscr{D}}'$, and $\tilde{\mathscr{F}}$ for the σ-field

$$\tilde{\mathscr{F}} = \{B : B = A \cap \tilde{\mathscr{X}}, A \in \mathscr{B}^\pi\}.$$

Any measure λ on $\mathscr{F}$ induces a measure $\tilde{\lambda}$ on $\tilde{\mathscr{F}}$ by

$$\tilde{\lambda}(\tilde{B}) = \lambda(\tilde{N}^{-1}(\tilde{B})), \qquad \tilde{B} \in \tilde{\mathscr{F}};$$

thus from (11.1) we can find a sequence of probability measures $\tilde{\lambda}_n$ on $\tilde{\mathscr{F}}$ such that

$$Q(A) = \lim_{n\to\infty} \int_{\tilde{\mathscr{X}}} N_x(A) \tilde{\lambda}_n(dx), \qquad A \in \mathscr{F}_{\mathscr{D}}. \tag{11.6}$$

For $x \in \tilde{N}(\mathscr{X})$, $N_x(A) = \pi_A(x)$, where π_A is the projection map at A; so we can write (11.6) as

$$Q(A) = \lim_{n\to\infty} \int_{\tilde{\mathscr{X}}} \pi_A(\omega) \tilde{\lambda}_n(d\omega) \qquad A \in \mathscr{F}_{\mathscr{D}}. \tag{11.7}$$

For each fixed $A \in \mathscr{F}_{\mathscr{D}}$, $\pi_A(\cdot)$ is a continuous function on $\Omega_{\mathscr{D}}'$ in the product topology; and $\tilde{\mathscr{X}}$ is closed, and hence compact, in the product topology.

Since $\tilde{\lambda}_n(\tilde{\mathscr{X}}) = 1$ for $n = 1, 2, \cdots$ and $\tilde{\mathscr{X}}$ is compact, $\{\tilde{\lambda}_n\}$ contains a subnet $\{\tilde{\lambda}_{n_\alpha}\}$ which converges *weakly* to a probability measure $\tilde{\lambda}$ on $\tilde{\mathscr{F}}$; that is, for all continuous functions h from $\tilde{\mathscr{X}}$ to the real line,

$$\lim_{n_\alpha} \int_{\tilde{\mathscr{X}}} h(\omega)\tilde{\lambda}_{n_\alpha}(d\omega) = \int_{\tilde{\mathscr{X}}} h(\omega)\tilde{\lambda}(d\omega) . \tag{11.8}$$

Since π_A is continuous for every $A \in \mathscr{F}_{\mathscr{D}}$, applying (11.8) to (11.7) shows that

$$Q(A) = \int_{\tilde{\mathscr{X}}} \pi_A(\omega)\tilde{\lambda}(d\omega) . \tag{11.9}$$

If $\omega \in \tilde{\mathscr{X}}$, then ω is a set function on $\mathscr{F}_{\mathscr{D}}$ (put $\omega(A) = \pi_A(\omega)$, $A \in \mathscr{F}_{\mathscr{D}}$) and whilst ω may not be σ-additive on each of the σ-fields $\mathscr{F}_\Delta$, $\Delta \in \mathscr{D}$, it is easy to see that ω must be a finitely additive set function on each $\mathscr{F}_\Delta$, since $\tilde{\mathscr{X}}$ is the closure of $\tilde{N}(\mathscr{X})$. We summarise the results of this section in

THEOREM 10. *Let Q be an r-invariant measure for $\{X_n\}$. Then there is a probability measure $\tilde{\lambda}$ on $(\tilde{\mathscr{X}}, \tilde{\mathscr{F}})$ such that, for any $A \in \mathscr{F}_{\mathscr{D}}$,*

$$Q(A) = \int_{\tilde{\mathscr{X}}} \omega(\cdot)\tilde{\lambda}(d\omega)$$

where $\omega(\cdot)$ is a finitely additive set function on $\mathscr{F}_{\mathscr{D}}$ for each $\omega \in \tilde{\mathscr{X}}$.

12. Equicontinuity and an r-invariant representation. We wish to find conditions on $\{X_n\}$ which will enable us to assert that the closure $\tilde{\mathscr{X}}$ of $\tilde{N}(\mathscr{X})$ in the product topology $\mathscr{T}$ on $\Omega_{\mathscr{D}}'$ defined by (11.4) lies in $\Omega_{\mathscr{D}}$ defined by (11.3). For each $\Delta \in \mathscr{D}$, let $\tilde{N}_\Delta : \mathscr{X} \to \Omega_\Delta'$ be given by

$$\tilde{N}_\Delta(x) = \Phi_\Delta(N_x(\cdot)) ,$$

where Φ_Δ is the natural embedding of Ω_Δ in Ω_Δ'; let $\mathscr{T}_\Delta$ be the product topology on Ω_Δ'. Clearly, $\tilde{\mathscr{X}} \subseteq \Omega_{\mathscr{D}}$ if and only if the $\mathscr{T}_\Delta$-closure of $\tilde{N}_\Delta(\mathscr{X})$ in Ω_Δ' is contained in Ω_Δ for every $\Delta \in \mathscr{D}$. The latter is equivalent to asking that $\{N_x(\cdot), x \in \mathscr{X}\}$ be relatively compact in Ω_Δ for all $\Delta \in \mathscr{D}$, when each Ω_Δ is equipped with the topology of setwise convergence on the sets of $\mathscr{F}_\Delta$. This topology on measure spaces has been studied by Topsøe (1970), who called this the *s-topology* on Ω_Δ, and Gänssler (1971), who called it the $\mathscr{T}_{\mathscr{F}_\Delta}$*-topology*.

Now let Δ be a fixed set in $\mathscr{D}$; from Proposition 8.1 $\sup_{x \in \mathscr{X}} N_x(\Delta) \leqq \beta_\Delta < \infty$. We shall say that $\{N_x, x \in \mathscr{X}\}$ is *equicontinuous* on Δ (cf. Gänssler (1971) 1.7) if, for any sequence of sets $\{A_k\}$, $A_k \in \mathscr{F}_\Delta$, such that $A_k \downarrow \varnothing$,

$$\lim_{j \to \infty} \sup_{x \in \mathscr{X}} N_x(A_j) = 0 . \tag{12.1}$$

The next result is a direct application of Gänssler (1971), Theorem 2.6, to the set $\{N_x, x \in \mathscr{X}\}$.

THEOREM G. *The following assertions are equivalent:*

(i) *the $\mathscr{T}_\Delta$-closure of $\tilde{N}_\Delta(\mathscr{X})$ in Ω_Δ' is a subset of Ω_Δ;*

(ii) *every sequence $\{\omega_n\}$ of measures in $\tilde{N}_\Delta(\mathscr{X})$ contains a subsequence $\{\omega_{n_k}\}$ which converges in the $\mathscr{T}_\Delta$-topology to a measure $\omega \in \Omega_\Delta$; that is*

$$\lim_{k \to \infty} \omega_{n_k}(A) = \omega(A) , \qquad A \in \mathscr{F}_\Delta ;$$

(iii) *$\{N_x, x \in \mathscr{X}\}$ is equicontinuous on Δ.*

We shall say that $\{N_x, x \in \mathscr{X}\}$ is $\mathscr{D}$-equicontinuous if $\{N_x\}$ is equicontinuous on each $\Delta \in \mathscr{D}$.

THEOREM 11. *Let* $\{N_x, x \in \mathscr{X}\}$ *be* $\mathscr{D}$-*equicontinuous. If* Q *is r-invariant for* $\{X_n\}$, *then there is a probability measure* $\tilde{\lambda}$ *on* $(\tilde{\mathscr{X}}, \tilde{\mathscr{F}})$ *such that*

$$G(A) = \int_{\tilde{\mathscr{X}}} \omega(A)\tilde{\lambda}(d\omega) \tag{12.2}$$

for every $A \in \mathscr{F}_{\mathscr{D}}$, *where for each* $\omega \in \tilde{\mathscr{X}}$,

(i) $\omega(\cdot)$ *is a measure on* $\mathscr{F}_\Delta$ *for each* $\Delta \in \mathscr{D}$ *(and hence can be extended as usual to a measure on* $\mathscr{F}$*);*

(ii) *if* $\mathscr{F}$ *is countably generated, for each* $\omega \in \tilde{\mathscr{X}}$ *there is a sequence* $\{\zeta(1), \zeta(2), \cdots\}$ *of points in* $\mathscr{X}$ *such that*

$$\omega(A) = \lim_{j\to\infty} N_{\zeta(j)}(A) \tag{12.3}$$

for every $A \in \mathscr{F}_{\mathscr{D}}$.

PROOF. Both (12.2) and (i) are consequences of Theorem 10 and the equivalence between (i) and (iii) of Theorem G. Suppose that $\mathscr{F}$ is generated by the countable collection of sets $\mathscr{F}^\circ = (F_1, F_2, \cdots)$ and write $\mathscr{F}_\Delta{}^\circ = \{F_1 \cap \Delta, F_2 \cap \Delta, \cdots\}$ for $\Delta \in \mathscr{D}$. The set of measures $\tilde{N}_\Delta(\mathscr{X}) \subseteq \Omega_\Delta$ is equicontinuous on Δ, and from Gänssler ((1971) 1.11) for this *equicontinuous* set the topology $\mathscr{T}_\Delta$ of setwise convergence on $\mathscr{F}_\Delta$ is equivalent to the topology $\mathscr{T}_\Delta{}^\circ$ of setwise convergence on the sets of $\mathscr{F}_\Delta{}^\circ$; and $\mathscr{T}_\Delta{}^\circ$ is metrizable. (One can set, for example,

$$d(\omega_1, \omega_2) = \sum_j 2^{-j}|\omega_1(F_j \cap \Delta) - \omega_2(F_j \cap \Delta)|$$

for $\omega_1, \omega_2 \in \tilde{N}_\Delta(\mathscr{X})$.) Thus the elements of $\tilde{\mathscr{X}}_\Delta$, the $\mathscr{T}_\Delta$-closure of $\tilde{N}_\Delta(\mathscr{X})$, are limits of sequences of points in $\tilde{N}_\Delta(\mathscr{X})$; a diagonal argument over $\Delta \in \mathscr{D}$ leads to (12.3). □

When $r = 1$, results similar to (12.2) have been proved when $\mathscr{X}$ and $\{X_n\}$ satisfy various topological considerations by, for example, Kunita and Watanabe (1967). However, the identification of points in $\tilde{\mathscr{X}}$ for arbitrary separable $\mathscr{F}$ with limits of measures $N_{\zeta(j)}$ when the equicontinuity condition is satisfied appears to be new for uncountable $\mathscr{X}$, even for $r = 1$.

Since identifying $\mathscr{D}$ may not always be possible, the assumption of $\mathscr{D}$-equicontinuity may seem difficult to check. However, if $\mathscr{K}$ is an arbitrary partition for which $\{N_x\}$ is $\mathscr{K}$-equicontinuous (equicontinuous on every $K \in \mathscr{K}$), then $\{N_x\}$ is also $\mathscr{D}'$-equicontinuous, where $\mathscr{D}' = \{K \cap \Delta, K \in \mathscr{K}, \Delta \in \mathscr{D}\}$, and the conclusions of the theorem continue to hold with $\mathscr{D}'$ in place of $\mathscr{D}$. We conclude this section with a sufficient condition for $\mathscr{D}'$-equicontinuity, in terms of the one-step transition probabilities $P(\cdot, \cdot)$, whose corollary can be checked for an arbitrary partition $\mathscr{K}$.

We define the measures $N_x^{(1)}(\cdot)$ by

$$N_x^{(1)}(A) = P(x, A)/P(x, \Delta'), \qquad A \in \mathscr{F}_{\Delta'}, x \in \mathscr{X} \tag{12.4}$$

for each $\Delta' \in \mathscr{D}'$, where (12.4) is taken to be zero if $P(x, \Delta') = 0$.

PROPOSITION 12.1. *If* $\{N_x^{(1)}, x \in \mathscr{X}\}$ *is* $\mathscr{D}'$*-equicontinuous, then* $\{N_x, x \in \mathscr{X}\}$ *is* $\mathscr{D}'$*-equicontinuous.*

PROOF. Let Δ' be an element of $\mathscr{D}'$, and let $\{A_k\}$, $A_k \downarrow \emptyset$, be a sequence of sets in $\mathscr{F}_{\Delta'}$. Put

$$\varepsilon_k = \sup_{x \in \mathscr{X}} N_x^{(1)}(A_k)\,; \tag{12.5}$$

by hypothesis, $\varepsilon_k \downarrow 0$. Write

$$N_x^{(n)}(A) = \sum_1^n P^j(x, A)r^j / \sum_1^n P^j(x, \Delta')r^j\,, \qquad A \in \mathscr{F}_{\Delta'} \tag{12.6}$$

where as in (12.4), $N_x^{(n)}(A) \equiv 0$ if the denominator of (12.6) is zero. Assume inductively that

$$\sup_{x \in \mathscr{X}} N_x^{(n)}(A_k) \leqq \varepsilon_k\,; \tag{12.7}$$

$$\begin{aligned} \sum_1^{n+1} P^j(x, A_k)r^j &= rP(x, A_k) + r \int_{\mathscr{X}} P(x, dy) \sum_1^n P^j(y, A_k)r^j \\ &= rP(x, A_k) + r \int_{\mathscr{X}} P(x, dy) N_y^{(n)}(A_k) \sum_1^n P^j(y, \Delta')r^j \\ &\leqq rP(x, A_k) + \varepsilon_k \sum_2^{n+1} P^j(x, \Delta')r^j \\ &\leqq \varepsilon_k \sum_1^{n+1} P^j(x, \Delta')r^j \end{aligned}$$

from (12.5) and (12.7); hence (12.7) holds for all n. Since

$$\begin{aligned} \lim_{n\to\infty} N_x^{(n)}(A) &= \sum_1^\infty P^j(x, A)r^j / \sum_1^\infty P^j(x, \Delta')r^j \\ &= N_x'(A) \end{aligned}$$

for each $A \in \mathscr{F}_{\Delta'}$ and $x \in \mathscr{X}$, there exists $n(j)$ (depending on x) such that

$$\begin{aligned} N_x'(A_j) &\leqq \varepsilon_j + N_x^{(n(j))}(A_j) \leqq 2\varepsilon_j\,; \\ N_x(A_j) &= N_x'(A_j)[G_r(x, \Delta')/G_r(x, D)] \\ &\leqq 2\varepsilon_j \beta_\Delta \end{aligned}$$

since $\Delta' \subseteqq \Delta$ for some $\Delta \in \mathscr{D}$. Hence $\{N_x\}$ is $\mathscr{D}'$-equicontinuous. □

COROLLARY. *If* $\mathscr{F}$ *is separable and Condition* I′ *holds, let* $p(x, y)$ *be the density of* $P(x, \cdot)$ *with respect to* M. *If there is a partition* $\mathscr{K}$ *of* $\mathscr{X}$ *and, for each* $K \in \mathscr{K}$, *a real number* $\nu(K)$ *such that for all* $x \in \mathscr{X}$

$$\sup_{y \in K} p(x, y) \leqq \nu(K) \inf_{y \in K} p(x, y)\,,$$

then $\{N_x\}$ *is* $\mathscr{D}'$*-equicontinuous.*

PROOF. Fix $\Delta' = K \cap \Delta$, $K \in \mathscr{K}$, $\Delta \in \mathscr{D}$, and let $A_k \downarrow \emptyset$, $A_k \in \mathscr{F}_{\Delta'}$; and assume $M(\Delta') > 0$ (otherwise $\{N_x^{(1)}\}$ is trivially equicontinuous on Δ'). Then for any $x \in \mathscr{X}$

$$\begin{aligned} P(x, A_k) &= \int_{A_k} p(x, y)M(dy) \\ &\leqq \sup_{y \in A_k} p(x, y)M(A_k) \\ &\leqq \nu(K) \inf_{y \in \Delta'} p(x, y)M(A_k) \\ &\leqq \nu(K)P(x, \Delta')[M(\Delta')]^{-1}M(A_k) \end{aligned}$$

and so $\sup_x P(x, A_k)/P(x, \Delta') \leqq \nu(K)[M(\Delta')]^{-1}M(A_k) \downarrow 0$, $A_k \downarrow \emptyset$. □

It is easy to construct examples of Markov chains on uncountable state spaces which satisfy the conditions of the corollary, and hence for which Theorem 11 holds (with $\mathscr{D}'$ in place of $\mathscr{D}$). The sufficient condition for $\mathscr{D}'$-equicontinuity of Proposition 12.1 seems to me to be considerably stronger than necessary, and I

CONJECTURE. A necessary and sufficient condition for $\{N_x, x \in \mathscr{X}\}$ to be $\mathscr{D}'$-equicontinuous for some refinement $\mathscr{D}'$ of $\mathscr{D}$ is that $\{P(x, \cdot), x \in \mathscr{X}\}$ be $\mathscr{K}$-equicontinuous for some partition $\mathscr{K}$ of $\mathscr{X}$.

13. A generalization of the Harris–Veech condition. In this section we shall assume that $\mathscr{D} = (\Delta(j))$ is such that

B(i) $\{N_x, x \in \mathscr{X}\}$ is $\mathscr{D}$-equicontinuous
B(ii) $\mathscr{F}$ is countably generated
B(iii) for each $\Delta \in \mathscr{D}$, if $A \in \mathscr{F}_\Delta^+$

$$ {}_AG_r(\zeta, A) \leqq 1 \qquad \text{for all} \quad \zeta \in A . \tag{13.1} $$

From Proposition 4.1 in I, if $\mathscr{D}$ satisfies B(i) then there is a refinement $\mathscr{D}'$ of $\mathscr{D}$ satisfying B(i), B(iii), and we could carry out the analysis above with $\mathscr{D}'$ in place of $\mathscr{D}$: thus the third assumption is trivial, and is merely to save notation.

Under these conditions we have

LEMMA 13.1. *A necessary and sufficient condition for the existence of an r-invariant measure for $\{X_n\}$ is the existence of a point $\omega \in \tilde{\mathscr{X}}$ such that $\omega(\cdot)$ is r-invariant for $\{X_n\}$.*

PROOF. Choose $\omega \in \tilde{\mathscr{X}}$; from B(i), $\omega(\cdot)$ is a measure and from B(ii) $\omega(\cdot)$ is given by (12.3) for some sequence $\{\zeta(j)\}$ of points in $\mathscr{X}$; and since N_x is r-subinvariant, $x \in \mathscr{X}$, $\omega(\cdot)$ is r-subinvariant (use Fatou's lemma). Write $\tilde{\mathscr{X}}_I$ for the set of r-invariant measures in $\tilde{\mathscr{X}}$. Suppose Q is r-invariant: from Theorem 11 there exists $\tilde{\lambda}$ such that (12.2) holds. It is easy to show that $\tilde{\lambda}(\tilde{\mathscr{X}} \backslash \tilde{\mathscr{X}}_I) = 0$, and so $\tilde{\mathscr{X}}_I$ is not empty, since $\tilde{\lambda}(\tilde{\mathscr{X}}) = 1$. This proves necessity, and sufficiency is trivial. □

Using this lemma, we prove a generalization of the celebrated theorem of Harris (1957) and Veech (1963) for 1-transient chains on $\mathscr{X} = \mathbb{Z}$. This was extended to r-transient chains on $\mathbb{Z}$ by Pruitt (1964), and the sufficiency part of the theorem was proved by Yang (1971) for $\mathscr{X}$ a σ-compact metric space and $r = 1$ under fairly weak continuity conditions on P. The direct connection between the Martin boundary construction and the Harris–Veech result, as contained in Lemma 13.1, seems to have been first noticed for $\mathscr{X} = \mathbb{Z}$ by Moy (1967).

THEOREM 12. *Under the conditions of this section, a necessary and sufficient condition for the existence of an r-invariant measure for $\{X_n\}$ is the existence of a sequence*

$\{\zeta(1), \zeta(2), \cdots\}$ *of states in* $\mathscr{X}$ *such that*

$$\lim_{j\to\infty} \lim_{k\to\infty} [P(\zeta(k), A) + \int_{\Delta^*(j)} {}_A G_r(\zeta(k), dy)P(y, A)]/G_r(\zeta(k), D) = 0 \tag{13.2}$$

for every $A \in \mathscr{F}_{\mathscr{D}}$, *where* $\Delta^*(j) = \bigcup_{n\geq j} \Delta(n)$.

PROOF. Let ω be a point in $\tilde{\mathscr{X}}$, and let $\{\zeta(k)\}$ be a sequence of points in $\mathscr{X}$ such that (12.3) holds. We show that (13.2) is necessary and sufficient for $\omega(\cdot)$ to be r-invariant, and the theorem then follows from Lemma 13.1.

Define $\Delta_*(j) = \bigcup_{n<j} \Delta(n)$, so that $\mathscr{X} = \Delta^*(j) \cup \Delta_*(j)$. For $j \geqq 1$, (12.3) implies that

$$\omega(A) = \lim_{k\to\infty} N_{\zeta(k)}(A)$$

for every $A \in \mathscr{F}_{\Delta_*(j)}$; this is equivalent to

$$\int_B N_{\zeta(k)}(dy)h(y) \to \int_B \omega(dy)h(y) \tag{13.3}$$

for any $B \in \mathscr{F}_{\Delta_*(j)}$ and any measurable bounded function h on B. (Gänssler (1971) 2.15). We shall use this equivalence twice.

Let A be a fixed set in $\mathscr{F}_{\mathscr{D}}$, and suppose J is such that $A \subseteqq \Delta_*(J)$. For $j > J$, a last exit decomposition gives

$$\begin{aligned} G_r(\zeta(k), A) &= rP(\zeta(k), A) + r \int_{\mathscr{X}} G_r(\zeta(k), dy)P(y, A) \\ &= rP(\zeta(k), A) + r \int_{\Delta_*(j)} G_r(\zeta(k), dy)P(y, A) \\ &\quad + r \int_{\Delta^*(j)} [{}_A G_r(\zeta(k), dy) + \int_A G_r(\zeta(k), dx)\, {}_A G_r(x, dy)]P(y, A)\,; \end{aligned}$$

dividing through by $G_r(\zeta(k), D)$, we get

$$\begin{aligned} N_{\zeta(k)}(A) &= [rP(\zeta(k), A) + r \int_{\Delta^*(j)} {}_A G_r(\zeta(k), dy)P(y, A)]/G_r(\zeta(k), D) \\ &\quad + r \int_{\Delta_*(j)} N_{\zeta(k)}(dy)P(y, A) \\ &\quad + r \int_A N_{\zeta(k)}(dx) \int_{\Delta^*(j)} {}_A G_r(x, dy)P(y, A)\,. \end{aligned} \tag{13.4}$$

Since $P(\cdot, A)$ is a bounded function on $\Delta_*(j)$, the second term in (13.4) tends as $k \to \infty$ to

$$r \int_{\Delta_*(j)} \omega(dy)P(y, A)$$

from (13.3); this in turn tends as $j \to \infty$ to $r \int_{\mathscr{X}} \omega(dy)P(y, A)$. For $x \in A$, the function

$$\begin{aligned} \int_{\Delta^*(j)} {}_A G_r(x, dy)P(y, A) &\leqq \int_{A^c} {}_A G_r(x, dy)P(y, A) \\ &\leqq {}_A G_r(x, A) \\ &\leqq 1 \end{aligned}$$

from B(iii); hence again by (13.3), the third term in (13.4) tends with k to

$$r \int_A \omega(dx) \int_{\Delta^*(j)} {}_A G_r(x, dy)P(y, A)\,. \tag{13.5}$$

But again by B(iii),

$$r \int_A \omega(dx) \int_{A^c} {}_A G_r(x, dy)P(y, A) \leqq r\omega(A) < \infty\,,$$

and so as $j \to \infty$, (13.5) tends to zero. Taking limits with k and j in (13.4) thus shows that

$$\omega(A) = r \int_{\mathscr{X}} \omega(dy)P(y, A)$$

if and only if (13.2) holds for A, and so ω is r-invariant if and only if (13.2) holds for all $A \in \mathscr{F}_{\mathscr{D}}$. $\square$

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